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Rational Contraction Mapping in *F*-Modular *b*-Metric Spaces

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Abstract

Our investigation specifically focuses on rational – type contractions within the framework of recently defined F-modular b-metric spaces (F.m.b.M.s.). Finally, to substantiate our findings, we present illustrative examples.

Subject Classification: *47H10, 54H25.*

Keywords: *R_μ- metric space, Invariant point, Rational contraction.*

1. Introduction and Foundational Concepts

Over time, many authors have improved and expanded upon the fixed-point theory [1] and Banach Contraction Principle(BCP), introducing logical contractions in complete metric spaces, and applications in various domains [2,3]. Since then, more findings in this approach have been made by the research community [5-8].

Tyagi et al. [9] improved on the work of Jleli and Samet [4] by applying Wardowski's approach to provide a new space, F-modular b-metric spaces (F.m.b.M.s). and its topological property. This work uses a rational contraction to study a self-mapping defined on a complete F.m.b.M.s. The results which are provided below, are essential elements required to understand the complexities and consequences of the main finding.

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Definition 1.1: [9] Let $\theta = \{\pi | \pi: (0, +\infty) \to \mathbb{R}\}$ s. t. $\forall\, u, v \in (0, +\infty)$
(F_1) $0 < u < v \;\Rightarrow\; \pi(u) \leq \pi(v)$.
(F_2) $\forall$ sequence $\{v_n\} \subset (0, +\infty)$, we have

Then function π is define as logarithmic like function., for e.g. $\pi(u) = \log u$.

Definition 1.2: [9] Assume S be a non-void set and $R_\mu: (0, \infty) \times S \times S \to [0, \infty)$ be a function which is continuous. If $\exists\, (\pi, \mu) \in \theta \times [0, \infty)$ s. t. $\forall\, u, v \in S$, it holds following axioms:
(P_1) $R_\mu\,(u, v) = 0 \Leftrightarrow u = v$,
(P_2) $R_\mu(u, v) = R_\mu(v, u)$,
(P_3) $\forall\, n \in N$, and $\forall\, \{v_i\}_{i=1}^{n} \subseteq$ S such that $v_1 = u\,, v_n = v$.
$R_\mu(u, v) > 0 \Rightarrow \pi\left(R_\mu(u, v)\right) \leq \pi\left\{\sum_{j=1}^{n-1} b^j R_{\mu_j}\left(v_j, v_{j+1}\right)\right\} + \lambda$
where $\mu = \sum_{j=1}^{n-1} \mu_j$ & $b \geq 1$. Then a set $R_\mu$define as F.m.b.M.s. and the pair (S, R_μ) is define as F.m.b.M.s..

Example 1.2: [9] Let S $= \mathbb{R}$ then, define a function
$R_\mu\,(u, v) = \frac{|u-v|^t}{\mu}$
with $t \geq 2$ and $u, v \in S$. Then (S, R_μ) is a F.m.b.M.s..

Definition 1.3: [9] Let (S, R_μ) is F.m.b.M.s. and $\{v_n\} \subset S$ be a sequence in S. Then

I. $\{v_n\}$ is define a F - modular b convergent to v, if $\lim_{n\to\infty} R_\mu\,(v_n, v) = 0$.
II. $\{v_n\}$ is defined a F - modular b Cauchy sequence, if $\lim_{n,m\to\infty} R_\mu\,(v_n, v_m) = 0$.
III. the space (S, R_μ) is said to be "F - modular b complete" , if every "F – modular b Cauchy sequence" $\{v_n\}$ in S is "F - modular b convergent" in S.

In next section, we define F - modular b rational contraction and proved an extension of the result of Dass and Gupta [10] in a complete F. m. b. M. s

2. F - Modular b Rational Contraction and related Theorem

Definition 2.1: Assume (S, R_μ) be F.m.b.M.s. A function $T : S \to S$ is define as F - modular b rational contraction if there exists $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that

$$R_\mu(Tv, Tw) \leq M_\mu(v, w), \tag{1}$$

Where $$M_\mu(v, w) = \alpha R_\mu(v, w) + \beta \frac{R_\mu(v, Tv) R_\mu(w, Tw)}{(1 + R_\mu(v, w))} \tag{2}$$

$\forall\, (v, w) \in S \times S$ and for all $\mu > 0$.

Theorem 2.1: *Suppose* (S, R_μ) *is a "F - modular b complete metric space". If there exist a function* $T : S \to S$ *which satisfies "F-modular b rational contraction" with,* $0 < bk < 1$*, then the map T has a fixed point(f.p.) in S, which is unique.*

Proof: Consider $v_0 \in S$. Construct a sequence $\{v_n\} \subset S$ s. t. for $n \in \mathbb{N} \cup \{0\}$

$$T^n(v_0) = T(v_n) = v_{n+1} \tag{3}$$

Since $T$is a F- modular b rational contraction, therefore using (1), we have

$$R_\mu(v_n, v_{n+1}) = R_\mu(Tv_{n-1}, Tv_n) \leq M_\mu(v_{n-1}, v_n) \tag{4}$$

where $M_\mu(v_{n-1}, v_n) = \alpha R_\mu(v_{n-1}, v_n) + \beta \frac{R_\mu(v_{n-1}, Tv_{n-1}) R_\mu(v_n, Tv_n)}{\left(1 + R_\mu(v_{n-1}, v_n)\right)}$

$$\leq \alpha R_\mu(v_{n-1}, v_n) + \beta R_\mu(v_n, Tv_n)$$

$$= \alpha R_\mu(v_{n-1}, v_n) + \beta R_\mu(v_n, v_{n+1})$$

$$\leq k R_\mu(v_{n-1}, v_n)$$

$$\leq k^n R_\mu(v_0, v_1), \text{ where, } k = \frac{\alpha}{1-\beta} < 1. \tag{5}$$

Now , We show that the sequence $\{v_n\}$ is a "F - modular b Cauchy sequence." W.l.o.g. $\forall\ n, m(\geq 2) \in N$.Consider a sequences $\{v_n\}$ & $\{v_m\}$ s.t.$v_n \neq v_m$ for $n \neq m$. Clearly, $R_\mu(v_n, v_m) > 0$ and so for all $m > n$, we have

$$R_\mu(v_n, v_m) \leq \sum_{j=n}^{m-1} b^j R_{\mu_j}(v_j, v_{j+1})$$

On using (1) and the fact $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \cdots = \mu_{m-n} = \frac{\mu}{m-n}$ in above inequality, we obtain

$$R_\mu(v_n, v_m)$$
$$\leq bk^n R_{\mu_1}(v_0, v_1) b^2 k^{n+1} R_{\mu_2}(v_0, v_1) + \ldots + b^{m-n} k^{m-1} R_{\mu_{(m-n)}}(v_0, v_1)$$
$$= \left(\frac{bk^n}{1-bk}\right) R_{\frac{\mu}{m-n}}(v_0, v_1)$$

which yields

$$R_\mu(v_n, v_m) \leq \sum_{j=n}^{m-1} b^j R_{\mu_j} \leq \left(\frac{bk^n}{1-bk}\right) R_{\frac{\mu}{m-n}}(v_0, v_1) \tag{6}$$

Since $\lim_{n\to\infty} \left(\frac{bk^n}{1-bk}\right) R_{\frac{\mu}{m-n}}(v_0, v_1) = 0$, therefore we can find $\epsilon > 0$ such that

$$0 \leq \left(\frac{bk^n}{1-bk}\right) R_{\frac{\mu}{m-n}}(v_0, v_1) < \epsilon \tag{7}$$

Due to property of π and λ ,we can find a non-decreasing g such that

$$g(p) < g(q) < g(\epsilon) - \lambda. \text{ with } 0 < p < q$$

$$g(R_\mu(v_n, v_m)) \leq g\left(\sum_{j=n}^{m-1} R_{\mu_j}(v_j, v_{j+1})\right) \leq g\left(\frac{bk^n}{1-bk}\right) R_{\frac{\mu}{m-n}}(v_0, v_i) < g(\epsilon) - \lambda$$

and so $g(R_\mu(v_n, v_m)) \leq g\left(\sum_{j=n}^{m-1} R_{\mu_j}(v_j, v_{j+1})\right) + \lambda < g(\epsilon)$

Hence (F_1) and the function g implies that $R_\mu(v_n, v_m) < \epsilon$. This proves that the sequence $\{v_n\}$ is a "F - modular - modular b Cauchy sequence". Due to completeness of space, we can find $\overline{v} \in S$ such that the sequence $\{v_n\}$ is "F-modular b convergent".

i.e.

$$\lim_{n\to\infty} R_\mu(v_n, \overline{v}) = 0.$$

Let us show that $\overline{v}$ is a f.p. of T. If $R\overline{v} = \overline{v}$ n. t. p.

Otherwise, $T\overline{v} \neq \overline{v}$, thus $R_\mu(T\overline{v}, \overline{v}) \neq 0$ and so $R_\mu(R\overline{v}, , \overline{v}) > 0$.

$$g\left(R_\mu(\overline{v}, T\overline{v})\right) \leq g\left(bR_{\frac{\mu}{2}}(\overline{v}, Tv_n) + bR_{\frac{\mu}{2}}(Tv_n, T\overline{v})\right) + \lambda$$

$$\leq g\left(bR_{\frac{\mu}{2}}(v_{n+1}, \overline{v}) + bM_{\frac{\mu}{2}}(\overline{v}, v_n)\right) + \lambda.$$

$$= g\left(\begin{array}{c} bR_{\frac{\mu}{2}}(v_{n+1}, \overline{v}) + \\ b\alpha R_{\frac{\mu}{2}}(\overline{v}, v_n) + \beta b \frac{R_{\frac{\mu}{2}}(\overline{v}, T\overline{v}) R_{\frac{\mu}{2}}(v_n, Tv_n)}{(1 + R_{\frac{\mu}{2}}(\overline{v}, v_n))} \end{array}\right) + \lambda. \tag{8}$$

For all $n \in \mathbb{N}$, and $n \geq 2$, on letting $n \to \infty$, we have

$$\left(bR_{\frac{\mu}{2}}(v_{n+1}, \overline{v}) + b\alpha R_{\frac{\mu}{2}}(\overline{v}, v_n) + \beta b \frac{R_{\frac{\mu}{2}}(\overline{v}, T\overline{v}) R_{\frac{\mu}{2}}(v_n, Tv_n)}{(1 + R_{\frac{\mu}{2}}(\overline{v}, v_n))}\right) \to 0.$$

Hence,

$$g(R_\mu(T\overline{v}, \overline{v})) \leq g\left(bR_{\frac{\mu}{2}}(v_{n+1}, \overline{v}) + b\alpha R_{\frac{\mu}{2}}(\overline{v}, v_n) + \beta b \frac{R_{\frac{\mu}{2}}(\overline{v}, T\overline{v}) R_{\frac{\mu}{2}} v_n, Tv_n)}{(1 + R_{\frac{\mu}{2}}(\overline{v}, v_n))}\right) + \lambda$$

$\to -\infty$.

Which is a contradiction and hence $T\overline{v} = \overline{v}$.

Uniqueness: If possible suppose u and v as distinct fixed points of T $\Rightarrow$ $Tu = u$ and $Tv = v$. Since $u \neq v \Rightarrow R_\mu(u, v) > 0$ we get.

$$R_\mu(u, v) = R_\mu(Tu, Tv)$$

$$= M_\mu(u, v)$$

$$\leq \alpha R_\mu(u, v) + \beta \frac{R_\mu(u, Tu) R_\mu(v, Tv)}{\left(1 + R_\mu(u, v)\right)}$$

$$= \alpha R_\mu(u, v) + \beta \frac{R_\mu(u, v) R_\mu(v, v)}{\left(1 + R_\mu(u, v)\right)}$$

$$= \alpha R_\mu(u, v) \Rightarrow (1 - \alpha) R_\mu(u, v) \leq 0.$$

This is possible only if $R_\mu(u, v) = 0$, Which contradicts to that $R_\mu(u, v) > 0$. This gives the uniqueness, and hence our result proved.

Example 2: Let $S = [0, 2)$. Define a metric by

$$R_\mu(v,w) = \begin{cases} 2^{|v-w|}, if\ v \neq w, \\ 0, if\ v = w. \end{cases}$$

Then $R_\mu(v,w)$ is an "F- modular b complete – metric" with set S.

Further let, $f(k) = \frac{-1}{k}$ and $\alpha = 1$.

Define map $T : S \to S$ by

$$T(k) = \begin{cases} \frac{1}{2}(k+1), if\ k \in [0,1], \\ 0, if\ otherwise. \end{cases}$$

Corollary 2.1: *By setting $\beta = 0$ in the above theorem, we can obtain the newly defined F- modular b contraction Principle.*

3. Conclusion

Recently, Tyagi et al. [9] developed the concept of F.m.b.M.s. and established a novel version of the BCP within the framework of these F.m.b.M.s. In this paper, we introduce the notion of rational contraction and prove a fixed point result in the context of F-modular b-metric spaces. Our findings extend, generalize, and unify various classic results in the existing literature. Additionally, we provide an illustrative example to highlight the significance of our main result. In future we will try to extend our result for pair of self-mappings in above metric spaces.

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