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Weighted statistical convergence of order $\tilde{\alpha}$ for double sequences in paranormed spaces

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Abstract

In this paper, the objective is to conceptualize weighted statistical convergence of order $\tilde{a}$ for double sequences in Paranormed spaces. Some correlations between the weighted (λ, μ)-statistical convergence of order $\tilde{a}$ of double sequences in Paranormed space and $[(\bar{N}_{(\lambda,\mu)}, u, v, a, b), g]_r^{\tilde{a}}$ - summability is examined with the help of an example. Further, we also discuss $[(\bar{N}, u, v, a, b), g]_r^{\tilde{a}}$ - summability in these spaces.

Subject Classification: *40A35, 40A05.*

Keywords: *Statistical convergence, Weighted statistical convergence, Double sequences, Paranormed space.*

1. Introduction and Preliminaries

Zygmund [34] had given the notion of statistical convergence in his the first edition monograph which was published in Warsaw in 1935. In 1951, the conceptualization of this notion was proposed by the Fast [18] and Steinhaus [31]. Furthermore, Schoenberg [32] reestablished this notion by investigating for both real and complex numbers. This notion has been applied in various fields, including measure theory, Fourier analysis, Banach spaces, number theory, and many more. With the help of 2D analogue of a natural density of subset of $\mathbb{N} \times \mathbb{N}$, Mursaleen and Edely [26] extended this concept to the double

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sequence. Several authors, including Hardy [22], Tripathy [33], Mursaleen and Mohiuddine [25], and many other studied statistical convergence for double sequences. This concept had also been studied in many spaces, including normed space, intuitionistic fuzzy normed linear spaces, probabilistic normed space, Housdroff topological space, and paranormed space. This idea was first conceptualized in paranormed space by Alotaibi and Alroqi [1] in 2012, who defined g-convergence and g-statistical convergence. Subsequently, Alghamdi and Mursaleen [2], Arani, Gordi, Soraya [3], and Ercan [12] investigated and expanded upon this idea.

Karakaya and Chishti [24] in 2009, proposed the idea of weighted statistical convergence. The definition of this concept is revised by Mursaleen et al. [28] in 2012 and establishes some connection between statistical summability and weighted statistical convergence. Ghosal [21], further altered the definition of weighted statistical convergence by introducing the constrain lim inf $p_n > 0$. The idea of weighted statistical convergence and $(\overline{N}_\lambda, p)$-summability was defined by Belen and Mohiuddine [7]. This concept had been discussed in paranormed along with $[(\overline{N}, p_n), g] -$ summability by Mikail Et, Karakas and Cinnar [17]. In 2014, Selma Altundog and Bayrom Sozbir [4] defined this notion for double sequences.Recently, Cinar and Mikail [16] studied and examined this respective notion in more generalized manner and established its correlation with summability of double sequences.

Further, the various work of weighted statistical convergence can be seen in [5-9,14-16,19,23,27,29].

Definition 1.1: [6] A function f defined as f: $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ or $\mathbb{C}$ is called double sequence. It is denoted as $y = \{y_{kl}\}$ where $k, l \in \mathbb{N}$

Definition 1.2: [11] Suppose the set $\mathcal{A} \subseteq \mathbb{N} \times \mathbb{N}$ and $|\mathcal{A}(s_1, s_2)| = \{(t_1, t_2) \in \mathcal{A}; t_1 \leq s_1$ and $t_2 \leq s_2\}$.Then, asymptotic density of the set $\mathcal{A}$ is expressed as

$$\delta^2(\mathcal{A}) = \lim inf \frac{\mathcal{A}(s_1, s_2)}{s_1 s_2}$$

When in Pringsheim [30] sense a sequence $\left\{\frac{\mathcal{A}(s_1, s_2)}{s_1 s_2}\right\}$ has a limit then double natural density of the set $\boldsymbol{\mathcal{A}}$ is expressed as

$$\delta^2(\mathcal{A}) = \lim \frac{\mathcal{A}(s_1, s_2)}{s_1, s_2}$$

In this paper, we take $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f} \in (0, 1]$, for the purpose of conciseness, we denote $\widetilde{\boldsymbol{\alpha}}$ $\boldsymbol{as}$ $(\boldsymbol{c}, \boldsymbol{d})$ and $\widetilde{\boldsymbol{\beta}}$ as $(\boldsymbol{e}, \boldsymbol{f})$. Also, define

- $\widetilde{\boldsymbol{\alpha}} \leq \widetilde{\boldsymbol{\beta}} \Leftrightarrow \boldsymbol{c} \leq \boldsymbol{e}$ and $\boldsymbol{d} \leq \boldsymbol{f}$
- $\widetilde{\boldsymbol{\alpha}} < \widetilde{\boldsymbol{\beta}} \Leftrightarrow \boldsymbol{c} < \boldsymbol{e}$ and $\boldsymbol{d} < \boldsymbol{f}$
- $\widetilde{\boldsymbol{\alpha}} \triangleq \widetilde{\boldsymbol{\beta}} \Leftrightarrow \boldsymbol{c} = \boldsymbol{e}$ and $\boldsymbol{d} = \boldsymbol{f}$
- $\widetilde{\boldsymbol{\alpha}} \in (0, 1] \Leftrightarrow \boldsymbol{c}, \boldsymbol{d} \in (0, 1]$
- $\widetilde{\boldsymbol{\beta}} \in (0, 1] \Leftrightarrow \boldsymbol{e}, \in (0, 1]$
- $\widetilde{\boldsymbol{\alpha}} \triangleq 1$ in case $\boldsymbol{c} = \boldsymbol{d} = 1$

- $\tilde{\beta} \leqq 1$ in case $e = f = 1$
- $\tilde{\alpha} > 1$ in case $c > 1$, $d > 1$

The $\tilde{\alpha}$ −double density of the set $\mathcal{A} \subseteq \mathbb{N} \times \mathbb{N}$ is

$$\delta^2 (\mathcal{A}) = \lim \frac{\mathcal{A}(s_1,s_2)}{s_1^c.s_2^d}$$

Definition 1.3: [10] Let $\{y_{t_1t_2}\}$ be the double sequence and $0 < \tilde{\alpha} \leq 1$.Then a sequence $y = \{y_{t_1t_2}\}$ is known as statistical convergence of double sequence of order $\tilde{\alpha}$ to limit T if for each $\xi > 0$

$$\lim_{s_1,s_2\to\infty} \frac{1}{s_1^c.s_2^d} \left|\{(t_1,t_2): t_1, \leq s_1 \; and \; t_2, \leq s_2: \left|y_{t_1t_2} - T\right| \geq \xi\}\right| = 0$$

Definition 1.4: [24] Let $\{u_s\}$ be a positive sequence in $\mathbb{R}$ such that $U_s = u_1 + u_2 + u_3 + \cdots + u_s$ where $u_s \neq 0 \; \forall \; s \in \mathbb{N}$ and $u_0 > 0$. Then sequence $\{y_t\}$ is known to be weighted statistically convergent to T if for each $\xi > 0$

$$\lim_{s\to\infty} \frac{1}{U_s} |\{t: t \leq s : u_t|y_t - T| \geq \xi|$$

Definition 1.5: [1] The function g is defined on a linear space Y such that g: $Y \to \mathbb{N}$ is called paranorm, $\forall \; \vartheta, \omega \; \in$ Y

- $g(\vartheta)$=0 when $\vartheta = \theta$
- $g(\vartheta)$= $g(-\vartheta)$
- $g(\vartheta + \omega)$≤ $g(\vartheta)$+ $g(\omega)$
- Suppose $\{\alpha_p\}$ is sequence of scalars with $\alpha_p \to \alpha_0$ as $p \to \infty$, $\vartheta_p \to \vartheta$ as where ϑ_p , $\vartheta \in$ Y . In the sense that $g(\vartheta_p - \vartheta)$→ 0 as $p \to \infty$,then $\alpha_p\vartheta_p \to \alpha_0\vartheta$ as $p \to \infty$,in the sense $g(\alpha_p\vartheta_p$- $\alpha_0\vartheta)$ →0 as $p \to \infty$.

When $g(\psi) = 0$ means that $\psi = \theta$ then paranorm g is known as total paranorm on $Y$and (Y, g) as total paranormed space.

Definition 1.6: [17] Let $\{u_s\}$ be a positive sequence in $\mathbb{R}$ such that $\lim\inf u_s > 0$ and $U_s = u_1 + u_2 + \ldots \cdots + u_s$where $u_s \neq 0 \; \forall \; u \in \mathbb{N}$ and $u_s > 0$. Then sequence $y = \{y_t\}$ is known to be weighted statistically convergent to T in (Y, g) for each $\xi > 0$

$$\lim_{s\to\infty} \frac{1}{U_s^\alpha} |\{t \leq U_s: u_t g(y_t - T) \geq \xi\}| = 0$$

2. Main Result

The conceptualization of weighted statistical convergence of order $\tilde{\alpha}$ for double sequences in paranormed space is studied and examined in this section

Definition 2.1: Let $\mathcal{A}\subset\mathbb{N} \times \mathbb{N}$, then $\tilde{\alpha}$-double weighted density is expressed as

$$\delta_{N}^{\tilde{\alpha}}(\mathcal{A}) = \lim_{s_1,s_2\to\infty} \frac{1}{U_{s_1}^{c} V_{s_2}^{d}} \left| \mathcal{A}_{U_{s_1}V_{s_2}}(s_1,s_2) \right| \text{ is finite}$$

where $\mathcal{A}_{U_{s_1}V_{s_2}}(s_1,s_2) = \{(t_1,t_2): t_1 \le U_{s_1} \text{ and } t_2 \le V_{s_2}: u_{t_1}v_{t_2}g(y_{t_1t_2} - T_1) \ge \xi\}$, lim inf $u_{t_1,} > 0$, lim inf $v_{t_2} > 0$

Definition 2.2: Let $\{u_{t_1}\} \ge 0$ and $\{v_{t_2}\} \ge 0$ be two sequences such that lim inf $u_{t_1} > 0$, lim inf $v_{t_2} > 0$ and

$$U_{s_1} = u_1 + u_2 + \dots \cdots + u_{s_1} > 0$$
$$V_{s_2} = v_1 + v_2 + \dots \cdots + v_{s_2} > 0$$

The double sequences $y = \{y_{t_1t_2}\}$ is known to be weighted statistically convergent of order $\tilde{\alpha}$ to T in $(Y, g)$for each $\xi > 0$

$$\lim_{s_1,s_2\to\infty} \frac{1}{U_{s_1}^{c} V_{s_2}^{d}} \left|\{(t_1,t_2): t_1 \le U_{s_1} \text{ and } t_2 \le V_{s_2}: u_{t_1}v_{t_2}g(y_{t_1t_2} - T) \ge \xi\}\right| = 0$$

It is denoted as

$$S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1t_2} = T$$

Or

$$y_{t_1t_2} \to T\left(S_{N_2}^{\tilde{\alpha}}(g)\right)$$

where, $S_{N_2}^{\tilde{\alpha}}(g)$ indicates the collection of every weighted statistically convergent of order $\tilde{\alpha}$ to T in paranormed space (Y, g).

When $\tilde{\alpha} = 1$, this concept simplifies to weighted statistical convergence for the double sequences in paranormed space whereas it is explicitly stated for $\tilde{\alpha} \in (0,1]$ but not for $\tilde{\alpha} > 1$. Let defined the sequence $\{y_{t_1t_2}\}$ such that

$$g(y_{t_1t_2}) = \begin{cases} 0, t_1+t_2 \text{ is odd} \\ 1, t_1+t_2 \text{ is even} \end{cases}$$

Then,

$$\lim_{s_1,s_2\to\infty} \frac{1}{U_{s_1}^{c} V_{s_2}^{d}} \left|\{(t_1,t_2): t_1 \le U_{s_1} \text{ and } t_2 \le V_{s_2}: u_{t_1}v_{t_2}g(y_{t_1t_2} - 1) \ge \xi\}\right| < \lim_{s_1,s_2\to\infty} \frac{\left(\frac{s_1}{2}+1\right)\left(\frac{s_1}{2}+1\right)}{{s_1}^{c}{s_2}^{d}}$$

$$\lim_{s_1,s_2\to\infty} \frac{1}{U_{s_1}^{c} V_{s_2}^{d}} \left|\{(t_1,t_2): t_1 \le U_{s_1} \text{ and } t_2 \le V_{s_2}: u_{t_1}v_{t_2}g(y_{t_1t_2} - 0) \ge \xi\}\right| < \lim_{s_1,s_2\to\infty} \frac{\left(\frac{s_1}{2}+1\right)\left(\frac{s_1}{2}+1\right)}{{s_1}^{c}{s_2}^{d}}$$

As double sequence $y = \{y_{t_1t_2}\}$ is weighted statistically convergent of order $\tilde{\alpha}$ in (Y, g) to both 0 and 1 which is not possible.

Theorem 2.3: *The double sequence $y = \{y_{t_1t_2}\}$ is weighted statistically convergent of order $\tilde{\alpha}$ in (Y, g), then is $S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1t_2}$ is unique.*

Proof: Suppose $S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1t_2} = T_1$ and $S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1t_2} = T_2$.

For given $\xi > 0$, we define

$$M_{U_{s_1}V_{s_2}}(s_1, s_2) = \lim_{s_1,s_2} \frac{1}{U_{s_1}^c V_{s_2}^d} \left|\{(t_1, t_2): t_1 \le U_{s_1} \,\&\, t_2 \le V_{s_2}: u_{t_1} v_{t_2} g(y_{t_1 t_2} - T_1) \ge \tfrac{\xi}{2}\}\right|$$

$$N_{U_{s_1}V_{s_2}}(s_1, s_2) = \lim_{s_1,s_2} \frac{1}{U_{s_1}^c V_{s_2}^d} \left|\{(t_1, t_2): t_1 \le U_{s_1} \;\; t_2 \le V_{s_2}: u_{t_1} v_{t_2} g(y_{t_1 t_2} - T_2) \ge \tfrac{\xi}{2}\}\right|$$

Since, $S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1 t_2} = T_1$ we have $\delta_{N_2}^{\tilde{\alpha}}\left(M_{U_{s_1}V_{s_2}}(s_1, s_2)\right) = 0$

Similarly, $S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1 t_2} = T_2$ we have $\delta_{N_2}^{\tilde{\alpha}}\left(N_{U_{s_1}V_{s_2}}(s_1, s_2)\right) = 0$

Let $\mathcal{A}_{U_{s_1}V_{s_2}}(s_1, s_2) = M_{U_{s_1}V_{s_2}}(s_1, s_2) \cup N_{U_{s_1}V_{s_2}}(s_1, s_2)$.

This means $\delta_{N_2}^{\tilde{\alpha}}\left(\mathcal{A}_{U_{s_1}V_{s_2}}(s_1, s_2)\right) = 0$ .Hence, $\mathcal{A}_{U_{s_1}V_{s_2}}(s_1, s_2) = 0$ is an non empty set and $\delta_{N_2}^{\tilde{\alpha}}\left(\mathcal{A}_{U_{s_1}V_{s_2}}^c(s_1, s_2)\right) = 1$

Then, we have, $g(T_1 - T_2) \le g(y_{t_1 t_2} - T_1) + g(y_{t_1 t_2} - T_2) \le \frac{\xi}{2} + \frac{\xi}{2} = \xi$

As $\xi > 0$ then $g(T_1 - T_2) = 0$.Thus $T_1 = T_2$. Hence, $S_{N_2}^{\tilde{\alpha}}(g) - lim y_{t_1 t_2}$ is unique.

Definition 2.4: Consider $\lambda = \{\lambda_{s_1}\}$ and $\mu = \{\mu_{s_2}\}$ two non-decreasing sequences of $\mathbb{R}^+$ such that $\lambda_{s_1}, \mu_{s_2} \to \infty$ as $s_1, s_2 \to \infty$

$$\lambda_{s_1+1} \le \lambda_{s_1} + 1, \lambda_1 = 0$$

$$\mu_{s_2+1} \le \mu_{s_2} + 1, \mu_1 = 0$$

Suppose $\{u_{t_1}\} \ge 0$ and $\{v_{t_2}\} \ge 0$ two sequences such that $\{u_0\} > 0$, $\{v_0\} > 0$ and

$$U_{\lambda_{s_1}} = \sum_{t_1 \in S_1} u_{t_1} \to \infty \; as \; t_1 \to \infty$$

$$V_{\mu_{t_2}} = \sum_{t_2 \in S_2} v_{t_2} \to \infty \; as \; t_2 \to \infty$$

where, $S_1 = [s_1 - \lambda_{s_1} + 1, s_1]$, $S_2 = [s_2 - \mu_{s_2} + 1, s_1]$

Then generalized weighted mean for order $\tilde{\alpha}$ is

$$\sigma_{s_1 s_2}^{ab} = \frac{1}{U_{\lambda_{s_1}}^c V_{\mu_{t_2}}^d} \sum\sum u_{t_1} v_{t_2} g(y_{t_1 t_2} - T)^r$$

$$\sigma_{s_1 s_2}^{11} = \frac{1}{U_{\lambda_{s_1}}^c V_{\mu_{t_2}}^d} \sum\sum u_{t_1} v_{t_2} g(y_{t_1 t_2})^r, \; \sigma_{s_1 s_2}^{01} = \frac{1}{U_{\lambda_{s_1}}^c} \sum_{t_1 \in S_1} u_{t_1} g(y_{t_1 s_1})^r,$$

$$\sigma_{s_1 s_2}^{01} = \frac{1}{V_{\mu_{t_2}}^d} \sum_{t_2 \in S_2} v_{t_2} g(y_{s_2 t_2})^r$$

where $r > 0$ be any real number and $(a, b) = (1,1), (0,1), (1,0)$

The double sequence $y = \{y_{t_1 t_2}\}$ is known as $\left[\left(\overline{N}_{(\lambda,\mu)}, u, v, a, b\right), g\right]_r^{\tilde{\alpha}} -$ summable of order $\tilde{\alpha}$ to T in (Y, g) if

$$\lim_{s_1,s_2\to\infty} \frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_q}} \sum\sum u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right)^r$$

we can write as

$$y_{t_1t_2} \to T\left[\left[\left(\overline{N}_{(\lambda,\mu)}, u, v, a, b\right), g\right]_r^{\tilde{\alpha}}\right]$$

Definition 2.5: The double sequence $y = \{y_{t_1t_2}\}$ is known to be weighted (λ,μ)-statistically convergent of order $\tilde{\alpha}$ to T in (Y, g) for each $\xi > 0$

$$\lim_{s_1,s_2\to\infty} \frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_q}} \left|\{(t_1,t_2): t_1 \le U_{s_1}\ and\ t_2 \le V_{s_2}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right) \ge \xi\}\right| = 0$$

It can also written as

$$S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}(g) - \lim y = T$$

where, $S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}(g)$ represents the collection of every weighted (λ,μ)-statistically convergent of order $\tilde{\alpha}$ to T in (Y, g)

Definition 2.7: Suppose r be a real positive number and $y = \{y_{t_1t_2}\}$ be the double sequence is known as $[(\overline{N}, u, v, a, b), g]_r^{\tilde{\alpha}} - summable$ of order $\tilde{\alpha}$ to T in (Y, g) when

$$\lim_{s_1,s_2\to\infty} \frac{1}{U^c_{s_1} V^d_{s_2}} \sum_{t_1=1}^{s_1} \sum_{t_2=1}^{s_2} u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right)^r$$

This can be represented as

$$y_{t_1t_2} \to T\left[[(\overline{N}, u, v, a, b), g]_r^{\tilde{\alpha}}\right]$$

where $[(\overline{N}, u, v, a, b), g]_r^{\tilde{\alpha}}$ denotes the collection of every $[(\overline{N}, u, v, a, b), g]_r^{\tilde{\alpha}} -$ summable order $\tilde{\alpha}$ to T in (Y, g)

Theorem 2.6: *If* $lim\left(\frac{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}}{U^c_{s_1} V^d_{s_2}}\right) > 0\ as\ s_1, s_2 \to \infty$, *then* $S^{\tilde{\alpha}}_{\overline{N_2}} \subset S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}$

Proof: Let $\{y_{t_1t_2}\} \to T\left[S^{\tilde{\alpha}}_{\overline{N_2}}\right]$

$$\lim_{s_1,s_2\to\infty} \frac{1}{U^c_{s_1} V^d_{s_2}} \left|\{(t_1,t_2): t_1 \le U_{s_1} \text{ and } t_2 \le V_{s_2}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right) \ge \xi\}\right| = 0$$

Now,

$$\lim_{s_1,s_2\to\infty} \frac{1}{U^c_{s_1} V^d_{s_2}} \left|\{(t_1,t_2): t_1 \le U_{s_1} \text{ and } t_2 \le V_{s_2}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right) \ge \xi\}\right|$$

$$\ge \lim_{s_1,s_2\to\infty} \frac{1}{U^c_{s_1} V^d_{s_2}} \left|\{(t_1,t_2): t_1 \le U_{\lambda_{s_1}}\ and\ t_2 \le V_{\mu_q}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right) \ge \xi\}\right|$$

$$\ge \lim_{s_1,s_2} \left(\frac{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}}{U^c_{s_1} V^d_{s_2}}\right) \cdot \frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}} \left|\{(t_1,t_2): t_1 \le U_{\lambda_{s_1}}\ \&\ t_2 \le V_{\mu_q}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right) \ge \xi\}\right|$$

Since, $\lim\left(\frac{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}}{U^c_{s_1} V^d_{s_2}}\right) > 0$, then by taking $s_1, s_2 \to \infty$

$$\{y_{t_1t_2}\} \to T\left[S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}(g)\right]$$

Hence,

$$S^{\tilde{\alpha}}_{\overline{N_2}} \subset S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}$$

Theorem 2.8: *Suppose the double sequence* $y = \{y_{t_1t_2}\}$ *is* $[(\overline{N}_{(\lambda,\mu)}, u, v, a, b), g]^{\tilde{\alpha}}_r$ $-$*summable to T, then* $\{y_{t_1t_2}\} \to T\left[S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}(g)\right]$ *and the inclusion is strict.*

Proof: Suppose for each $\epsilon > 0$, the sequence $y = \{y_{t_1t_2}\}$ be $[(\overline{N}_{(\lambda,\mu)}, u, v, a, b), g]^{\tilde{\alpha}}_r$ summable to T.
we have

$$\left|\frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_q}} \sum_{t_1 \in S_1} \sum_{t_2 \in S_2} u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right)^r\right|$$

$$\left|\frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}} \sum^{t_1 \in S_1}_{(t_1,t_2)\in} \sum^{t_2 \in S_2}_{\mathcal{A}_{U_{\lambda_{s_1}} V_{\mu_{s_2}}}} u_{t_1} v_{t_2} g(y_{t_1t_2} - T)^r + \right.$$
$$\left.\frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}} \sum^{t_1 \in S_1}_{(t_1,t_2)\notin} \sum^{t_2 \in S_2}_{\mathcal{A}_{U_{\lambda_{s_1}} V_{\mu_{s_2}}}} u_{t_1} v_{t_2} g(y_{t_1t_2} - T)^r\right|$$

$$\geq \frac{1}{U^c_{\lambda_{s_1}} V^d_{\mu_{s_2}}} \left|\{(t_1, t_2): t_1 \leq U_{s_1} \ and\ t_2 \leq V_{s_2}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right)^r \geq \xi\}\right|$$

and by taking limit as $s_1, s_2 \to \infty$

$$S^{\tilde{\alpha}}_{\overline{N(\lambda,\mu)}}(g) - \lim y = T$$

Consider an example

Suppose $\lambda = s_1, \mu = s_2, u_{t_1} = t_1, v_{t_2} = t_2\ \forall\ t_1, t_2, s_1, s_2 \in \mathbb{N}$ and define a sequence by

$$y_{t_1t_2} = \begin{cases} \sqrt{t_1t_2}, & t_1 \text{ and } t_2 \text{ are square} \\ 0, & \text{otherwise} \end{cases}$$

Then, $U^c_{s_1} = \left[\sum^{s_1}_{t_1=1} u_{t_1}\right]^c = \left[\frac{s_1(s_1+1)}{2}\right]^c, V^d_{s_2} = \left[\sum^{s_2}_{t_2=1} v_{t_2}\right]^d = \left[\frac{s_2(s_2+1)}{2}\right]^d$

Now, we have

$$\frac{1}{U^c_{s_1} V^d_{s_2}} \left|\{(t_1, t_2): t_1 \leq U_{s_1} \text{ and } t_2 \leq V_{s_2}: u_{t_1} v_{t_2} g\left(y_{t_1t_2} - T\right)^r \geq \xi\}\right|$$
$$\leq \frac{1}{\left(\frac{s_1(s_1+1)}{2}\right)^{\frac{c}{2}} \left(\frac{s_2(s_2+1)}{2}\right)^{\frac{d}{2}}} \to \infty \text{ by taking limit } s_1, s_2 \to \infty$$

But,

$$\frac{1}{U^c_{s_1} V^d_{s_2}} \sum^{s_1}_{t_1=1} \sum^{s_2}_{t_2=1} u_{t_1} v_{t_2} g\left(y_{t_1t_2}\right)^r \to \infty$$

Theorem 2.9: *Let* $u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r \leq M \ \forall \ t_1,t_2 \in \mathbb{N}$.*If the double sequence* $y=\{y_{t_1t_2}\}$ *is weighted* (λ,μ)*-statistically convergent to T, then, it is also* $\left[\left(\overline{N}_{(\lambda,\mu)},u,v,a,b\right),g\right]_r^{\tilde{\alpha}}$*-summable to T and hence* $[(\overline{N},u,v,a,b),g]_r^{\tilde{\alpha}}$*-summable to T.*

Proof: The first statement is evident. Now,

$$\frac{1}{U_{s_1}^c V_{s_2}^d}\sum_{t_1=1}^{s_1}\sum_{t_2=1}^{s_2} u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r$$

$$=\frac{1}{U_{s_1}^c V_q^d}\sum_{t_1=1}^{s_1-\lambda_{s_1}}\sum_{t_2=1}^{s_2-\mu_{s_2}} u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r + \frac{1}{U_{s_1}^c V_{s_2}^d}\sum_{t_1\in s_1}\sum_{t_2\in s_2} u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r$$

$$\leq \frac{1}{U_{\lambda_{s_1}}^c V_{\mu_{s_2}}^d}\sum_{t_1=1}^{s_1-\lambda_{s_1}}\sum_{t_2=1}^{s_2-\mu_{s_2}} u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r + \frac{1}{U_{\lambda_p}^c V_{\mu_{s_2}}^d}\sum_{t_1\in s_1}\sum_{t_2\in s_2} u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r$$

$$\leq \frac{2}{U_{\lambda_{s_1}}^c V_{\mu_{s_2}}^d}\sum_{t_1\in s_1}\sum_{t_2\in s_2} u_{t_1}v_{t_2}g\left(y_{t_1t_2}-T\right)^r \to \infty \ as \ s_1,s_2\to\infty$$

Hence, $\{y_{t_1t_2}\}\to T\left[[(\overline{N},u,v,a,b),g]_r^{\tilde{\alpha}}\right]$

Conclusion

The conceptualization of weighted statistical convergence of order $\tilde{\alpha}$ for double sequence and weighted (λ,μ)- statistical convergence of order $\tilde{\alpha}$ for double sequence in paranormed space for $\tilde{\alpha}\epsilon(0,1]$ has been introduced which is more generalized than the notion of weighted (λ,μ)-statistical convergence of order $\tilde{\alpha}$ for double sequence. The correlation between these two conceptualizations discussed in Theorem 3.7. The relation between $[(\overline{N}_{(\lambda,\mu)},u,v,a,b),g]_r^{\tilde{\alpha}}$ − summable and $S_{\overline{N(\lambda,\mu)}}^{\tilde{\alpha}}$ is examined with the help of an example.

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