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On λ-Fibonacci statistical convergence of order- α in intuitionistic fuzzy normed spaces (IFNS)

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Abstract

The main goal of this paper is to study and develop the theory of λ-Fibonacci statistical convergence of order-α in IFNS. Also λ-Fibonacci statistical Cauchy sequence of order-α and λ-Fibonacci statistical completeness of order-α in IFNS have been defined and studied.

Subject Classification: *40A35, 40A05.*

Keywords: *λ-statistical convergence, Fibonacci sequence, Intuitionistic fuzzy normed space.*

1. Introduction

The notion of statistical convergence was first presented by Zygmund's [4] monograph in 1935, and Steinhaus [1]subsequently introduce the statistical convergence for real and complex sequences. Fast [2]expands the usual concept of sequential limit and called it statistical convergence.

Definition 1.1: A sequence $w = (w_z)$ is statistically convergent to the number w_0 if $\forall \varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{m} |\{z \leq m: |w_z - w_0| \geq \varepsilon\}| = 0$$

The vertical bars represent the number of elements contained within the enclosed set. Here, we would write $S - \lim w_z \to w_0 = w_0$ or $w_z \to w_0(S)$.

Take the set of all statistically convergent sequences is denoted by S.

Various remarkable generalizations of statistical convergence have been proposed in the literature. Among these is the λ-statistical convergence provided

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by Mursaleen [7] using a increasing sequence $\lambda = (\lambda_m)$. Assume that, $\lambda = (\lambda_m)$be an increasing sequence of negative numbers that tends to ∞ s.t $\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1$."Generalised de la Vallee-Pousin mean" is the definition given by

$$t_n(w) = \frac{1}{\lambda_m} \sum_{z \in I_n} w_z \qquad \text{where } I_n = [m - \lambda_m + 1, \vartheta]$$

Definition 1.2: A sequence $w = (w_z)$ is λ-statistically convergent or S_λ-convergent to w_0 if $\forall\ \varepsilon > 0$

$$\lim_{m\to\infty} \frac{1}{\lambda_m} |\{z \in I_n : |w_z - w_0| \geq \varepsilon\}| = 0.$$

In this case, we write S_λ-convergent to w_0 or $w_z \to w_0(S_\lambda)$. Let

$$S_\lambda = \{w: \exists\, w_0 \in \mathbb{R},\ S_\lambda - \lim w = w_0\}$$

For $\alpha \in (0,1]$ Çolak and Bektas [16]generalized the λ-statistical convergence in terms of λ-statistical convergence of order-α and obtained some analogous results.

Definition 1.3: A sequence $w = (w_z)$ is λ-statistically convergent of order-α or S_λ^α-convergent to w_0, if $\forall\ \varepsilon > 0$, where $0 < \alpha < 1$

$$\lim_{m\to\infty} \frac{1}{\lambda_m^\alpha} |\{z \in I_n : |w_z - w_0| \geq \varepsilon\}| = 0.$$

So, we write as S_λ^α-convergent to w_0 or $w_z \to w_0(S_\lambda^\alpha)$. Let

$$S_\lambda^\alpha = \{w: \exists\, w_0 \in \mathbb{R},\ S_\lambda - \lim w = w_0\}$$

In 1202, the Fibonacci sequence was initially appeared in Fibonacci's book Liber Abraci,. While the sequence began with 1 in Liber Abaci, [6] it now starts with either $f_0 = 0$ or $f_1 = 1$.

Definition 1.4: The Fibonacci numbers are defined by the linear recurrence equation

$$f_l = f_{l+1} + f_{l+2} \quad l \geq 2, \qquad \text{where } l = 1,2, \ldots.$$

it denotes that the Fibonacci sequence's first two numbers are either 1 and 1 or (0 and 1).

The following are the characteristics of Fibonacci numbers:

$$\lim_{m\to\infty} \frac{f_{m+1}}{f_m} = \frac{1+\sqrt{5}}{2} = \alpha, \quad \text{(Golden ratio)}$$

$$\sum_{m=0}^{l} f_m = f_{l+2} - 1 \quad (l \in N),$$

$$\sum_m \frac{1}{f_m} \quad \text{converges},$$

$$f_{l-1} f_{l+1} - f_l^2 = (-1)^{(l+1)} (l \geq 1) \quad \text{(Cassini formula)}$$

Kara and Başarir [17], introduced the Fibonacci sequence for first time in the theory of sequence spaces. The Fibonacci matrix $\hat{F}$ defined by Afterward Kara [21].

Let f_l be the l-th Fibonacci number $\forall l \in \mathbb{N}$. Then, we define the infinite matrix $\hat{\mathrm{F}} = (f_{lm})$ by

$$\hat{F}_{lm}= \begin{cases} -\frac{f_{l}+1}{f_l}, \quad (m = l-1) \\ \frac{f_l}{f_{l}+1}, \quad (m = l) \\ 0\,, (0 \leq m < l-1 \; or \; m > l) \end{cases}$$

On intuitionistic fuzzy normed space, Murat Kirişci [19] researched Fibonacci statistical convergence.

Atanossov [5] presented an interesting generalization of fuzzy sets which were initially introduced by Zadeh [3] for those problems of real world which cannot be framed by via crisp set theory. Actually, applications of fuzzy sets were found useful in many areas like as , [10], [8], [12] [15] etc.

The concept of an IFMS was introduced by Park [9]. Moreover, Saadati and Park [11]in 2006, used the concept of Atanossov to define the notion of an IFNS, whereas Karakus [14]studied statistical convergence in these spaces. For further, development of these spaces from summability point of view we refer [13], [18],[20].

We review some concepts and definitions used in the paper.

Definition 1.5: Let $\star$ be a binary operation as $\star$: $[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous triangular-conorm. If the properties are satisfied as follows:

a) $\star$ is associative and commutative

b) $\star$ is continuous

c) $\mathrm{w} \star \nu \leq \mathrm{z} \star \zeta$ when $\mathrm{w} \leq \nu$ and $\mathrm{z} \leq \zeta$ $\mathrm{w}, \nu, \mathrm{z}, \zeta \in [0,1]$

d) $\mathrm{w} \star 1 = \mathrm{w} \, \forall \, \mathrm{w} \in [0,1]$

Definition 1.6: Let ∘ be a binary operation as ∘ : [0, 1] × [0, 1] → [0, 1] is a continuous triangular-norm. If the properties are satisfied as follows:

a) ∘ is associative and commutative

b) ∘ is continuous

c) $\mathrm{w} \circ \nu \leq \mathrm{z} \star \zeta$ when $\mathrm{w} \leq \nu$ and $\mathrm{z} \leq \zeta \; \forall \; \mathrm{w}, \mathrm{y}, \mathrm{z}, \zeta \in [0, 1]$

d) $\mathrm{w} \circ 0 = \mathrm{w} \; \forall \; \mathrm{w} \in [0, 1]$

Definition 1.7: The five tuples $(\mathrm{W}, \varphi, \varpi, \star, \circ)$ are considered intuitionistic fuzzy normed spaces (IFNS) when $\mathrm{s}, \zeta > 0$ and $\mathrm{w}, \nu \in \mathrm{w}$. Considering that w is vector space, $\star$ is the continuous triangular-norm, and ∘ is the continuous triangular-conorm. Fuzzy sets on $\mathrm{w} \times (0, \infty)$ with (φ, ϖ) satisfying the following properties

a) $\lim_{\varsigma \rightarrow \infty} \varpi \; (\mathrm{w}, \zeta) = 0$ and $\lim_{\zeta \rightarrow \infty} \varpi \, (\mathrm{w}, \zeta) = 1$

b) $\varpi \, (\mathrm{w}, .) : (0,\infty) \rightarrow [0, 1]$ is continuous

c) $\varpi \, (\mathrm{w}, \zeta) \circ \varpi \, (\mathrm{y}, \mathrm{s}) \geq \varpi \, (\mathrm{w} + \mathrm{v}, \zeta + \mathrm{s})$

d) $\varpi \, (\chi \mathrm{w}, \zeta) = \varpi \left(w, \frac{\varsigma}{|\chi|} \right) \; \forall \; \chi \neq 0$

e) $\varpi\,(\mathrm{w},\zeta) < 1$

f) ϖ (w, ζ) = 0 iff w=0

g) $\varphi\,(\mathrm{w},\zeta) + \varpi\,(\mathrm{w},\zeta) \leq 1$

h) $\lim_{\varsigma\to\infty} \varphi\,(w,\zeta) = 1 and \lim_{\zeta\to\infty} \varphi\,(w,\zeta) = 0$

i) $\varphi\,(\mathrm{w},.) : (0,\infty) \to [0,1]$ is continuous

j) $\varphi\,(\mathrm{w},\zeta) \star \varphi\,(\mathrm{y},\mathrm{s}) \leq \varphi\,(\mathrm{w}+\mathrm{v},\zeta+\mathrm{s})$

k) $\varphi\,(\mathrm{w},\zeta) > 0$

l) $\varphi\,(\mathrm{w},\zeta) = 1$ iff $\mathrm{w} = 0$

m) $\varphi\,(\chi\mathrm{w},\zeta) = \varphi\left(w,\frac{\varsigma}{|\chi|}\right) \forall\, \chi \neq 0$

2. Main Results

In this section, we aim to define λ-Fibonacci statistical convergence of order-α in IFNS. We begin with the following definition.

Definition 2.1: Let $(W,\varphi,\varpi,\star,\circ)$be an IFNS and $\lambda = (\lambda_m)$ be an increasing sequence as defined above. A sequence $w = w_z$ in w is called λ-Fibonacci statistical convergent of order α to w_0 w.r.t IFNS(φ,ϖ) if $\forall$ $\varepsilon > 0$ and $\vartheta > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_m^\alpha} \left|\left\{z \in I_n : \varphi\left(\hat{F}w_z - w_0,\vartheta\right) \leq 1-\varepsilon \;or\; \varpi\left(\hat{F}w_z - w_0\right) \geq \varepsilon\right\}\right| = 0$$

Or equivalently, $S\,\alpha\,\lambda\,(K_\varepsilon) = 0$, where

$$K_\varepsilon = \left\{z \in I_n : \varphi\left(\hat{F}w_z - W_0,\vartheta\right) \leq 1-\epsilon \;or\; \varpi\left(\hat{F}w_z - W_0\right) \geq \varepsilon\right\}$$

Or

$$w_z \to w_0$$

$$S_\lambda^\alpha(\varphi,\varpi) - \lim_{k\to\infty} \hat{F}w_z = w_0$$

Remark 2.2:

a) For the choice $(\lambda_m) = m$ and $\alpha = 1$, λ-Fibonacci statistical convergence of order- α in IFNS coincides with the λ-Fibonacci statistical convergence in IFNS.

Lemma 2.3: *Consider an IFNS* $(W,\varphi,\varpi,\star,\circ)$. *The subsequently requirements are equivalent* $\forall\,\varepsilon > 0$ *and* $\vartheta > 0$:

a) $S_\lambda^\alpha(\varphi,\varpi) - \lim \hat{F}w_z = w_0$

b) $\lim_{n\to\infty} \frac{1}{\lambda_m^\alpha} \left|\left\{z \in I_n : \varphi\left(\hat{F}w_z - w_0,\vartheta\right) \leq 1-\varepsilon\right\}\right|$ and $\lim_{n\to\infty} \frac{1}{\lambda_m^\alpha} \left|\left\{\mathrm{z} \in \mathrm{I_n} : \varpi\left(\mathrm{\hat{F}w_z} - \mathrm{w_0},\vartheta\right) \geq \varepsilon\right\}\right| = 0$

c) $\lim_{n\to\infty} \frac{1}{\lambda_m^\alpha} \left|\left\{z \in I_n : \varphi\left(\hat{F}w_z - w_0,\vartheta\right) > 1-\varepsilon\right\}\right|$ and $\lim_{n\to\infty} \frac{1}{\lambda_m^\alpha} \left|\left\{z \in I_n : \varpi\left(\hat{F}w_z - w_0,\vartheta\right) < \varepsilon\right\}\right| = 1$

d) $\lim_{n\to\infty}\frac{1}{\lambda_m^\alpha}\left|\{z\in I_n:\varphi(\hat{F}w_z-w_0,\vartheta)>1-\varepsilon\}\right|=\lim_{n\to\infty}\frac{1}{\lambda_m^\alpha}\left|\{z\in I_n:\varpi(\hat{F}w_z-w_0,\vartheta)<\varepsilon\}\right|=1$

e) $S_\lambda^\alpha-\lim\varphi\left(\hat{F}w_z-w_0,\vartheta\right)=1$ and $S_\lambda^\alpha-\lim\varpi\left(\hat{F}w_z-w_0,\vartheta\right)=0$

Theorem 2.4: Let$(W,\varphi,\varpi,\star,\circ)$ *be an IFNS. If* $(\hat{F}w_n)$ *is* $\lambda-$ *Fibonacci statistically convergent to* w_0, *then*

$$S_\lambda^\alpha(\varphi,\varpi)-\lim\hat{F}w_z=w_0 \qquad \text{is distinct.}$$

Proof: Suppose that $\exists$ two distinct points $w_0',w_0''\in w$ s.t $S_\lambda^\alpha(\varphi,\varpi)-\lim\hat{F}w_z=w_0'$ and $S_\lambda^\alpha(\varphi,\varpi)-\lim\hat{F}w_z=w_0''$ choose $\varrho>0$ s.t $(1-\varrho)>1-\varepsilon$ and $\varrho\circ\varrho<\varepsilon$ we have $\varepsilon>0\ \forall\ \xi>0$ The following set define as:

$$K_{\varphi,1}(\varrho,\xi)=\left\{z\in I_n:\varphi\left(\hat{F}w_z-w_0',\frac{\xi}{2}\right)\le 1-\varrho\right\}$$

$$K_{\varphi,2}(\varrho,\xi)=\left\{z\in I_n:\varphi\left(\hat{F}w_z-w_0'',\frac{\xi}{2}\right)\le 1-\varrho\right\}$$

$$K_{\varpi,1}(\varrho,\xi)=\left\{z\in I_n:\varphi\left(\hat{F}w_z-w_0',\frac{\xi}{2}\right)\ge \varrho\right\}$$

$$K_{\varpi,2}(\varrho,\xi)=\left\{z\in I_n:\varphi\left(\hat{F}w_z-w_0'',\frac{\xi}{2}\right)\ge \varrho\right\}$$

Since $S_\lambda^\alpha(\varphi,\varpi)-\lim\hat{F}w_z=w_0'$, we have **lemma 2.3**,we get

$$S_\lambda^\alpha\left(k_{\varpi,1}(\varrho,\xi)\right)=S_\lambda^\alpha\left(k_{\varphi,1}(\varrho,\xi)\right)=0\forall\xi>0$$

By using $S_\lambda^\alpha(\varphi,\varpi)-\lim\hat{F}w_z=w_0'$, we get

$$S_\lambda^\alpha\left(k_{\varpi,2}(\varrho,\xi)\right)=S_\lambda^\alpha\left(k_{\varphi,1}(\varrho,\xi)\right)=0\forall\xi>0$$

Now let

$$K_{\varphi,\varpi}(\varrho,\xi)=\left(K_{\varphi,1}(\varrho,\xi)\right)\cup\left(K_{\varphi,2}(\varrho,\xi)\right)$$
$$\left(K_{\varpi,1}(\varrho,\xi)\right)\cup\left(K_{\varpi,2}(\varrho,\xi)\right)$$

Hence observed that

$$S_\lambda^\alpha\left(k_{\varphi,\varpi}(\varrho,\xi)\right)=0$$

which implies

$$S_\lambda^\alpha\left(I_n/k_{\varpi,1}(\varrho,\xi)\right)=1$$

If $k\in K_{\varphi,\varpi}(\varrho,\xi)$, then we have two possible cases.

Case 1: $z\in I_n/\left(K_{\varphi,1}(\varrho,\xi)\cup K_{\varphi,2}(\varrho,\xi)\right)$ and

Case 2: $z\in I_n/\left(K_{\varpi,1}(\varrho,\xi)\cup K_{\varpi,2}(\varrho,\xi)\right)$

Firstly, consider **Case 1,** then

$$\varphi(w_0' - w_0'', \xi) \geq \varphi\left(\hat{F}w_z - w_0', \frac{\xi}{2}\right) \star \varphi\left(\hat{F}w_z - w_0'', \frac{\xi}{2}\right) > (1-\varrho) \star (1-\varrho)$$

Given that $(1-\varrho) \star (1-\varrho) > 1 - \varepsilon$, if follows that

$$\varphi(w_0' - w_0'', \xi) > 1 - \varepsilon, \text{where } \varepsilon > 0, \forall\, \xi > 0$$

we get, $\varphi(w_0' - w_0'', \xi) = 1$,
from this, $w_0' = w_0''$ is observed.

In other case, if $k \in N/\left(K_{\varphi,1}(\varrho,\xi) \cup K_{\varphi,2}(\varrho,\xi)\right)$, then we can write

$$\varpi(w_0' - w_0'', \xi) < \varpi\left(\hat{F}w_z - w_0', \frac{\xi}{2}\right) \circ \varpi\left(\hat{F}w_z - w_0'', \frac{\xi}{2}\right) < \varrho \circ$$

we see that $\varphi(w_0' - w_0'', \varrho) < \varepsilon$
by using fact $\varrho \circ \varrho$

$$\forall \varrho > 0, \text{where } \varepsilon > 0 \text{ is arbitrary we get}$$

$$\varphi(w_0' - w_0'', \varrho) = 0$$

Thus, $w_0' = w_0''$

Hence, we determined that $S_\lambda^\alpha(\varphi,\varpi) - \lim \hat{F}w_z = w_0$ is distinct.

Theorem 2.5: *Let $(W, \varphi, \varpi, \star, \circ)$ be an IFNS and $\lambda = (\lambda_m)$ be an increasing sequence. If $(\varphi,\varpi) - \lim \hat{F}\, w_z = w_0$, then*

$$S_\lambda^\alpha(\varphi,\varpi) - \lim \hat{F}w_z = w_0.$$

Proof: Suppose, $(\varphi,\varpi) - \lim \hat{F}w = w_0$ then $\exists$ a number $B \in N\ \forall\, \varepsilon > 0$ and $\xi > 0$

s.t, $\varphi\big(\hat{F}w_z - w_0, \xi\big) > 1 - \varepsilon$ and $\varpi\big(\hat{F}w_z - w_0, \xi\big) < \varepsilon$,

$\forall\, k \geq B$. It shows that the set

$$\left\{z \in I_n : \varphi\big(\hat{F}w_z - w_0, \xi\big) \leq 1 - \varepsilon \ or\ \varpi\big(\hat{F}w_z - w_0, \xi\big) \geq \varepsilon\right\}$$

finite numbers of terms. Since density of every finite subset of *N* is zero and hence.

$$S_\lambda^\alpha\left(\left\{z \in I_n : \varphi\big(\hat{F}w_z - w_0, \vartheta\big) \leq 1 - \varepsilon\, or\ \varpi\big(\hat{F}w_z - w_0\big) \geq \varepsilon\right\}\right) = 0$$

that is, $S_\lambda^\alpha(\varphi,\varpi) - \lim \hat{F}w_z = w_0$

Definition 2.6: Let $(W, \varphi, \varpi, \star, \circ)$ be an IFNS. A sequence $w = \big(\hat{F}w_z\big)$ is said to be $\boldsymbol{\lambda^\alpha}$ - Fibonacci Statistical Cauchy w.r.t IFN(φ,ϖ) if $\forall\, \varepsilon > 0$ and q > 0, $\exists$ N=N(ε) s.t

$$S_\lambda^\alpha\left(\left\{z \in I_n : \varphi\big(\hat{F}w_z - w_N, q\big) \leq 1 - \varepsilon \ or\ \varpi\big(\hat{F}w_z - w_N, q\big) \geq \varepsilon\right\}\right)$$

Theorem 2.7: *Take $(W, \varphi, \varpi, \star, \circ)$ be an IFNS.A sequence $w = \big(\hat{F}w_z\big)$ is λ^α-Fibonacci statistical Cauchy w.r.t IFN (φ,ϖ) if only if sequence is λ^α-Fibonacci statistical convergent in IFNS.*

Proof: Let $S_\lambda^\alpha(\varphi,\varpi)-\lim \hat{F}\, w_z=w_0$, if $\varepsilon>0$ select $\omega>0$ s. t $(1-\varepsilon)\star(1-\varepsilon)>1-\omega$ and $\varepsilon\circ\varepsilon<\omega$. Then, for $q>0$, we have

$$S_\lambda^\alpha\big(B(\varphi,\varpi)\big)=$$
$$S_\lambda^\alpha\left(\left\{z\in I_n:\varphi\left(w_z-w_0,\frac{q}{2}\right)\le 1-\varepsilon \text{ } and \text{ } \varpi\left(w_z-w_0,\frac{q}{2}\right)\ge\varepsilon\right\}\right)=0$$

which implies that

$$S_\lambda^\alpha\big(B^c(\varphi,\varpi)\big)=$$
$$S_\lambda^\alpha\left(\left\{z\in I_n:\varphi\left(\hat{F}w_z-w_0,\frac{q}{2}\right)>1-\varepsilon \text{ and } \varpi\left(\hat{F}w_z-w_0,\frac{q}{2}\right)<\varepsilon\right\}\right)=1$$

Let $n\in B^c(\varphi,\varpi)$. Then

$$\varphi\big(\hat{F}w_n-w_0,q\big)>1-\varepsilon \text{ and } \varpi\big(\hat{F}w_n-w_0,q\big)<\varepsilon$$

Now let

$$D(\varphi,\varpi)=\left\{z\in I_n:\varphi\big(\hat{F}w_z-w_n,q\big)\le 1-\omega \text{ } or \text{ } \varpi\big(\hat{F}w_z-w_n,q\big)\ge\right\}$$

right to prove that $D(\varphi,\varpi)\subset B(\varphi,\varpi)$

Let $k\in D(\varphi,\varpi)-A(\varphi,\varpi)$

Then $\varphi\big(\hat{F}w_z-w_n,q\big)\le 1-\omega$ and $\varphi\big(\hat{F}w_z-w_0,q\big)>1-\varepsilon$ in particular $\varphi\big(\hat{F}w_n-w_0,q\big)>1-\varepsilon$, then

$$1-\omega\ge\varphi\big(\hat{F}w_z-w_n,q\big)\ge\varphi\left(\hat{F}w_n-w_0,\frac{q}{2}\right)\star\varphi\left(\hat{F}w_n-w_0,\frac{q}{2}\right)>$$
$$(1-\varepsilon)\star(1-\varepsilon)>1-\omega$$

which is not possible.

and $\varpi\left(\hat{F}w_n-w_0,\frac{q}{2}\right)<\varepsilon,$ then

$$\omega\le\varpi\big(\hat{F}w_z-w_n,q\big)\le\varpi\left(\hat{F}w_n-w_0,\frac{q}{2}\right)\star\varphi\left(\hat{F}w_n-w_0,\frac{q}{2}\right)<\varepsilon\circ\varepsilon<\omega$$

which is not possible.

Hence $D(\varphi,\varpi)\subset B(\varphi,\varpi)$

Therefore, $S_\lambda^\alpha\big(B(\varphi,\varpi)\big)=0$

Hence, W is λ-Fibonacci statistically Cauchy of order-α w.r.t the IFN(φ, ϖ).

Definition 2.8: If all $\boldsymbol{S_\lambda^\alpha}$- Cauchy sequence w.r.t the IFN $(\boldsymbol{\varphi},\boldsymbol{\varpi})$ is $\boldsymbol{S_\lambda^\alpha}$-convergent w.r.t the IFN$(\boldsymbol{\varphi},\boldsymbol{\varpi})$. Then, an IFNS $(\boldsymbol{W},\boldsymbol{\varphi},\boldsymbol{\varpi},\star,\circ)$ is becomes complete.

Theorem 2.9: *Let $\lambda^\alpha=(\lambda_n^\alpha)$ be a sequence. Then every IFNS $(W,\varphi,\varpi,\star,\circ)$ is $S_\lambda^\alpha-$ complete.*

Proof: Let $w=\big(\hat{F}w_z\big)$ S_λ^α – Cauchy but not convergent w.r.t $IFN(\varphi,\varpi)$. If $\varepsilon>0$, choose $\varrho>0$ $s.t$ $(1-\varrho)\star(1-\varrho)>1-\varepsilon$ and $\varrho\circ\varrho<\varepsilon$

Now

$$\varphi(\hat{F}w_z - w_n, q) \geq \varphi\left(\hat{F}w_n - w_0, \frac{q}{2}\right) \star \varphi\left(\hat{F}w_n - w_0, \frac{q}{2}\right) > (1-\varepsilon) \star (1-\varepsilon) > 1 - \varrho$$

and

$$\varpi(\hat{F}w_z - w_n, q) \leq \varpi\left(\hat{F}w_z - w_0, \frac{q}{2}\right) \star \varphi\left(\hat{F}w_n - w_0, \frac{q}{2}\right) < \varepsilon \circ \varepsilon < \varrho$$

Given that, w is not $S_\lambda^\alpha(\varphi, \varpi)$ – convergent.

Therefore, $S_\lambda^\alpha\big(A(\varphi, \varpi)\big) = 0$, where

$$A(\varepsilon, q) = \{z \in I_n : \varpi(\hat{F}w_z - w_n) \geq 1 - \omega\}$$

and so $S_\lambda^\alpha\big(A(\varepsilon, q)\big) = 1$

which is a illogical,

Given that, w.r.t the $IFN(\varphi, \varpi)$ λ was $S_\lambda^\alpha -$ Cauchy. Consequently, w has to be convergent with regards to the IFN(φ, ϖ). Thats why, all IFNS are $\boldsymbol{S_\lambda^\alpha} -$ complete.

3. Conclusion

The concept of λ -Fibonacci statistical convergence of order-α in these spaces is more generalized than the Fibonacci statistical convergence of order-α. For the choice $(\lambda_m) = m$ and $\alpha = 1$, λ-Fibonacci statistical convergence of order- α in IFNS coincides with the Fibonacci statistical convergence of order-α in same spaces. These are most generalized in the terms as for particular case $(\lambda_m) = m$ these notions coincide with the Fibonacci statistical convergence of order-α define by Murat Kirişci [19].

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