Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 28 (2025), No. 1, pp. 161–167
DOI : 10.47974/JIM-1806

A modern secant-kind technique with quadratic of convergence

Yoksal A. Laylani [†]
Suaad M. Abdullah [§]
Department of Mathematics
College of Sciences
University of Kirkuk
Kirkuk 36001
Iraq

Basim Abbas Hassan *
Department of Mathematics
College of Computers Sciences and Mathematics
University of Mosul
Mosul 41002
Iraq

Abstract

The advancement of the second approximation is employed to introduce a modified version of the popular Secant technique, which is used to solve nonlinear optimization problems involving a single variable with no constraints. By utilizing the Taylor series expansion, an iterative formula is constructed, incorporating an approximation of $\ell(\chi)$ second derivative. It is proven that these novel methods achieve quadratic convergence. when evaluated against the Newton technique using seven types of test function, A new approach performs well, as observed through real-world application.

Subject Classification: *90C30, 65K05, 49M37.*

Keywords: *Modern secant-kind method, Iteration technique, Functions test.*

[†] *ORCID-ID:* **0000-0003-4898-5083**
[§] *ORCID-ID:* **0009-0001-9205-414X**
* *ORCID-ID:* **0000-0003-3510-9818**

[†] *E-mail:* **yuksel@uokirkuk.edu.iq**
[§] *E-mail:* **suaad_m@ uokirkuk.edu.iq**
* *E-mail:* **basimah@uomosul.edu.iq; (Corresponding Author)**
basimabas39@gmail.com

1. Introduction

The numerical function $\ell(\varkappa)$, which depends only on $\varkappa$, one independent variable. Let us assume, [1], that we want to find the value of $\varkappa$ where $\ell(\varkappa)$, is the least value. Locating a local minimum using gradient-based minimization techniques involves searching for locations that meet the optimality requirement. Discovering $\varkappa^*$ that meets the requirements and conditions for first order optimality, this as:

$$\ell'(\varkappa) = 0. \tag{1}$$

is identical to locating the roots of first derivative of the function that is to be reduced. Therefore, root finding techniques prove to be valuable in reducing functions and can be utilized to identify stationary locations. For more details, go to [2-3]. Among the several approximation techniques, Newton's approach has the best convergence, as seen in [4]. Newton's method employs a quadratic function that concurs with the objective function at a particular point to locally estimate it. You can keep using this method until you get an efficient approximation function. Newton's method has gained widespread recognition due to its requirement for assessing the initial and subsequent derivatives in each iteration, as seen by the following iteration formula:

$$\varkappa_{\iota+1} = x_\iota - \frac{\ell'(x_\iota)}{\ell''(x_\iota)}. \tag{2}$$

See[5]. Secant methods are a class of Newton-type techniques that circumvent this need by not requiring the 2nd derivative. Because of the possibility of challenges and time-consuming nature in determining the second derivative of a function, the iteration formula, commonly known as:

$$\varkappa_{\iota+1} = \varkappa_\iota - \frac{\ell'(x_\iota)(\varkappa_\iota - \varkappa_{\iota-1})}{\ell'(x_\iota) - \ell'(x_{\iota-1})}. \tag{3}$$

have been developed; in [6]. The computational effort is often dominated by the calculation of f and its derivatives. Despite the fact that several new amendment Secant-kind algorithms with greater emphasis on achieving convergence have been presented (e.g. [7-10]), some studies have concentrated on tackling nonlinear optimization problems where the second derivative is not necessary (see [4, 5, 11-19]).

We present a method called secant technique that is derived and explained using the second-order Taylor expansion. In order to evaluate the effectiveness of this new approach, the number of iterations, a commonly used criterion, is utilized and compared to the Newton method, employs seven test functions.

2. A New Secant Kind Technique

In nonlinear optimization, the class of the second-order Taylor expansion holds great importance. Regarding the second-order Taylor expansion centered of point $\varkappa_\iota$, the following equation is applicable[13]:

$$\ell(\varkappa) \cong \ell(\varkappa_\iota) + \ell'(\varkappa_\iota)(\varkappa - \varkappa_\iota) + \frac{1}{2}\ell''(\varkappa_\iota)(\varkappa - \varkappa_\iota)^2 \tag{4}$$

The derivative of may be calculated at the smallest location when it equals zero by:

$$\ell'(x) \cong \ell'(\varkappa_\iota) + \ell''(\varkappa_\iota)(\varkappa - \varkappa_\iota) = 0 \tag{5}$$

Using Equations (5) and (4), we were able to achieve the following results:

$$\ell''(\varkappa_\iota)(\varkappa - \varkappa_\iota)^2 \cong \ell(\varkappa_\iota) - \ell(\varkappa) - \frac{1}{2}\ell'(\varkappa_\iota)(\varkappa - \varkappa_\iota) \tag{6}$$

This leads to:

$$\ell''(\varkappa_\iota) = \frac{\ell(\varkappa_\iota) - \ell(\varkappa) - 1/2\ell'(\varkappa_\iota)(\varkappa - \varkappa_\iota)}{(\varkappa - \varkappa_\iota)^2} \tag{7}$$

It is possible to simplify the previous equation by replacing $\varkappa_{\iota-1}$ for $\varkappa$:

$$\ell''(\varkappa_\iota) = \frac{\ell(\varkappa_\iota) - \ell(\varkappa_{\iota-1}) - 1/2\ell'(\varkappa_\iota)(\varkappa_{\iota-1} - \varkappa_\iota)}{(\varkappa_{\iota-1} - \varkappa_\iota)^2} \tag{8}$$

Simple algebraic operations on (8), we have:

$$\ell''(\varkappa_\iota) = \frac{\frac{\ell(\varkappa_\iota) - \ell(\varkappa_{\iota-1})}{(\varkappa_{\iota-1} - \varkappa_\iota)} - 1/2\ell'(\varkappa_\iota)}{(\varkappa_{\iota-1} - \varkappa_\iota)} \tag{9}$$

The result of plugging Eq. (9) into Eq. (1) is:

$$\varkappa_{\iota+1} = \varkappa_\iota - \frac{\ell'(\varkappa_\iota)(\varkappa_{\iota-1} - \varkappa_\iota)}{\frac{f(\varkappa_\iota) - f(\varkappa_{\iota-1})}{(\varkappa_{\iota-1} - \varkappa_\iota)} - 1/2\ell'(\varkappa_\iota)} \tag{10}$$

Presented below is an algorithm that is based on the new secant-kind method mentioned above:

Stage 1. Put $\boldsymbol{\varepsilon}, \boldsymbol{\varkappa_0}$ and $\ell(\boldsymbol{x_0})$, set $\iota = 0$.
Stage 2. Allow $\iota = \iota + 1$.
Stage 3. Establish $\varkappa_\iota = \varkappa_{\iota-1} - \frac{\ell'(\varkappa_{\iota-1})}{\ell''(\varkappa_{\iota-1})}$.
Stage 4. Determine $\varkappa_{\iota+1} = \varkappa_\iota - \frac{\ell'(\varkappa_\iota)(\varkappa_{\iota-1} - \varkappa_\iota)}{\frac{f(\varkappa_\iota) - f(\varkappa_{\iota-1})}{(\varkappa_{\iota-1} - \varkappa_\iota)} - 1/2\ell'(\varkappa_\iota)}$.
Stage 5. Put an end to it if $|\varkappa_{\iota+1} - \varkappa_\iota| \leq \varepsilon$.

3. Analysis of convergence

As it is essential to the creation of optimization algorithms, we investigate the convergence of the new secant type iteration in greater detail.

Theorem 3.1: *To begin with, let $\varkappa^* \in I$ be a zero of an open interval Isufficiently smooth function $\ell: I \to \mathbb{R}$. Iterative methods with second order convergence have $\varkappa_0$ sufficiently near to $\varkappa^*$.*

Proof: The type method of the new secant is as follows:

$$\varkappa_{\iota+1} = \varkappa_\iota - \frac{\ell'(\varkappa_\iota)(\varkappa_{\iota-1} - \varkappa_\iota)}{\frac{f(\varkappa_\iota) - f(\varkappa_{\iota-1})}{(\varkappa_{\iota-1} - \varkappa_\iota)} - 1/2\ell'(\varkappa_\iota)} \tag{11}$$

By taking x^* off of both sides of equation (11), we may obtain the following outcome:

$$e_{\iota+1} = e_\iota - \frac{\ell'(\varkappa_\iota)(e_{\iota-1}-e_\iota)}{\frac{\ell(\varkappa_\iota)-\ell(\varkappa_{\iota-1})}{(e_{\iota-1}-e_\iota)} - 1/2\ell'(\varkappa_\iota)} \tag{12}$$

where $e_\iota = \varkappa_\iota - \varkappa^*$, as the approximate minimal error following ι iterations. Now, utilizing the mean value form of Taylor's theorem:

$$\ell(\varkappa^*) = \ell(\varkappa_\iota) + \ell'(\varkappa_\iota)(\varkappa^* - \varkappa_\iota) + \frac{1}{2!}\ell''(\varkappa_\iota)(\varkappa^* - \varkappa_\iota)^2 + \frac{1}{3!}\ell'''(\varsigma)(\varkappa^* - \varkappa_\iota)^2 \tag{13}$$

Since the derivative of the following equation is zero, for some ς between $\varkappa^*$ and $\varkappa_\iota$, we obtain:

$$-\ell'(\varkappa_\iota) = -e_\iota \ell''(\varkappa_\iota) + \frac{e_\iota^2}{2}\ell'''(\varsigma) \tag{14}$$

By entering Eq. (14) into Eq. (11), we may arrive at the following outcome:

$$e_{\iota+1} = e_\iota^2 \frac{1}{2}\frac{\ell'''(\varsigma)}{\ell''(\varkappa_\iota)} \tag{15}$$

It suggests a quadratic convergence order. The task of proving is completed.

4. Application Examples

To solve seven test functions (previously employed in [7]), tables will be utilized which consist of the functions and their respective minimum points, $\varkappa^*$. The competence of the new technique is assessed by comparing it to the Newton approach. The method explained in [7,14] is employed to generate the initial point, $\varkappa_\iota$, for the numerical experiments. The calculations are performed utilizing double accuracy on a Pentium processor. In a numerical examination, the search is halted whenever $|\varkappa_{\iota+1} - \varkappa_\iota| \le \varepsilon$ where $\varepsilon = 10^{-10}$occurs. The superiority of the new method over the Newton method is evident when applied to any function that requires fewer iterations for obtaining results. For ease, we use E.T. instead of Execution time, N.I. instead of Number of iterations, N.M. instead of Newton method and S.K. Newton method Secant kind.

Example 1: Initial Approximation $\varkappa_0 = 2$, Function $\ell(\varkappa) = cos(\varkappa) + (\varkappa - 2)^2$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	4	2.3542	0.5790
S.K.	4	2.3288	0.1720

Example 2: Initial Approximation $\varkappa_0 = 0.25$, Function $\ell(\varkappa) = e^{\varkappa} - 3\varkappa^2$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	4	0.2045	0.2660
S.K.	3	0.2036	0.2030

Example 3: Initial Approximation $\varkappa_0 = 1$, Function $\ell(\varkappa) = e^{-\varkappa} + \varkappa^2$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	5	0.3517	0.3130
S.K.	4	0.4004	0.2190

Example 4: Initial Approximation $\varkappa_0 = 0$, Function $\ell(\varkappa) = -\varkappa e^{-\varkappa}$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	7	1	0.3900
S.K.	5	0.8564	0.3120

Example 5: Initial Approximation $\varkappa_0 = 0.1$, Function $\ell(\varkappa) = 0.65 - 0.75/(1 + \varkappa^2) - 0.65\varkappa\, tan^{-1}(1/\varkappa)$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	6	0.4809	0.7970
S.K.	4	0.4872	0.3750

Example 6: Initial Approximation $\varkappa_0 = 2$, Function $\ell(\varkappa) = 0.5\varkappa^2 - sin(\varkappa)$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	5	0.7391	0.4210
S.K.	3	0.7427	0.1720

Example 7: Initial Approximation $\varkappa_0 = 1$, Function $\ell(\varkappa) = \varkappa^4 + 2\varkappa^2 - \varkappa - 3$.

Method	N.I.	$\varkappa_\iota$	E.T.
N.M.	7	0.2367	0.6870
S.K.	4	0.2847	0.2190

Conclusion

The target of this paper is to establish a new second derivative by constructing a new secant equation and second-order Taylor expansion. It is acceptable to conclude that, in terms of efficiency, the new secant kind technique performs better than Newton technique. It is possible for certain methods may not reach to converge to the global minimum and for others to do so only occasionally.

References

[1] S. S. Rao, Engineering Optimization: Theory and Practice (2019). [Online]. Available: https://doi.org/10.1002/9781119454816.

[2] E. S. Al-Rawi, B. A. Hassan, and B. M. Sulaiman, "A new secant type method for solving one-variable functions," *Italian J. Pure Appl. Math.,* vol. 47 (2022).

[3] E. Kahya, "Modified Secant-type methods for unconstrained optimization," *Appl. Math. Comput.,* vol. 181, no. 2 (2006). [Online]. Available: https://doi.org/10.1016/j.amc.2006.03.003.

[4] A. Y. Özban, "Some new variants of Newton's method," *Appl. Math. Lett.,* vol. 17, no. 6 (2004). [Online]. Available: https://doi.org/10.1016/S0893-9659(04)90104-8.

[5] S. Weerakoon and T. G. I. Fernando, "A variant of Newton's method with accelerated third-order convergence," *Appl. Math. Lett.,* vol. 13, no. 8 (2000). [Online]. Available: https://doi.org/10.1016/S0893-9659(00)00100-2.

[6] B. M. Sulaiman, B. A. Hassan, and R. M. Sulaiman, "A new secant method for minima one-variable problems," *J. Interdiscip. Math.,* vol. 26, no. 6, pp. 1015-1021 (2023). [Online]. Available: https://doi.org/10.47974/JIM-1601.

[7] B. A. Hassan and E. S. Al-Rawi, "A modified Newton's method for solving functions of one variable," *Italian J. Pure Appl. Math*., vol. 46, pp. 577–582 (2021).

[8] H. H. H. Homeier, "A modified Newton method with cubic convergence: The multivariate case," *J. Comput. Appl. Math.*, vol. 169, no. 1 (2004). [Online]. Available: https://doi.org/10.1016/j.cam.2003.12.041.

[9] H. H. H. Homeier, "On Newton-type methods with cubic convergence," *J. Comput. Appl. Math.*, vol. 176, no. 2 (2005). [Online]. Available: https://doi.org/10.1016/j.cam.2004.07.027.

[10] Y. X. Yuan, "A modified BFGS algorithm for unconstrained optimization," *IMA J. Numer. Anal.*, vol. 11, no. 3, pp. 325-338 (1991). [Online]. Available: https://doi.org/10.1093/imanum/11.3.325.

[11] M. Frontini and E. Sormani, "Some variant of Newton's method with third-order convergence," *Appl. Math. Comput.,* vol. 140, no. 2-3 (2003). [Online]. Available: https://doi.org/10.1016/S0096-3003(02)00238-2.

[12] Z. F. Salih, B. A. Hassan, and B. M. Sulaiman, "A new secant type method with quadratic-order convergence," *Eur. J. Pure Appl. Math.,* vol. 15, no. 3 (2022). [Online]. Available: https://doi.org/10.29020/nybg.ejpam.v15i3.4460.

[13] M. R. Farahani, S. Jafari, S. A. Mohiuddine, and M. Cancan, "Intuitionistic fuzzy stability of generalized additive set-valued functional equation via fixed point method," *Mathematical Statistician and Engineering Applications*, vol. 71, no. 3s3, pp. 142-154 (2022). [Online]. Available: https://www.philstat.org.ph/index.php/MSEA/article/view/355.

[14] M. R. Farahani, J. E. N. Valdes, M. Cancan, S. Jafari, and C. Park, "Fuzzy stability in the sense of Hyers-Ulam-Rassias of a functional equation on quadratic forms," *J. Interdiscip. Math.*, vol. 26, no. 6, pp. 1355-1365 (2023). [Online]. Available: https://doi.org/10.47974/JIM-1633.

[15] S. M. Kadham and M. A. Mustafa, "Medical applications of the new-transform," *J. Interdiscip. Math.*, vol. 26, no. 6, pp. 1341-1353 (2023). [Online]. Available: https://doi.org/10.47974/JIM-1632.

[16] A. H. Alwan, "Small intersection graph of subsemimodules of a semi¬module," *Commun. Combinatorics Cryptogr. Comput. Sci.*, vol. 2022, no. 1, pp. 15-22 (2022).

[17] S. Ediz, M. Cancan, and M. R. Farahani, "Eccentric connectivity and connective eccentricity indices of generalized Petersen graphs," *Commun. Combinatorics Cryptogr. Comput. Sci.*, vol. 2021, no. 1, pp. 1-6 (2021).

[18] R. Ponraj, S. Prabhu, and M. Sivakumar, "Pair mean cordial labeling of total graph of some graphs and prism-related graphs," *Commun. Combinatorics Cryptogr. Comput. Sci.*, vol. 2024, no. 2, pp. 181-192 (2024).

[19] F. O. Oduol and I. O. Okoth, "Counting vertices among all noncrossing trees by levels and degrees," *Commun. Combinatorics Cryptogr. Comput. Sci.*, vol. 2024, no. 2, pp. 193-212 (2024).

Received June, 2024