

On some fixed-point results in bipolar b-metric space

Saijal Chandel *

Surjeet Singh Chauhan (Gonder) [†]

Department of Mathematics

Chandigarh University

Gharuan 140301

Mohali, Punjab

India

Abstract

This paper introduces a novel metric space called bipolar b-metric space that is also an expansion of b-metric space and bipolar metric space. Further develop the result of uniqueness and existence of the fixed point in the bipolar b-metric space and also provide the example with respect to the main result.

Subject Classification: 47H10.

Keywords: Fixed point, Bipolar b-metric space, Unique fixed point, Bi-sequence, Cauchy bi-sequence, Bi-convergent.

1. Introduction

In the last three decades fixed point theory has grown immensely due to its various applications. In 1922, Banach [1] provided a broad criterion that ensures the process of iterating a function will result in a fixed point if it is met. Fixed point (fp) in a space remains unchanged under a given mapping or operator. Finding fixed points corresponds to finding solutions that remain unchanged under certain transformations or operations [2]. Fixed point theory investigates the suitable conditions which are broadly divided into three main categories fixed point theory (fpt) topological spaces, fpt in metric spaces, and fpt in discrete space. Fréchet [3] introduced metric space in 1902, has since been extended by numerous authors through the relaxation of certain conditions and alterations to the metric functions (refers to references [4]-[7]). In metric space and in its variants, the focus is on assessing the separation in points within a set. Nevertheless, issue of measuring the distance arise when considering elements

* E-mail: surjeetschauhan@yahoo.com (Corresponding Author)

[†] E-mail: sejalchandel11002@gmail.com

from distinct sets. These challenges are related to distance evaluation and encountered across diverse mathematical disciplines and other scientific domains. Mutlu and Gurdal [3] introduced bipolar metric space (bms) in 2016. Since then, theory of fp results using several conditions in bipolar b-metric spaces (bbms), have been used by many researches ([8]-[19]) in this paper introduces new space called bipolar b-metric space and develop two new results based on this space. This extends the results on bipolar b- metric space. Some supporting illustrations are also given. The remaining sections are arranged as follow: In Section 2 we give the results based on which our theorem is proved. In Section 3 we inaugurate bipolar b-metric space (BBMS). In context of BBMS, we prove fixed point results in order to illustrate our key findings. In Section 4 we give the examples based on which our theorem is proved.

2. Preliminaries

Definition 2.1: [15] A metric M on a set F is a function $M: F \times F \rightarrow R^+$ which satisfies the following properties: –

- (a) $M(\mu, \rho) \geq 0$ and $M(\mu, \rho) = 0$ iff $\mu = \rho$.
- (b) $M(\mu, \rho) = M(\rho, \mu)$ (symmetry).
- (c) $M(\mu, \rho) \leq M(\mu, h) + M(h, \rho)$ [triangular inequality]

A metric space $(F; M)$ is a set F with metric space defined on F .

Definition 2.2: [2] Let (F, G) be two sets where $F, G \neq \emptyset$ and $M: F \times G \rightarrow R^+$

- (a) $M(\mu, \rho) = 0$ iff $\mu = \rho \forall (\mu, \rho) \in F \times G$.
- (b) $M(\mu, \rho) = M(\mu, \rho) \forall (\mu, \rho) \in F \cap G$.
- (c) $M(\mu, \rho) \leq M(\mu, z) + M(w, z) + M(w, \rho) \forall \mu, w \in F$ and $z, \rho \in G$

The triplet $(F; G; M)$ is known as bipolar metric space (BMS).

Definition 2.3: [2] Let (F_1, G_1) and (F_2, G_2) be two pairs of sets. A map $S: F_1 \cup G_1 \rightarrow F_2 \cup G_2$ is called-

- (1) Covariant if $S(F_1)$ is subset of F_2 and $S(G_1)$ is subset of G_2 and it is denoted as $S: (F_1, G_1) \Rightarrow (F_2, G_2)$
- (2) Contra-variant if $S(F_1)$ is subset of G_2 and $S(G_1)$ is subset of F_2 and it is denoted as $S: (F_1, G_1) \Leftarrow (F_2, G_2)$
- (1) Set of G =right points; set of $F \cap G$ =central points, set of F =left points. The space is disjoint if $F \cap G = \emptyset$, otherwise joint.
- (2) (μ_n) on the set F is left sequence and (ρ_n) on G is right sequence. In a bms, a right and left sequence is called simply a sequence.
- (3) (μ_n) is convergent to a point μ iff (μ_n) is a left sequence, (μ_n) is a right sequence and $\mu \rightarrow F$, $\lim_{n \rightarrow \infty} \aleph(\mu, \mu_n) = 0$ or $\lim_{n \rightarrow \infty} \aleph(\mu_n, \mu) = 0$ and $\mu \rightarrow G$

- (4) A bi-sequence (μ_n, ρ_n) on (F, G, μ) is a set $F \times G$. Furthermore, if the sequences (μ_n) and (ρ_n) are convergent, then (μ_n, ρ_n) bi-sequence is convergent. If (μ_n) and (ρ_n) converge to a common point $v \in F \cap G$ then (μ_n, ρ_n) is bi-convergent.
- (5) bi-sequence (μ_n, ρ_n) cauchy bi-sequence only if $\lim_{n \rightarrow \infty} \aleph(\mu_n, \rho_n) = 0$.

3. Main Results

Definition 3.1: Suppose (F, G) two sets where $F, G \neq \varnothing$ and $M: F \times G \rightarrow R^+$

- (a) $M(\mu, \rho) = 0 \Leftrightarrow \mu = \rho \forall (\mu, \rho) \in F \times G$.
- (b) $M(\mu, \rho) = M(\mu, \rho) \forall (\mu, \rho) \in F \cap G$.
- (c) $M(\mu, \rho) \leq b[M(\mu, z) + M(w, z) + M(w, \rho)] \forall \mu, w \in F; z, \rho \in G$

The triplet $(F; G; M)$ is known as bipolar b-metric space (bbms).

Theorem 3.1: Suppose $(F; G; M)$ complete bbms. Consider the mapping $W: (F, G, M) \Rightarrow (F, G, M)$ such that $M(W_\mu, W_\rho) \leq \alpha M(\mu, \rho) \forall \mu \in F$ and $\rho \in G$ where $0 < \alpha < 1$. Suppose that

$$\sup_{\beta \geq 1} \lim_{\gamma \rightarrow \infty} \frac{M(\mu_{\gamma+1}, \rho_{\gamma+2})}{M(\mu_\gamma, \rho_{\gamma+1})} M(\mu_{\gamma+1}, \rho_\beta) \leq \frac{1}{\alpha} \quad (1)$$

$$\sup_{\beta \geq 1} \lim_{\gamma \rightarrow \infty} \frac{M(\mu_{\gamma+1}, \rho_{\gamma+1})}{M(\mu_\gamma, \rho_\gamma)} M(\mu_{\gamma+1}, \rho_\beta) \leq \frac{1}{\alpha} \quad (2)$$

Then $W: F \cup G \rightarrow G \cup F$ has Unique Fixed Point (ufp).

Proof: Suppose $\mu_0 \in F, \rho_0 \in G$

Here, $W(\mu_\delta) = \mu_{\delta+1}; W(\rho_\delta) = \rho_{\delta+1}, \forall \delta \in N$.

Then $(\{\mu_\delta\}, \{\rho_\delta\})$ is a bi-sequence on (F, G, M) and

$$\begin{aligned} M(\mu_\delta, \rho_\delta) &= M(W(\mu_{\delta-1}), W(\rho_{\delta-1})) \\ &\leq \alpha M(\mu_{\delta-1}, \rho_{\delta-1}) \end{aligned}$$

$$\vdots$$

$$\leq \alpha^\delta M(\mu_0, \rho_0)$$

Now furthermore,

$$\begin{aligned} M(\mu_\delta, \rho_{\delta+1}) &= b[M(W(\mu_{\delta-1}), W(\rho_\delta))] \\ &\leq \alpha M(\mu_{\delta-1}, \rho_\delta) \end{aligned}$$

$$\vdots$$

$$\leq \alpha^\delta M(\mu_0, \rho_1)$$

For all natural numbers $\delta \leq \beta$ and $b \geq 1$, we have

$$M(\mu_\delta, \rho_\beta) \leq b[M(\mu_\delta, \rho_{\delta+1}) + M(\mu_{\delta+1}, \rho_{\delta+1}) + M(\mu_{\delta+1}, \rho_\beta)]$$

$$M(\mu_\delta, \rho_\beta) \leq b[M(\mu_\delta, \rho_{\delta+1}) + M(\mu_{\delta+1}, \rho_{\delta+1}) + M(\mu_{\delta+1}, \rho_\beta)M(\mu_{\delta+1}, \rho_{\delta+2})]$$

$$+M(\mu_{\delta+1}, \rho_\beta)M(\mu_{\delta+2}, \rho_{\delta+2}) + M(\mu_{\delta+1}, \rho_\beta)M(\mu_{\delta+1}, \rho_\delta)]$$

.

.

.

$$\begin{aligned} M(\mu_\delta, \rho_\beta) &\leq b[M(\mu_\delta, \rho_{\delta+1}) \\ &+ M(\mu_{\delta+1}, \rho_{\delta+1}) + \sum_{\gamma=\delta+1}^{\beta-1} \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_{\gamma+1}) \\ &+ \sum_{\gamma=\delta+1}^{\beta-1} \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_\gamma) + \prod_{u=\delta+1}^\beta M(\mu_\mu, \rho_\beta) M(\mu_\beta, \rho_\beta) \\ M(\mu_\delta, \rho_\beta) &\leq b[M(\mu_\delta, \rho_{\delta+1})\alpha^\delta M(\mu_0, \rho_0) + M(\mu_{\delta+1}, \rho_{\delta+1})\alpha^{\delta+1}M(\mu_0, \rho_0)] \\ &+ \sum_{\gamma=\delta+1}^{\beta-1} \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_{\gamma+1})\alpha^\gamma M(\mu_0, \rho_1) \\ &+ \sum_{\gamma=\delta+1}^{\beta-1} \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_\gamma) \alpha^\gamma M(\mu_0, \rho_0) \\ &+ \prod_{u=\delta+1}^\beta M(\mu_\mu, \rho_\beta) \alpha^\beta M(\mu_0, \rho_1) \\ M(\mu_\delta, \rho_\beta) &\leq b[M(\mu_\delta, \rho_{\delta+1})\alpha^\delta M(\mu_0, \rho_0) + M(\mu_{\delta+1}, \rho_{\delta+1})\alpha^{\delta+1}M(\mu_0, \rho_0)] \\ &+ \sum_{\gamma=\delta+1}^\beta \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_{\gamma+1})\alpha^\gamma M(\mu_0, \rho_1) \\ &+ \sum_{\gamma=\delta+1}^{\beta-1} \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_\gamma) \alpha^\gamma M(\mu_0, \rho_0) \\ &+ \prod_{u=\delta+1}^\beta M(\mu_\mu, \rho_\beta) \alpha^\beta M(\mu_0, \rho_1) \\ M(\mu_\delta, \rho_\beta) &\leq b[M(\mu_\delta, \rho_{\delta+1})\alpha^\delta M(\mu_0, \rho_0) + M(\mu_{\delta+1}, \rho_{\delta+1})\alpha^{\delta+1}M(\mu_0, \rho_0)] \\ &+ \sum_{\gamma=\delta+1}^\beta \prod_{i=\delta+1}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_\gamma)\alpha^\gamma M(\mu_0, \rho_1) \\ &+ \sum_{\gamma=\delta+1}^{\beta-1} \prod_{i=0}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_\gamma) \alpha^\gamma M(\mu_0, \rho_0) \end{aligned}$$

Let,

$$Xe = b \sum_{\gamma=0}^e \prod_{i=0}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_{\gamma+1}) \alpha^\gamma$$

And,

$$Ye = b \sum_{\gamma=0}^e \prod_{i=0}^\gamma M(\mu_i, \rho_\beta) M(\mu_\gamma, \rho_\gamma) \alpha^\gamma$$

Hence, we have,

$$\begin{aligned} M(\mu_\delta, \rho_\beta) &\leq b[M(\mu_0, \rho_1)[M(\mu_\delta, \rho_{\delta+1})\alpha^{\delta+(X_\beta - X_\delta)} \\ &+ M(\mu_0, \rho_0) + M(\mu_{\delta+1}, \rho_{\delta+1})\alpha^{\delta+1}(Y_{\beta-1} - Y_\beta)] \end{aligned} \quad (3)$$

From the definition (2.1) and ratio test we get that, $\lim_{\delta \rightarrow \infty} X_\delta$ and $\lim_{\delta \rightarrow \infty} Y_\delta$ exists

Therefore, $\{X_\delta\}$ and $\{Y_\delta\}$ are Cauchy Sequences.

As $\delta, \beta \rightarrow \infty$ implies $\log_{\delta, \beta \rightarrow \infty} M(\mu_\delta, \rho_\beta) = 0$ Similarly, we can derive

$$\lim_{\beta, \delta \rightarrow \infty} M(\mu_{g_\beta}, \rho_\delta) = 0 \quad (4)$$

Hence $(\{\mu_\delta\}, \{\rho_\beta\})$ is a CB.

By completeness property $\{\mu_\delta\} \rightarrow e, \{\rho_\delta\} \rightarrow e$ where $e \in F \cap G$ and

$$\{W(\mu_\delta)\} = \{\rho_{\delta+1}\} \rightarrow e \in F \cap G.$$

Since W is continuous,

$W(\rho_\delta) \rightarrow W(e)$, $W(e) = z$. Hence W has a fixed point z .

Suppose ∂ is some other fp of W , then $W(\partial) = \partial$ implies $\partial \in P \cap Q$ and

$$M(z, \partial) = d(W(z), W(\partial)) \leq \alpha M(z, \partial)$$

here $0 < \alpha < 1$, implies $M(z, \partial) = 0$ so that $z = \partial$.

4. Example

Suppose $F = [0, 1]$, $G = [1, 2]$ equipped with $M(\mu, \rho) = |\mu - \rho|^2$ and $M(\mu, \rho) = \mu\rho + 1 \forall \mu \in F$ and $\rho \in G$. Then (F, G, M) is complete bbms.

Solution: Let us consider $W: F \cup G \rightarrow W: F \cup G$ by

$$W(p) = \begin{cases} \frac{\mu}{4}, & \text{if } p \in (0, 1) \\ 1, & \text{if } \mu \in [1, 2] \end{cases}, \forall \mu \in F \cup G$$

Let $\alpha = \frac{1}{2} \in (0, 1)$, $\mu_0 = \frac{1}{2} \in F$, $\rho_0 = 2 \in G$ then

$$\mu_\delta = W(\mu_{\delta+1}) = \frac{1}{4s} \quad \forall \delta \in \mathbb{N}$$

$$\mu_\delta = W(\mu_{\delta+1}) = 1 \quad \forall \delta \in G$$

We get,

$$\sup_{\beta \geq 1} \lim_{\gamma \rightarrow \infty} \frac{M(\mu_{\delta+1}, \rho_{\delta+2})}{M(\mu_\delta, \rho_{\delta+1})} M(\mu_{\delta+1}, \rho_\beta) \leq \frac{1}{\alpha}$$

$$\sup_{\beta \geq 1} \lim_{\gamma \rightarrow \infty} \frac{M(\mu_{\delta+1}, \rho_{\delta+1})}{M(\mu_\delta, \rho_\delta)} M(\mu_{\delta+1}, \rho_\beta) \leq \frac{1}{\alpha}$$

Let $\mu \in F$ and $\rho \in G$ then

$$\begin{aligned} P(W_\mu, W_\rho) &= \left| \frac{\mu}{4} - 1 \right|^2 \\ &\leq \frac{1}{2} |\mu - \rho|^2 \end{aligned}$$

Thus, the theorem is satisfied and W has a ufp $p = 1$

5. Conclusion

This research paper presents a new variant of metric space known as bipolar b-metric space. Under this space, we prove a new fixed point theorem that satisfies the condition of bbms, and an example is given in support of it.

References

- [1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales," *Fund. Math.*, vol. 13, pp. 133–181 (1922).
- [2] K. Mutlu and G. U. Özkan, "Coupled fixed point theorems on bipolar metric spaces," *Eur. J. Pure Appl. Math.*, vol. 10, pp. 655–667 (2017).

- [3] M. M. Fréchet, "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo* (1884-1940), vol. 22, no. 1, pp. 1-72 (1906).
- [4] S. S. Chauhan, K. Utreja, M. Imdad, and M. Ahmadullah, "Strong convergence theorems for a quasi-contractive type mapping employing a new iterative scheme with an application," *Honam Math. J.*, vol. 39, no. 1, pp. 1-25 (2017).
- [5] S. S. Chauhan, M. Imdad, G. Kaur, and A. Sharma, "Some fixed point theorems for sf-contraction in complete fuzzy metric spaces," *Afrika Matematika*, vol. 30, no. 3-4, pp. 651-662 (2019).
- [6] R. Mani, J. G. Arul, S. S. Fauzi, Z. M. Rahman, and R. S. Ramaswamy, "Application of fixed point results in the setting of f-contraction and simulation function in bipolar metric space," *AIMS Math.*, vol. 8, pp. 3269-3285 (2023).
- [7] G. N. V. Rao and B. S. Kishore, "Geraghty-type contraction and common coupled fixed point theorems in bipolar metric spaces with applications to homotopy," *Int. J. Math. Trends Technol.*, vol. 63, pp. 25-34 (2018).
- [8] I. Bakhtin, "The contraction mapping principle in quasimetric spaces," *Functional Analysis*, vol. 30, pp. 26-37 (1989).
- [9] A. Bartwal, R. Dimri, and G. Prasad, "Some fixed point theorems in fuzzy bipolar metric spaces," *J. Nonlinear Sci. Appl.*, vol. 13, pp. 196-204 (2020).
- [10] R. F. Brown, *Fixed Point Theory and Its Applications*, American Mathematical Society (1988).
- [11] V. Gupta, S. S. Chauhan, and I. K. Sandhu, "Banach contraction theorem on extended fuzzy cone b-metric space," *Thai J. Math.*, vol. 20, no. 1, pp. 177-194 (2022).
- [12] R. P. Kishore, B. S. Rao, G. N. V. Rao, and R. V. N. S. Agarwal, "Caristi-type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications," *Fixed Point Theory Appl.* (2018).
- [13] E. Chávez, G. Navarro, R. Baeza-Yates, and J. L. Marroquín, "Searching in metric spaces," *ACM Comput. Surv.*, vol. 33, no. 3, pp. 273-321 (2001).
- [14] S. Czerwik, "Contraction mappings in b-metric spaces," *Acta Math. Inform. Univ. Ostraviensis*, vol. 1, no. 1, pp. 5-11 (1993).

- [15] S. S. Chauhan, P. Garg, and K. Thakur, "Study of metric space and its variants," *Journal of Mathematics*, vol. 2022, no. 1, p. 7142651 (2022).
- [16] M. I. Pasha, K. R. K. Rao, G. Mani, A. J. Gnanaprakasam, and S. Kumar, "Solving a fractional differential equation via the bipolar parametric metric space," *J. Math.* (2024).
- [17] M. Kumar, Ö. Ege, V. Mor, P. Kumar, and M. De la Sen, "Boyd-Wong type contractions in generalized parametric bipolar metric space," *Heliyon*, vol. 10, no. 2 (2024).
- [18] E. Karapınar and M. Cvetković, "An inevitable note on bipolar metric spaces," *AIMS Math.*, vol. 9, no. 2, pp. 3320–3331 (2024).
- [19] G. Mani, B. Ramalingam, S. Etemad, İ. Avcı, and S. Rezapour, "On the Menger probabilistic bipolar metric spaces: Fixed point theorems and applications," *Qualitative Theory Dyn. Syst.*, vol. 23, no. 3, pp. 1–20 (2024).

Received July, 2024