

Study of fixed point theorems using C-class function in partially ordered b-metric spaces

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Abstract

This article demonstrates the existence of fixed points for two pairs of self-maps in the context of a partially ordered b-metric space, using the C-class function. Furthermore, by the use of compatibility condition on maps, we have proved the uniqueness of fixed points. Our

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findings expand upon and provide a broader application for several established outcomes in the existing literature.

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1 Introduction

Czerwik [2] established and provided a formal definition for the notion of b-metric space. Additionally, Czerwik proved the contraction mapping theorem specifically in (sense of) b-metric spaces. It is important to observe that “the domain of b-metric spaces is much more boarder than the class of metric spaces”. Some application of authors have also mentioned a discussion on relaxing the triangle inequality and introduced a new distance measure termed as non-linear elastic matching.

In 1965, Marr [6] introduced partially ordered set and as well defined partially ordered metric spaces with help of metric. The notion of convergence in a partially ordered metric space was introduced by Marr [6]. Some notable results on fixed point can be found in [3,8-14].

2 Preliminaries

Let us see the basic concepts like lemmas and definitions mentioned below:

Definition 2 1: [1] Let $\mathfrak{B}: [0, \infty)^2 \rightarrow R$ be a mapping “known as C-class function”, if it assuring continuity along with following properties:

- (i) $\mathfrak{B}(\zeta, \eta) \leq \zeta$ for all that $\zeta, \eta \in [0, \infty)$
- (ii) $\mathfrak{B}(\zeta, \eta) = \zeta \Rightarrow \zeta = 0$ or $\eta = 0$; for all that $\zeta, \eta \in [0, \infty)$

In 1984, Khan [5] defined a new mapping known as “control function”. This concept was an intriguing aspect in the domain of metric fixed point theory, guaranteed the existence of fixed point. These functions are also known as “altering distance functions”.

Definition 2 2: [5] A continuous, non- decreasing function $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is referred as altering distance function provided $\psi(t) = 0$ implies $t = 0$.

Lemma 2 3: [4] Let (Ω, ρ) is a b-metric space and $s \geq 1$. Then a sequence $\{\xi_n\}$ is said to be b - Cauchy sequence in (Ω, ρ) , if $d(\xi_n, \xi_{n+1}) \leq kd(\xi_{n-1}, \xi_n)$ for all $n = 1, 2, 3 \dots$ and for some $0 < k < \frac{1}{s}$.

For the concepts like comparable elements, well-ordered set, monotone non-decreasing, monotone non-increasing refer the [7]

3 Main Result:

Theorem 3 1: Let $(\Omega, \rho, \leq)$ be partially ordered b – complete b metric space and $\varpi, \varrho, \varsigma, v: \Omega \rightarrow \Omega$ be self-mappings on Ω such that $\varpi(\Omega) \subseteq v(\Omega)$ and $\varrho(\Omega) \subseteq \varsigma(\Omega)$ If for every two comparable elements $\zeta, \mu \in \Omega$

$$\psi(s^4 d(\varpi\zeta, \varrho\mu)) \leq \mathfrak{W}[\psi(\aleph(\zeta, \mu), \varphi(\aleph(\zeta, \mu))], \quad (1)$$

where

$$\aleph(\zeta, \mu) = \max \left\{ \rho(\varsigma\zeta, v\mu), \frac{1}{2s} [\rho(\varpi\zeta, \varsigma\zeta) + d(\varrho\mu, v\mu)], \frac{1}{2s} [d(\varsigma\zeta, \varrho\mu) + d(\varpi\zeta, v\mu)] \right\}, \quad (2)$$

C- $\mathfrak{W}$ is a class function and ψ, φ are altering distance functions Assuming non-increasing sequence $\{\xi_n\}$ and $\{\eta_n\}$ with $\eta_n \leq \xi_n \forall n$ and if $\eta_n \rightarrow u$ implies $u \leq \xi_n$

Moreover, if

- (i) (ϖ, ς) is compatible, ϖ or ς continuous and (ϱ, v) is weakly compatible pair or
- (ii) (ϱ, v) is compatible, ϱ or v continuous and (ϖ, ς) is weakly compatible pair

Then maps preserved a common fixed point of $\varpi, \varrho, \varsigma, v$ Moreover, the set of fixed point of $\varpi, \varrho, \varsigma, v$ is well ordered iff $\varpi, \varrho, \varsigma, v$ have unique common fixed point

Proof: Let ζ_0 be an arbitrary point in Ω Since $\varpi(\Omega) \subseteq v(\Omega)$, and $\varrho(\Omega) \subseteq \varsigma(\Omega)$

We can define inductively sequence $\{\xi_n\}$ and $\{\eta_n\}$ in Ω by

$$\begin{cases} \eta_{2n+1} = \varpi\xi_{2n} = v\xi_{2n+1} \\ \eta_{2n+2} = \varrho\xi_{2n+1} = \varsigma\xi_{2n+2} \end{cases} \text{ for } n = 0, 1, 2, 3 \dots \quad (3)$$

Since $\xi_{2n+1} \leq v\xi_{2n+1} = \varpi\xi_{2n} \leq \xi_{2n}$ and $\xi_{2n} \leq \varsigma\xi_{2n} = \varrho\xi_{2n-1} \leq \xi_{2n-1}$
 $\xi_{n+1} \leq \xi_n$ for all $n \geq 0$

Claim that $\rho(\eta_{n+1}, \eta_{n+2}) \leq \lambda \rho(\eta_n, \eta_{n+1})$ where $\lambda \in \left[0, \frac{1}{s}\right) \forall n \in N$

Eq (1) concludes,

$$\begin{aligned} \psi[\rho(\eta_{2k+1}, \eta_{2k+2})] &\leq \psi(s^4 \rho(\eta_{2k+1}, \eta_{2k+2})) = \psi(s^4 \rho(\varpi\xi_{2k}, \varrho\xi_{2k+1})) \\ &\leq \mathfrak{W}[\psi(\aleph(\xi_{2k}, \xi_{2k+1}), \varphi(\aleph(\xi_{2k}, \xi_{2k+1}))] \end{aligned} \quad (4)$$

where

$$\aleph(\xi_{2k}, \xi_{2k+1}) = \max \left\{ \begin{aligned} &\rho(\varsigma\xi_{2k}, v\xi_{2k+1}), \\ &\frac{1}{2s} [\rho(\varpi\xi_{2k}, \varsigma\xi_{2k}) + \rho(\varrho\xi_{2k+1}, v\xi_{2k+1})], \\ &\frac{1}{2s} [\rho(\varsigma\xi_{2k}, \varrho\xi_{2k+1}) + \rho(\varpi\xi_{2k}, v\xi_{2k+1})] \end{aligned} \right\}$$

From Eq (3) and using triangle inequality,

$$\aleph(\xi_{2k}, \xi_{2k+1}) \leq \max \left\{ \rho(\eta_{2k}, \eta_{2k+1}), \frac{1}{2} [\rho(\eta_{2k}, \eta_{2k+1}) + \rho(\eta_{2k+1}, \eta_{2k+2})] \right\}$$

There are two alternatives that may occur

Alternative 1: If $\aleph(\xi_{2k}, \xi_{2k+1}) = \rho(\eta_{2k}, \eta_{2k+1})$, from Eq (4) we have

$$\psi[s^4 \rho(\eta_{2k+1}, \eta_{2k+2})] \leq \mathfrak{B}[\psi(\rho(\eta_{2k}, \eta_{2k+1})), \varphi(\rho(\eta_{2k}, \eta_{2k+1}))]$$

Assume $\mathfrak{B}(\zeta, \mu) \leq \zeta$ [because $\mathfrak{B}$ is $\mathcal{C}$ - class function]

$$\psi[s^4 \rho(\eta_{2k+1}, \eta_{2k+2})] \leq \psi(\rho(\eta_{2k}, \eta_{2k+1}))$$

Continuity of ψ gives that

$$s^4 \rho(\eta_{2k+1}, \eta_{2k+2}) \leq \rho(\eta_{2k}, \eta_{2k+1}) \quad (5)$$

Alternative 2: Secondly, if $\aleph(\xi_{2k}, \xi_{2k+1}) = \frac{1}{2} [\rho(\eta_{2k}, \eta_{2k+1}) + \rho(\eta_{2k+1}, \eta_{2k+2})]$

Here again we have two options,

Option-1 Let if $\rho(\eta_{2k+1}, \eta_{2k+2}) \leq \rho(\eta_{2k}, \eta_{2k+1})$ then

$$\aleph(\xi_{2k}, \xi_{2k+1}) \leq \frac{1}{2} [\rho(\eta_{2k}, \eta_{2k+1}) + \rho(\eta_{2k}, \eta_{2k+1})] \leq \rho(\eta_{2k}, \eta_{2k+1})$$

Thus, it reduces Alternative 1

Option-2 Otherwise assume that $\rho(\eta_{2k}, \eta_{2k+1}) \leq \rho(\eta_{2k+1}, \eta_{2k+2})$

Then

$$\aleph(\xi_{2k}, \xi_{2k+1}) \leq \max \{ \rho(\eta_{2k+1}, \eta_{2k+2}), \rho(\eta_{2k+1}, \eta_{2k+2}) \} \leq \rho(\eta_{2k+1}, \eta_{2k+2})$$

$$\text{And hence } s^4 \rho(\eta_{2k+1}, \eta_{2k+2}) \leq \rho(\eta_{2k+1}, \eta_{2k+2})$$

As $s \geq 1$ which is a contradiction Thus, only Alternative 1 is true

$$\text{Therefore, from Eq (5), } \rho(\eta_{2k+1}, \eta_{2k+2}) \leq \frac{1}{s^4} \rho(\eta_{2k}, \eta_{2k+1})$$

$$\text{Likewise, we may infer that, } \rho(\eta_{2k+2}, \eta_{2k+3}) \leq \frac{1}{s^4} \rho(\eta_{2k+1}, \eta_{2k+2})$$

By continuing in this manner, we may as certain

$$\rho(\eta_{n+1}, \eta_{n+2}) \leq \lambda \rho(\eta_n, \eta_{n+1}), \text{ with } \lambda \in (0, \frac{1}{s}] \text{ for wholly } n \in N$$

It follows that (by Lemma 2 3), $\{\eta_n\}$ is a b - Cauchy sequence Since Ω is b - complete, $\eta \in \Omega$, subsequently

$$\lim_{n \rightarrow \infty} \varpi \xi_{2n} = \lim_{n \rightarrow \infty} \upsilon \xi_{2n+1} = \lim_{n \rightarrow \infty} \varrho \xi_{2n+1} = \lim_{n \rightarrow \infty} \varsigma \xi_{2n+2} = \eta$$

Claim: To prove that $\varpi, \varrho, \varsigma, \upsilon$ have a common fixed point η

Continuity of ς achieves that

$$\lim_{n \rightarrow \infty} \varsigma^2 \xi_{2n+2} = \varsigma [\lim_{n \rightarrow \infty} \varsigma \xi_{2n+2}] = \varsigma \eta$$

$$\lim_{n \rightarrow \infty} \varsigma \varpi \xi_{2n} = \varsigma [\lim_{n \rightarrow \infty} \varpi \xi_{2n}] = \varsigma \eta$$

Triangle inequality in b - metric space implies

$$\rho(\varpi \varsigma \xi_{2n}, \varsigma \eta) \leq s [\rho(\varpi \varsigma \xi_{2n}, \varsigma \varpi \xi_{2n}) + \rho(\varsigma \varpi \xi_{2n}, \varsigma \eta)] \quad (6)$$

As the pair (ϖ, ς) is compatible $\lim_{n \rightarrow \infty} \rho(\varpi \varsigma \xi_{2n}, \varsigma \varpi \xi_{2n}) = 0$

Take upper limit as $n \rightarrow \infty$ in Eq (6), $\lim_{n \rightarrow \infty} \varpi \varsigma \xi_{2n} = \varsigma \eta$

As $\varsigma \xi_{2n+2} = \varrho \xi_{2n+1} \preccurlyeq \xi_{2n+1}$ then from Eq (1),

$$\psi[s^4 \rho(\varpi \varsigma \xi_{2n+2}, \varrho \xi_{2n+1})] \leq \mathfrak{W}[\psi(\aleph(\varsigma \xi_{2n+2}, \xi_{2n+1})), \varphi(\aleph(\varsigma \xi_{2n+2}, \xi_{2n+1}))]$$

Where

$$\aleph(\varsigma \xi_{2n+2}, \xi_{2n+1}) = \max \left\{ \begin{array}{l} \rho(\varsigma^2 \xi_{2n+2}, v \xi_{2n+1}), \\ \frac{1}{2s} [\rho(\varpi \varsigma \xi_{2n+2}, \varsigma^2 \xi_{2n+2}) + \rho(\varrho \xi_{2n+1}, v \xi_{2n+1})], \\ \frac{1}{2s} [\rho(\varsigma^2 \xi_{2n+2}, \varrho \xi_{2n+1}) + d(\varpi \varsigma \xi_{2n+2}, v \xi_{2n+1})] \end{array} \right\}$$

Apply upper limit in above equation and use the fact that $\mathfrak{W}$ is C - class function and continuity of ψ implies that

$$s^4 \rho(\varsigma \eta, \eta) \leq s^2 \rho(\varsigma \eta, \eta)$$

This is a contradiction, therefore, $\varsigma \eta = \eta$

Next to prove, $\varpi \eta = \eta$, let us consider

$$\frac{1}{s} \rho(\eta, \varpi \eta) \leq \rho(\eta, \varrho \xi_{2n+1}) + \rho(\varpi \eta, \varrho \xi_{2n+1}) \quad (7)$$

Since $\varrho \xi_{2n+1} \preccurlyeq \xi_{2n+1}$ and $\varrho \xi_{2n+1} \rightarrow \eta$ as $n \rightarrow \infty$ then $\eta \preccurlyeq \xi_{2n+1}$ and from Eq (1),

$$\psi[s^4 \rho(\varpi \eta, \varrho \xi_{2n+1})] \leq \mathfrak{W}[\psi(\aleph(\eta, \xi_{2n+1})), \varphi(\aleph(\eta, \xi_{2n+1}))]$$

$$\text{where } \aleph(\eta, \xi_{2n+1}) = \max \left\{ \begin{array}{l} d(\varsigma \eta, v \xi_{2n+1}), \\ \frac{1}{2s} [\rho(\varpi \eta, \varsigma \eta) + \rho(\varrho \xi_{2n+1}, v \xi_{2n+1})], \\ \frac{1}{2s} [\rho(\varsigma \eta, \varrho \xi_{2n+1}) + d(\varpi \eta, v \xi_{2n+1})] \end{array} \right\}$$

Using the property of v and continuity of ψ , we obtain from two inequalities

$$\rho(\varpi \eta, \varrho \xi_{2n+1}) \leq \frac{1}{s^4} \aleph(\eta, \xi_{2n+1})$$

Hence from Eq (7) and applying limit as $n \rightarrow \infty$

$$\frac{1}{s} \rho(\eta, \varpi \eta) \leq \lim_{n \rightarrow \infty} \rho(\eta, \varrho \xi_{2n+1}) + \frac{1}{s^4} \lim_{n \rightarrow \infty} \aleph(\eta, \xi_{2n+1}) \leq \frac{1}{s^4} \lim_{n \rightarrow \infty} \aleph(\eta, \xi_{2n+1}) \quad (8)$$

$$\text{where } \lim_{n \rightarrow \infty} \aleph(\eta, \xi_{2n+1}) = \frac{\rho(\varpi \eta, \eta)}{2s}$$

and therefore from Eq (8), we obtain, $\frac{1}{s} \rho(\eta, \varpi \eta) \leq \frac{1}{s^4} \frac{\rho(\varpi \eta, \eta)}{2s}$

This is possible only if $\rho(\eta, \varpi \eta) = 0$ that is $\eta = \varpi \eta$

As $\varpi(\Omega) \subseteq v(\Omega)$ implies existence of $\sigma \in \Omega$, so that $\varpi \eta = v\sigma$

Suppose that $\varrho \sigma \neq v\sigma$ Since $\sigma \preccurlyeq v\sigma = \varpi \eta \preccurlyeq \eta$, so from Eq (1),

$$\psi[\rho(v\sigma, \varrho \sigma)] = \psi[\rho(\varpi \eta, \varrho \eta)] \leq \mathfrak{W}[\psi(\aleph(\eta, \sigma)), \varphi(\aleph(\eta, \sigma))]$$

As $\mathfrak{W}$ is C-class function,

$$\psi[\rho(v\sigma, \varrho \sigma)] \leq \psi[\aleph(\eta, \sigma)] \quad (9)$$

$$\text{wherever } \aleph(\eta, \sigma) = \max \left\{ \begin{array}{l} \rho(\varsigma \eta, v\sigma), \\ \frac{1}{2s} [\rho(\varpi \eta, \varsigma \eta) + \rho(\varrho \sigma, v\sigma)], \\ \frac{1}{2s} [\rho(\varsigma \eta, \varrho \sigma) + \rho(\varpi \eta, v\sigma)] \end{array} \right\}$$

But $\varpi\eta = \zeta\eta = \eta = v\sigma$

$$\aleph(\eta, \sigma) = \max \left\{ \frac{1}{2s} [\rho(\varrho\sigma, v\sigma)], \frac{1}{2s} [\rho(v\sigma, \varrho\sigma)] \right\} = \frac{\rho(v\sigma, \varrho\sigma)}{2s}$$

Thus from Eq (9) and continuity of ψ implies $\rho(v\sigma, \varrho\sigma) \leq \frac{\rho(v\sigma, \varrho\sigma)}{2s} \Rightarrow \varrho\sigma = v\sigma$

Since the pair (ϱ, v) is weakly compatible

Therefore $\varrho\eta = \varrho\varpi\eta = \varrho v\sigma = v\varrho\sigma = v\varpi\eta = v\eta$

Concludes ϱ and v has the coincidence point η

Since $\zeta\xi_{2n} \leq \xi_{2n}$ and $\zeta\xi_{2n} \rightarrow \eta$ as $n \rightarrow \infty$ infers $\eta \leq \xi_{2n}$ Hence from Eq (1), Assumptions on $\aleph$ and ψ implies that

$$s^4 \rho(\varpi\xi_{2n}, \varrho\eta) \leq \aleph(\xi_{2n}, \eta) \quad (10)$$

Using triangle inequality and Eq (10) we can have

$$\frac{1}{s} \rho(\eta, \varrho\eta) \leq \rho(\eta, \varpi\xi_{2n}) + \frac{1}{s^4} \aleph(\xi_{2n}, \eta)$$

Assume the limit as $n \rightarrow \infty$,

$$\frac{1}{s} \rho(\eta, \varrho\eta) \leq \lim_{n \rightarrow \infty} \rho(\eta, \varpi\xi_{2n}) + \frac{1}{s^4} \lim_{n \rightarrow \infty} \aleph(\xi_{2n}, \eta) \quad (11)$$

$$\text{where } \lim_{n \rightarrow \infty} \aleph(\xi_{2n}, \eta) = \max \left\{ \begin{array}{l} \rho(\eta, v\eta), \\ \frac{1}{2s} [\rho(\eta, \eta) + \rho(\varrho\eta, v\eta)], \\ \frac{1}{2s} [\rho(\eta, \varrho\eta) + \rho(\eta, v\eta)] \end{array} \right\} = \rho(\eta, \varrho\eta)$$

Substitute in Eq (10), we obtain $\eta = \varrho\eta$

Consequently $\varpi\eta = \varrho\eta = \zeta\eta = v\eta = \eta$ A similar case for continuous functions ϖ Let the set of common fixed points $\varpi, \varrho, \zeta, v$ is well ordered that is every non empty subset has at least one point To assure the uniqueness, Assume, for the sake of contradiction, that τ is another fixed point But $\tau \neq \sigma$, by assumption, from Eq (1), simplifying $\psi[\rho(\tau, \sigma)] \leq \psi[\rho(\tau, \sigma)]$ Since ψ is continuous function implies that $\rho(\tau, \sigma) \leq \rho(\tau, \sigma)$ a paradox Therefore, $\tau = \sigma$ The converse is obvious This established the proof of the Theorem 3 1

4. Conclusion

In this study, we established a fixed point theorem for pairs of mapping with C-class functions in a “partially ordered, b-complete b-metric space” We further demonstrated that these pairs of mappings have a “unique fixed point” These findings generalize and expand certain well-known results from the literature

References

- [1] S. Chandok, V. Ozturk, and S. Radenovic, “On fixed points in the context of b-metric spaces”, *Matematički Vesnik*, vol. 71, no. 1-2, pp. 23–30, Mar. (2019).

- [2] S. Czerwik, "Contraction mapping in b-metric spaces", *Acta Math. Inform. Univ. Ostraviensis*, vol. 1, no. 1, pp. 5–11 (1993).
- [3] Y. Errai, E. M. Marhrani, and M. Aamri, "Related fixed-point theorems in partially ordered b-metric spaces and applications to integral equations", *Abstract Appl. Anal.*, Article ID 6672724, 14 pages (2021).
- [4] N. Jamal, T. Abdeljawad, M. Sarwar, N. Mlaiki, and S. K. Panda, "Some valid generalizations of Boyd and Wong inequality and (ψ, ϕ) -weak contraction in partially ordered b-metric spaces", *Int. J. Math. Math. Sci.* (2020).
- [5] M. S. Khan, "Fixed point theorem by altering distance between the points", *Bull. Aust. Math. Soc.*, vol. 30, pp. 1–9 (1984).
- [6] R. Marr, "Partially ordered space and metric spaces", *Am. Math. Mon.*, vol. 72, no. 6, pp. 628–631 (1965).
- [7] N. S. Rao and K. Kalyani, "Unique fixed point theorems in partially ordered metric spaces", *Heliyon*, vol. 6, pp. 55–63 (2020).
- [8] B. A. Leila, K. Berrah, T. E. Oussaeif, and S. Manro, "Common fixed point theorems in intuitionistic Menger space using property EA and an application to Fredholm integral equations", *J. Interdiscip. Math.*, vol. 25, no. 8, pp. 2187–2207 (2022).
- [9] N. M. Nikhate, K. L. Bondar, and B. A. Kiwne, "Fixed point theorems for self-mappings in generalized metric spaces", *J. Dyn. Syst. Geom. Theor.*, vol. 21, no. 2, pp. 121–126 (2023).
- [10] V. Gupta, Ramandeep, N. Mani, and A. Tripathi, "Some fixed point result involving generalized altering distance function", *Procedia Comput. Sci.*, vol. 79, pp. 112–117 (2016).
- [11] V. Bhardwaj, V. Gupta, and N. Mani, "Common fixed point theorems without continuity and compatible property of maps", *Bol. Soc. Paran. Mat.*, vol. 35, no. 3, pp. 67–77 (2017).
- [12] N. Mani, "Generalized $C\beta\psi$ -rational contraction and fixed point theorem with application to second-order differential equation", *Math. Moravica*, vol. 22, no. 1, pp. 43–54 (2018).
- [13] A. H. Ansari, V. Gupta, and N. Mani, "C-class functions on some coupled fixed point theorems in partially ordered S-metric spaces", *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, vol. 68, no. 2, pp. 1694–1708 (2019).
- [14] P. Tyagi, S. S. Chauhan (Gonder), N. Mani, and R. Shukla, "F-Modular b-Metric Spaces and Some Analogies of Classical Fixed Point Theorems", *Int. J. Anal. Appl.*, vol. 22, pp. 66 (2024).