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Some characterizations on screen generic lightlike submanifolds

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Abstract

In this paper, we examine the geometry of screen generic lightlike submanifolds of a semi-Riemannian product manifold. Here we find a characterization theorem for the induced connection on a screen generic lightlike submanifold to be a metric connection. We investigate the integrability of various distributions associated with screen generic lightlike submanifolds. Then we obtain necessary and sufficient conditions for a screen generic lightlike submanifold to be a screen generic lightlike product manifold.

Subject Classification: *53C40, 53C50.*

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1. Introduction

Many researchers have been investigated the geometry of lightlike submanifolds thoroughly because of the geometry of lightlike can be utilized to explore the theory of black holes in general theory of relativistic models. Bejancu [1] in 1978 firstly introduced and developed the concept of CR-submanifolds of an indefinite Kaehler manifold as a generalization of lightlike real hypersurfaces. To advance this endeavor, other mathematicians delved into the diverse geometric properties of CR-submanifolds within Kaehler manifolds ([2]-[5]). Additionally, Duggal explored the interplay between CR-structure and Lorentzian geometry, notably relevant in relativistic contexts (for comprehensive insights, refer to [6]). It is a well-established notion that a submanifold of a semi-Riemannian manifold earns the designation of a lightlike submanifold when its induced metric becomes degenerate. This degeneracy entails the intersection of the normal vector bundle with the tangent vector bundle, a distinctive attribute intricately entwining the study of lightlike submanifolds. Recent investigations have unveiled numerous significant applications of lightlike submanifolds in mathematical physics and relativity.

Remarkably, semi-Riemannian product manifolds extend the scope of Riemannian product manifolds into the semi-Riemannian realm, boasting a plethora of geometric intricacies. Consequently, exploring lightlike submanifolds within semi-Riemannian product manifolds becomes an intriguing pursuit. The profound analysis of lightlike submanifolds and their extensive applications in mathematical physics have motivated our present undertaking. Thus, this paper introduces the concept of screen generic lightlike submanifolds within a semi-Riemannian product manifold. We proceed to investigate the requisite conditions for the induced connection on such a submanifold to qualify as a metric connection. Furthermore, we establish criteria for the integrability of various distributions and delineate necessary and sufficient conditions for a screen generic lightlike submanifold to transform into a screen generic lightlike product manifold.

2. Preliminaries

2.1 *Basic results on lightlike submanifolds*

Suppose that (K, g) be an m-dimensional submanifold of a $(m+n)$-dimensional semi-Riemannian manifold $(\bar{K}, \bar{g})$ with q (constant index) such that with condition $m, n \geq 1, 1 \leq q \leq m + n - 1$. If $\bar{g}$ is degenerate on $TK$of K, then, T_pK and $T_pK^{\perp}$ both are degenerate and there exists a radical (null) subspace $Rad(T_pM)$such that $Rad(T_pK) = T_pK \cap T_pK^{\perp}$.If $Rad(TK) : p \in K \rightarrow Rad(T_pK)$ is smooth distribution on K with rank $r > 0$ and $1 \leq r \leq m$. Then, K is said to be r-lightlike distribution on $\bar{K}$. Let $S(TK)$ be the screen distribution in TK such that $TK = Rad(TK) \perp S(TK)$ and $S(TK^{\perp})$ is complementary vector sub-bundle to $Rad(TK)$ in $TK^{\perp}$. Furthermore, there exists $\{N_i\}$, local null frame of null sections with values in the orthogonal complement of $S(TK^{\perp})$ and $S(TK^{\perp})^{\perp}$ such that $\bar{g}(N_i, \xi_j) = \delta_{ij}$, $\bar{g}(N_i, N_j) = 0$, for any $i, j \in \{1, 2, .., r\}$, where$\{\xi_i\}$ is any local basis of $\Gamma(Rad(TK)$. Let $tr(TK)$ and $ltr(TK)$ be complementary vector bundles (but non orthogonal) to TK in $T\bar{K}|_M$ and to $Rad(TK)$ in $S(TK^{\perp})^{\perp}$ respectively.Then, we have

$$tr(TK) = ltr(TK) \perp S(TK^{\perp}). \tag{1}$$

$$T\bar{K}|_M = TK \oplus tr(TK) = (Rad(TK) \oplus ltr(TK)) \perp S(TK) \perp S(TK^{\perp}). \tag{2}$$

Consider $\bar{\nabla}$ be the Levi-Civita connection on $\bar{K}$, then from the decomposition (2), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h(Y_1, Y_2) \text{ and } \bar{\nabla}_{Y_1}U = -\mathrm{A}_U Y_1 + \nabla^{\perp}_{Y_1}U, \tag{3}$$

$\forall\, Y_1, Y_2 \in \Gamma(TK), U \in \Gamma(tr(TK))$, where, $\{\nabla_{Y_1}Y_2, \mathrm{A}_U Y_1\}$ and $\{h(Y_1, Y_2), \nabla^{\perp}_{Y_1}U\}$ belong to $\Gamma(TK)$, and $\Gamma(tr(TK))$, respectively. Here ∇ is a torsion-free linear connection on K, h is second fundamental form with symmetric bilinear form on $\Gamma(TK)$, A_U is known as shape operator, which is linear operator on M. In view of Eq. (1), considering the projection morphisms L and S of $tr(TK)$ on $ltr(TK)$ and $S(TK^{\perp})$, respectively", then Eq. (3) become

$\bar{\nabla}_{Y_1}Y_2 = \nabla_{Y_1}Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2)$ and

$$\bar{\nabla}_{Y_1}U = -\mathrm{A}_U Y_1 + \mathrm{D}^l_{Y_1}U + \mathrm{D}^s_{Y_1}U, \tag{4}$$

where $h^l(Y_1, Y_2) = L(h(Y_1, Y_2))$, $h^s(Y_1, Y_2) = S(h(Y_1, Y_2))$, $\mathrm{D}^l_{Y_1}U = \mathrm{L}(\nabla^{\perp}_{Y_1}U)$, $\mathrm{D}^s_{Y_1}U = \mathrm{S}(\nabla^{\perp}_{Y_1}U)$. As h^l and h^s are $ltr(TK)$ and $S(TK^{\perp})$-valued respectively. More Precisely,

$\bar{\nabla}_{Y_1}N = -\mathrm{A}_N Y_1 + \nabla^l_{Y_1}U + D^s(Y_1, N)$ and

$$\bar{\nabla}_{Y_1}V = -\mathrm{A}_V Y_1 + \nabla^s_{Y_1}V + D^l(Y_1, V) \tag{5}$$

where $Y_1 \in \Gamma(TK), N \in \Gamma(ltr(TK)), V \in \Gamma(S(TK^{\perp}))$.

2.2 *Semi-Riemannian Product Manifolds*

Suppose that (K_1, g_1) and (K_2, g_2) be two semi-Riemannian manifolds having dimension m_1 and m_2 with constant indices q'_1 and q'_2 respectively. Consider $\pi: K_1 \times K_2 \to K_1$ and $\sigma : K_1 \times K_2 \to K_2$ be the two projection mapping given by $\pi(x_{1,} y_1) = x_1$ and $\sigma\,(x_{1,}y_1) = y_1$ for any $(x_{1,}y_1) \in K_1 \times K_2$. then we denote the product manifold by $(\bar{K}, \bar{g}) = (K_1 \times K_2, \bar{g})$, where $\bar{g}(X_{1,} Y_1) = g_1(\pi_* X_{1,}, \pi_* Y_1) + g_2(\eta_* X_{1,}, \eta_* Y_1)$, for any $X_1, Y_1 \in \Gamma(T\bar{K})$, here * denotes the differential mapping.

Screen Generic Lightlike Submanifolds

Firstly, we define screen generic lightlike submanifolds following Dogan et al. [7] as:

Definition 3.1: [7] Let $(K, g, S(TK))$ be a real r-lightlike submanifold of a semi-Riemannian product manifold $(\bar{K}, \bar{g})$. Then, K is said to be a screen generic lightlike submanifold if the following conditions are satisfied: (i) $Rad(TK)$ is invariant with respect to F, that is $F\,Rad(TK) = Rad(TK)$.

(ii) There exists a subbundle D_0 of $S(TK)$ such that $D_0 = F\, S(TK) \cap S(TK)$, where D_0 is a non-degenerate almost complex distribution on K and TK of K have following decomposition $TK = D' \oplus D_0$, where D is invariant and $D = D_0 \oplus_{orth} Rad(TK)$.

Consider P_0, P_1 and Q be the be the projection morphisms on D_0, $Rad(TK)$ and D', respectively. Then for any $X \in \Gamma(TK)$, we have

$$X = P_0X + P_1X + QX = PX + QX, \tag{6}$$

Further applying F to Eq. (6), we have,

$$FX = \emptyset X + \omega X, \tag{7}$$

where $\emptyset X \in \Gamma(TK)$ and $\omega X \in \Gamma tr(TK)$. It is clear that $FD' \neq D'$In particular, we acquire for $\in D'$, $FY = \emptyset Y + \omega Y$, where $\emptyset Y \in \Gamma(D')$ and $\omega Y \in \Gamma(S(TK^{\perp}))$. Moreover, for $V \in \Gamma tr(TK)$, we attain,

$$FV = \emptyset V + \omega V, \tag{8}$$

where BV and CV are the tangential and transversal component of FV. Since F is parallel on K then using the Eqs. (4), (5) and (7), we drive

$$\nabla_{X_1}\emptyset Y_1 - A_{\omega Y_1}X_1 = \emptyset\nabla_{X_1}Y_1 + Bh^s(X_1, Y_1), \tag{9}$$

$$h^l(X_1, \emptyset Y_1) + D^l(X_1, \omega Y_1) = Ch^l(X_1, Y_1), \tag{10}$$

$$h^s(X_1, \emptyset Y_1) + \nabla^s_{X_1}\omega Y_1 = \omega\nabla^s_{X_1}Y_1 + Ch^s(X_1, Y_1). \tag{11}$$

Note: In the forthcoming part of the paper, we shall write $s.gc.l.s.$ for a screen generic lightlike submanifold and $\overline{K}$ taken for semi-Riemannian product manifold**.**

Theorem 3.2: *Let K be a $s.gc.l.s.$ of a semi-Riemannian product manifold $\overline{K}$. Then the induced connection is a metric connection if and only if the following conditions are holds :$-\emptyset A^*_{F\xi_1}X_1 + \nabla^{*t}_{X_1}F\xi_1 \in \Gamma(Rad(TK))$ and $Bh^s(X_1, F\xi_1) = 0$ for $X_1 \in \Gamma(TK)$ and $\xi_1 \in \Gamma(Rad(TK))$.*

Proof: Since F is an almost product structure of $\overline{K}$ thus we say that $\overline{\nabla}_{X_1}\xi_1 = \overline{\nabla}_{X_1}F^2\xi_1$ for any $\xi_1 \in \Gamma(Rad(TK))$ and $X_1 \in \Gamma(TK)$. Then from the Eq. (4), we get $\overline{\nabla}_{X_1}\xi_1 = F\overline{\nabla}_{X_1}F\xi_1$and using the Eqs. (3), (7) and (8), we get $\nabla_{X_1}\xi_1 + h^l(X_1, \xi_1) + h^s(X_1, \xi_1) = F(\nabla_{X_1}F\xi_1 + h(X_1, F\xi_1))$. Moreover, on equating the tangential parts of the above Eq. and using the Eqs. (7) and (8), we get, $\nabla_{X_1}\xi_1 = -\emptyset A^*_{F\xi_1}X_1 + \nabla^{*t}_{X_1}F\xi_1 + Bh^s(X_1, F\xi_1)$. We know that if $Rad(TK)$ is a parallel distribution, then ∇ is metric connection and vice versa (4). Thus $\nabla_{X_1}\xi_1 \in \Gamma(Rad(TK))$, if and only if $-\emptyset A^*_{F\xi_1}X_1 + \nabla^{*t}_{X_1}F\xi_1 \in \Gamma(Rad(TK))$ and $Bh^s(X_1, F\xi_1)$=0 for $X_1 \in \Gamma(TK)$ and $\xi_1 \in \Gamma(Rad(TK))$. Hence, the result holds.

Theorem 3.3: *Let K be a $s.gc.l.s.$ of a semi-Riemannian product manifold $\overline{K}$. Then the distribution D_0 is integrable for any $X_1, Y_1 \in D_0$, if*

$$\bar{g}\left(\nabla^*_{X_1}FY_1 - \nabla^*_{Y_1}FX_1, \emptyset Z_1\right) = -g(B(h^s(X_1, FY_1) - h^s(Y_1, FX_1)), FZ_1) \tag{12}$$

$$h^*(X_1, F\mathrm{Y}_1) = h^*(Y_1, F\mathrm{X}_1) \tag{13}$$

Proof: We know that distribution D_0 is integrable if an only if for any $X_1, Y_1 \in D_0$ then $[X_1, Y_1] \in D_0$ that is $g[X_1, Y_1], Z_1] = \bar{g}[X_1, Y_1], N_1] = 0$, where $Z_1 \in D'$ and $N_1 \in \Gamma(ltr(TK))$. If we employing the Eqs. (4), (7) and (8), we obtain, $g([X_1, Y_1], Z_1) = \bar{g}(F\overline{\nabla}_{X_1}Y_1 - F\overline{\nabla}_{Y_1}X_1, F\,Z_1)$, therefore

$$\bar{g}\left(\nabla^*_{X_1}\mathrm{FY}_1 - \nabla^*_{Y_1}\mathrm{FX}_1, \emptyset Z_1\right) = -g(\mathrm{B}(h^s(X_1, F\mathrm{Y}_1) - h^s(Y_1, F\mathrm{X}_1)), FZ_1).$$

Which holds (12) and similarly, for $N_1 \in \Gamma(ltr(TK))$ and $X_1, Y_1 \in D_0$, we obtain,

$$\begin{aligned} g([X_1, Y_1], N_1) &= \bar{g}\left(F\overline{\nabla}_{X_1}Y_1 - F\overline{\nabla}_{Y_1}X_1, F\,N_1\right) \\ &= \bar{g}(h^*(X_1, F\mathrm{Y}_1) - h^*(Y_1, F\mathrm{X}_1), F\,N_1)) \end{aligned}$$

therefore $\bar{g}(h^*(X_1, F\mathrm{Y}_1) - h^*(Y_1, F\mathrm{X}_1), F\,N_1)) = 0$, Which holds (13).

Theorem 3.4: *Let K be a $s.gc.l.s.$ of $\overline{K}$. Then the distribution D' is integrable if and only if $\nabla_{Z_1}\emptyset Z_2 - \nabla_{Z_2}\emptyset Z_1 - A_{\omega Z_2}Z_1 + A_{\omega Z_1}Z_2 \in \Gamma(D')$, for any $Z_1, Z_2 \in D'$.*

Proof: We assume that distribution D' is integrable. Then for any $Z_1, Z_2 \in D'$, $X_1 \in D_0$ and $N_1 \in \Gamma(ltr(TK))$, then $g([Z_1, Z_2], X_1) = \bar{g}([Z_1, Z_2], N_1) = 0$. Now we employing the Eqs. (4), (5) and (7), we proceed as above we obtain the required result.

4. Screen Generic Ligtlike Product manifolds

In this section we will investigate screen generic lightlike product submanifolds of a semi-Riemannian manifolds.

Theorem 4.1: *Let K be a totally geodesic $s.gc.l.s.$ of a semi-Riemannian product manifold $\overline{K}$. Suppose that there exists a transversal vector bundle of K, which is parallel along D' with respect to the Levi-Civita connection on K, that is, $\bar{\nabla}_{X_1}V_1 \in \Gamma(tr(TK))$ for any $X_1 \in \Gamma(D')$ and $V_1 \in \Gamma(tr(TK))$. Then K is screen generic lightlike product manifold.*

Proof: Since K be a totally geodesic $s.gc.l.s.$ then $Bh(\mathrm{X}_1, \mathrm{Y}_1) = 0$ for any $X_1, Y_1 \in \Gamma(D)$. Thus, distribution D defines a totally geodesic foliation in K. Now, $\overline{\nabla}_{X_1}\mathrm{V}_1 \in \Gamma(tr(TK))$ for any $\mathrm{X}_1 \in \Gamma(D')$ and $\mathrm{V}_1 \in \Gamma(tr(TK))$ therefore the Eq. (5) implies that $\mathrm{A}_{V_1}\mathrm{X}_1 = 0$. Then from the Eq. (9) we obtain $\emptyset\nabla_{X_1}\mathrm{Y}_1 = 0$ for any $X_1, \mathrm{Y}_1 \in \Gamma(D')$, which further gives $\nabla_{X_1}\mathrm{Y}_1 \in \Gamma(D')$. Hence, the distribution D' defines a totally geodesic foliation in K and K becomes a screen generic lightlike product manifold.

References

[1] A. Bejancu, "CR-submanifolds of a Kähler manifold I," *Trans. Amer. Math. Soc.,* vol. 69, no. 1, pp. 135–142 (1978).

[2] A. Bejancu, "CR-submanifolds of a Kähler manifold II," *Trans. Amer. Math. Soc.,* vol. 250, pp. 333–345 (1979).

[3] A. Bejancu and K. L. Duggal, "Lightlike submanifolds of semi-Riemannian manifolds," *Acta Appl. Math.,* vol. 38, pp. 197–215 (1995).

[4] D. N. Kupeli, Singular Semi-Riemannian Geometry, vol. 366, Mathematics and its Applications. Dordrecht, The Netherlands: Kluwer Academic Publishers (1996).

[5] S. Kumar, "Some characterizations on minimal lightlike submanifolds," *Int. J. Geom. Methods Mod. Phys.,* vol. 14, no. 7 (2017).

[6] K. L. Duggal, "CR-structures and Lorentzian geometry," *Acta Appl. Math.,* vol. 7, no. 3, pp. 211–223 (1986).

[7] B. Dogan, B. Sahin, and E. Yasar, "Screen generic lightlike submanifolds," *Mediterr. J. Math.,* vol. 16, no. 4, Article ID 104 (2019).

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