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0-Semigroup of *g*-transformation

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Abstract

In this paper, we introduce a novel type of semi group which is called the g-C_0-semigroup. We get the new relation between C0-semigroup and integral transformations by using g-transformation to induce certain type of c0-semigroup. We give some results and some theorems on this concept.

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Introduction

One-parameter semi group theory of linear operators in Banach spaces was started in this century [3, 5]. Semigroups are highly useful for solving equations and studying the behavior of their solutions.. The semigroup method is succeeded in solving import problems [2, 3]. Also, it is considered one of the methods used for solving the differential equation by integral transformations [6, 7]. There are several types of integral transformations like Laplace, Foureir

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transformations, which are celebrated works of Laplace (1749-1827) [13] on mathematical application within 1780. Foureir's Laplace methods are based on the kerenal of their transformations [14, 15], but there are also other integral transformations such as Sumudu, Millin, Elzaki, and Temimi [1, 10-13]. There are a lot of integral transformations that can be used to solve certain types of linear problems, but only a few of them can be used to solve nonlinear problems directly [9]. In this paper, one parameter semi group is defined by using generalized transformation as a link between the theory of semi groups and integral transformations to facilitate finding the solution and applying the integral transformation [4,8].

1. Fundamental Concepts

Definition 1.1. [6]: If h(x) is piecewise continuous function and satisfies $|h(x)| \leq Me^{kx}, x \in [0,\infty)$ then g-transformation g(h(x)) can be defined as following integral

$$H_g(s) = g\big(h(x)\big) = \rho(s)\int_0^\infty e^{-\beta(s)x}h(x)dx, \rho(s) \neq 0$$

Such that integral is convergent and $\big\|g\big(h(x)\big)\big\| \leq \frac{\rho(s)M}{k-\beta(s)}$

1.2 *g-Transformation of some selected function [6]:*

ID	h(x)	$g\big(h(x)\big) = \rho(s)\int_0^\infty e^{-\beta(s)x}h(x)dx$
1	A, A constant	$A\frac{\rho(s)}{\beta(s)}$
2	sin(ax)	$\frac{a\rho(s)}{(\beta(s))^2 + a^2}$
3	Cos(ax)	$\frac{\beta(s)\rho(s)}{(\beta(s))^2 + a^2}$
4	x^n	$\frac{n!\,\rho(s)}{(\beta(s))^{n+1}}$

Proposition 1.3. [14]: Let h(x) be a piecewise function, where $x \in [0,\infty), |h(x)| \leq Me^{kx}, g\big(h(x)\big) = H_g(s)$ exists, then

1. $g\big(h(x+r)\big) = e^{\beta(s)r}[H_g(s) - \rho(s)\int_0^r e^{-\beta(s)x}h(x)dx$
2. $g\big(e^{rx}h(x)\big) = \rho(s)L(h(x), \beta(s) - r)$
   where L(.)stands for the Laplace transformation.

Proposition 1.4. [6]:

1. $g(h'(x)) = \beta(s)g\big(h(x)\big) - \rho(s)h(0)$
2. $g(h''(x)) = \beta^2(s)g\big(h(x)\big) - \beta(s)\rho(s)h(0) - \rho(s)h'(0)$

3. $g(h^{(n)}(x)) = \beta^n(s)g(h(x)) - \rho(s)\sum_{k=0}^{n-1}\beta^{n-1-k}(s)\,h^{(k)}(0)$

Proposition 1.5: Let h(x) be a piecewise function where $x \in [0,\infty)$, $|h(x)| \leq Me^{kx}$, $g(h(x)) = H_g(s)$ then

1. $g^{-1}\left(\frac{H_g(s)}{\beta(s)}\right) = \int_0^\infty h(x)\,dx$
2. $g^{-1}\left(\beta(s)H_g(s)\right) = h'(x) + h(0)\delta(x)$
3. $g(U(x-a)h(x-a)) = e^{-\beta(s)a}\,H_g(s)$

Where $U(x-a)$ and $\delta(x)$ are Unite-step function and Delta-Dirac function, respectively?

Proof:

1. Let $r(x) = \int_0^x h(t)dt$ then $r'(x) = h(x)$, Now $g\left(r'(x)\right) = \beta(s)g(r(x)) - \alpha(s)r(0)$.
   Since $r(0) = \int_0^0 h(t)dt$ and $r'(x) = h(x)$ thus $g(h(x)) = H_g(s) = \beta(s)g(r(x))$
   $\frac{H_g(s)}{\beta(s)} = g(r(x)) = g(\int_0^x h(t)dt)$ therefore $g^{-1}\left(\frac{H_g(s)}{\beta(s)}\right) = \int_0^\infty h(x)\,dx$
2. Since $g\left(h'(x)\right) = \beta(s)g(h(x)) - \rho(s)h(0)$ then
   $$g^{-1}\left(\beta(s)H_g(s)\right) = h'(x) + g^{-1}(\rho(s)h(0)) = h'(x) + h(0)g^{-1}(\rho(s)) = h'(x) + h(0)\delta(x)$$
3. $g(U(x-a)h(x-a)) = \rho(s)\int_0^\infty e^{-\beta(s)x}U(x-a)h(x-a)dx$
   $= \rho(s)\int_a^\infty e^{-\beta(s)x}h(x-a)dx$, put $u = x - a$ then we get
4. $g(U(x-a)h(x-a)) = \rho(s)\int_0^\infty e^{-\beta(s)(u+a)}h(u)du =$
   $$\rho(s)e^{-\beta(s)a}\int_0^\infty e^{-\beta(s)u}h(u)du$$
   $= e^{-\beta(s)a}\,H_g(s)$

Definition 1.6. [5]: Let $W(t), 0 \leq t \leq \infty$ be a family of linear operators in Banach space X, then we say that W(t) is a semigroup on X if:

1. $W(t)\theta(x) = \theta(x)$
2. $W(t+s)\theta(x) = W(t)W(s)\theta(x)$ for every $t, s > 0$.
   If we added the condition:
3. $\lim_{t\to 0} W(t)\ \theta(x) = \theta(x), \forall\ \theta(x) \in X$ then W(t) is called a C0-semigroup.

Definition 1.7. [5]: The linear operator A defined by:

$$A\theta(x) = \lim_{t\to 0}\frac{W(t)\theta(x) - \theta(x)}{t} = \left.\frac{\partial W(t)\theta(x)}{\partial t}\right|_{t=0}, \forall\ \theta(x) \in D(A)$$

Where $D(A) = \left\{\theta(x) \in X\colon \lim_{t\to 0}\frac{W(t)\theta(x)-\theta(x)}{t} \text{ exists}\right\}$is called generator for the semigroup W(t) on the domain D(A).

Proposition 1.8. [3]: We define the resolvent set of A on Banch space X over complex field C,as following:

$\rho(A) = \{\mu \in C\colon \mu I - A\colon \mathrm{Dom}(A) \to X \text{ bijective}\}$.The operator $R(\mu, A) = (\mu I - A)^{-1}$ is defined as resolvent operator of A at μ.

2. g-C0-semigroup.

Definition 2.1: Let $X = C^1([0,\infty))$ be a Banach space and $\|\theta(x)\| = \max_{x \in X}|\theta(x)|$ such that every function in X, has g-transformation then a family of operators:

$T_g(z)\theta(x) = g^{-1}\left(e^{-\beta(s)z}\Theta_g(s)\right), z > 0$ is worked on a space X.

Proposition 2.2: A family of operators $T_g(z)$ is C0-semigroup with addition operation. It is called g-C0-semigroup.

Proof: $T_g(0)\theta(x) = g^{-1}\left(e^{-\beta(s)0}\Theta_g(s)\right) = \theta(x)$

$$T_g(z_1)T_g(z_2)\theta(x) = T_g(z_1)g^{-1}\left(e^{-\beta(s)z_2}\Theta_g(s)\right)$$

$$= g^{-1}\left(e^{-\beta(s)z_1}g\left(g^{-1}\left(e^{-\beta(s)z_2}\Theta_g(s)\right)\right)\right) = g^{-1}\left(e^{-\beta(s)z_1}e^{-\beta(s)z_2}\Theta_g(s)\right)$$

$$= g^{-1}\left(e^{-\beta(s)(z_1+z_2)}\Theta_g(s)\right) = T_g(z_1+z_2)\theta(x)$$

3) From definition of $T_g(z)$ we have that

$\lim_{z\to 0} T_g(z)\ \theta(x) = \theta(x), \forall\ \theta(x) \in X$.Therefor $T_g(z)$ is C0-semigroup.

Proposition 2.3: A generator of the g-C0-semigroup $T_g(z)$ is $A_g\theta(x) = -\left(\theta'(x) + \theta(0)\delta(x)\right)$

Proof:

$$A_g\theta(x) = \frac{\partial T_g(z)\theta(x)}{\partial z}\bigg|_{z=0} = \frac{\partial}{\partial z}g^{-1}\left(e^{-\beta(s)z}\Theta_g(s)\right)\bigg|_{z=0}$$

$$= -g^{-1}\left(\beta(s)e^{-\beta(s)z}\Theta_g(s)\right)\bigg|_{z=0}$$

$$A_g\theta(x) = -\left(\theta'(x) + \theta(0)\delta(x)\right)$$

Proposition 2.4: The solution of the following partial differential equation $\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = 0, u(x,0) = \theta(x), \theta(0) = 0$. Can be expressed by g-C0-semigroup $T_g(z)$ as following: $u(x,t) = T_g(t)\theta(x)$

Proof: $\frac{\partial u(x,t)}{\partial t} = \frac{\partial T_g(t)\theta(x)}{\partial t} = \frac{\partial}{\partial t} g^{-1}\left(e^{-\beta(s)t}\Theta_g(s)\right) = -g^{-1}\left(\beta(s)e^{-\beta(s)t}\Theta_g(s)\right)$

By Proposition(2.3,1,2) we get:

$$\frac{\partial u(x,t)}{\partial x} = -U(x-t)\theta'^{(x)} = -\frac{\partial u(x,t)}{\partial t}$$

Therefore $\frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = 0.$

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