

New results in bipolar controlled b-dislocated metric space and its applications for solving fractional differential equations

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Abstract

In this paper, a fixed point theorem is proved in bipolar controlled b dislocated metric space first time. Also, the findings of this result are applied to Caputo Fractional Differential Equation for finding its unique solution. Additionally, the example is provided to support the finding, which illustrate that the proved result is very simple and efficient.

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1. Introduction

According to fixed point theory, a function has unique fixed point when certain conditions are satisfied [1]. In 1922, Banach [2] introduced the notion of the Banach contraction principle (BCP). It illustrates that if T is contraction mapping and R is a metric space which is complete having a unique fixed point. Recently N. Hussain [3] used b metric space and DMS and introduced BDMS. In 2016, Mutlu and Gürdal [5] found the concept of BMS and investigated some established contractive conditions ([4],[5],[6]-[11]). Since then, other scholars have presented fixed point outcomes in bipolar metric spaces under various contractive circumstances ([5]-[12],[16-18]). In this paper we investigate fixed points for given mappings in the framework of bipolar controlled b-dislocated metric space. We discuss few results related to fixed point theorem from the

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mapping R to S on g_1 and g_2 . We provide appropriate examples to support the findings. We also demonstrate how the derived result can be used to solve caputo fractional differential equation analytically and numerically. The remaining sections are arranged as follow: In Section 2 we give the basic definitions based on which our main theorem is proved. In Section 3 we introduce BCBDS and investigate various topological characteristics. In relation to bipolar controlled b-dislocated metric spaces, we prove fixed point theorems. In Section 4 we give an application in support of the main result.

2. Preliminaries

Definition 2.1: [12] Bipolar Controlled Metric Space

Let $R \neq \emptyset$ and $S \neq \emptyset$ be two sets and $g_1: R \times S \rightarrow [0, +\infty)$, $g_2: R \times S \rightarrow [0, +\infty)$ s.t

- (a) $g_1(\eta, \mu) = 0$ iff $\eta = \mu \forall (\eta, \mu) \in R \times S$.
- (b) $g_1(\eta, \mu) = g_1(\mu, \eta) \forall (\eta, \mu) \in R \cap S$.
- (c) $g_1(\eta, \mu) \leq g_2(\eta, u)g_1(\eta, u) + g_2(v, u)g_1(v, u) + g_2(v, \mu)g_1(v, \mu)$
 $\forall \eta, v \in R$ and $\mu, u \in S$.

Then $(R; S; g_1; g_2)$ is known as bipolar controlled metric space (BCMS).

Definition 2.2: [13] b-Dislocated Metric Space

Let $R \neq \emptyset$ be a set $g: R \times R \rightarrow [0, +\infty)$ and a constant $k \geq 1$ s.t

- (a) $g(\eta, \mu) = 0 \Rightarrow \eta = \mu \forall \eta, \mu \in R$.
- (b) $g(\eta, \mu) = g(\mu, \eta) \forall \eta, \mu \in R$.
- (c) $g(\eta, \mu) \leq k[g(\eta, \kappa) + g(\kappa, \mu)] \forall \eta, \mu, \kappa \in R$.

Then (R, g) is called b-dislocated metric space (BDMS). Moreover, if $k = 1$ the BDMS becomes DMS ([14], [15]).

3. Main Results

In order to support our primary findings, we must first establish certain fundamental terminology.

Definition 3.1: Let $R \neq \emptyset$ and $S \neq \emptyset$ be two sets and $g_1: R \times S \rightarrow [0, +\infty)$, $g_2: R \times S \rightarrow [0, +\infty)$ $\forall \eta, v \in R$ and $\mu, u \in S$ and a constant $k \geq 1$ s.t

- (a) $g_1(\eta, \mu) = 0$ then $\eta = \mu \forall (\eta, \mu) \in R \times S$.
- (b) $g_1(\eta, \mu) = g_1(\mu, \eta) \forall (\eta, \mu) \in R \cap S$.
- (c) $g_1(\eta, \mu) \leq k[g_2(\eta, u)g_1(\eta, u) + g_2(v, u)g_1(v, u) + g_2(v, \mu)g_1(v, \mu)]$
 $\forall \eta, v \in R$ and $\mu, u \in S$ and $k \geq 1$.

Then $(R; S; g_1; g_2)$ is called bipolar controlled b-dislocated metric space (BCBDMS).

3.2 Completeness Property

A BCBDS is complete if every cauchy bisequence in $(R; S; g_1; g_2)$ is convergent.

Theorem 3.1: Let $(R; S; g_1; g_2)$ be a complete BCBDS. Consider $T: (R; S; g_1; g_2) \rightarrow (R; S; g_1; g_2)$ and $g_1(T_\eta, T_\mu) \leq \alpha g_1(\eta, \mu)$ for all $\eta \in R$ and $\mu \in S$ where $0 < \alpha < 1$. Take

$$\sup_{\beta \geq 1} \lim_{\gamma \rightarrow \infty} \frac{g_2(\eta_{\gamma+1}, \kappa_{\gamma+2})}{g_2(\eta_\gamma, \kappa_{\gamma+1})} g_2(\eta_{\gamma+1}, \kappa_\beta) \leq \frac{1}{\alpha}, \sup_{\beta \geq 1} \lim_{\gamma \rightarrow \infty} \frac{g_2(\eta_{\gamma+1}, \kappa_{\gamma+1})}{g_2(\eta_\gamma, \kappa_\gamma)} g_2(\eta_{\gamma+1}, \kappa_\beta) \leq \frac{1}{\alpha} \quad (1)$$

Then the function $T: S \cup R \rightarrow S \cup R$ has a unique fixed point.

Proof: Consider $\eta \in R$, $\mu \in S$. Now, determine $T(\eta_\delta) = \eta_{\delta+1}$ and $T(\mu_\delta) = \mu_{\delta+1}$, $\forall \delta \in \mathbb{N}$.

Then $(\{\eta_\delta\}, \{\mu_\delta\})$ is a bisequence on (R, S, g_1) and

$$g_1(\eta_\delta, \mu_\delta) = g_1(T(\eta_{\delta-1}), T(\mu_{\delta-1})) \leq \alpha g_1(\eta_{\delta-1}, \mu_{\delta-1})$$

$$\vdots$$

$$\leq \alpha^\delta g_1(\eta, \mu)$$

Furthermore, $g_1(\eta_\delta, \mu_{\delta+1}) = g_1(T(\eta_{\delta-1}), T(\mu_\delta)) \leq \alpha g_1(\eta_{\delta-1}, \mu_\delta)$

$$\vdots$$

$$\leq \alpha^\delta g_1(\eta_0, \mu_1)$$

$\forall$ natural numbers $\delta \leq \beta$, we have $k \geq 1$ and

$$g_1(\eta_\delta, \mu_\beta) \leq k[g_2(\eta_\delta, \mu_{\delta+1})g_1(\eta_\delta, \mu_{\delta+1}) + g_2(\eta_{\delta+1}, \mu_{\delta+1})g_1(\eta_{\delta+1}, \mu_{\delta+1})$$

$$+ g_2(\eta_{\delta+1}, \mu_\beta)g_1(\eta_{\delta+1}, \mu_\beta)]$$

$$g_1(\eta_\delta, \mu_\beta) \leq k[g_2(\eta_\delta, \mu_{\delta+1})g_1(\eta_\delta, \mu_{\delta+1}) + g_2(\eta_{\delta+1}, \mu_{\delta+1})g_1(\eta_{\delta+1}, \mu_{\delta+1})$$

$$+ g_2(\eta_{\delta+1}, \mu_\beta)k[g_2(\eta_{\delta+1}, \mu_{\delta+2})g_1(\eta_{\delta+1}, \mu_{\delta+2})]$$

$$+ g_2(\eta_{\delta+1}, \mu_\beta)k[g_2(\eta_{\delta+2}, \mu_{\delta+2})g_1(\eta_{\delta+2}, \mu_{\delta+2})]$$

$$+ g_2(\eta_{\delta+1}, \mu_\beta)k[g_2(\eta_{\delta+2}, \mu_\beta)g_1(\eta_{\delta+2}, \mu_\beta)]]$$

$$g_1(\eta_\delta, \mu_\beta) \leq k[g_2(\eta_\delta, \mu_{\delta+1})\alpha^\delta g_1(\eta_0, \mu_1) + g_2(\eta_{\delta+1}, \mu_{\delta+1})\alpha^{\delta+1} g_1(\eta_0, \mu_0)$$

$$+ k \sum_{\gamma=\delta+1}^\beta \prod_{p=\delta+1}^\gamma g_2(\eta_p, \mu_\beta)g_2(\eta_\gamma, \mu_\gamma)\alpha^\gamma g_1(\eta_0, \mu_1)$$

$$+ k \sum_{\gamma=\delta+1}^{\beta-1} \prod_{p=\delta+1}^\gamma g_2(\eta_p, \mu_\beta)g_2(\eta_\gamma, \mu_\gamma)\alpha^\gamma g_1(\eta_0, \mu_0)$$

$$\text{Let } G_e = k^{n+1} \sum_{\gamma=0}^e \prod_{p=0}^\gamma g_2(\eta_p, \mu_\beta)g_2(\eta_\gamma, \mu_{\gamma+1})\alpha^\gamma,$$

$$Q_e = k^{n+1} \sum_{\gamma=0}^e \prod_{p=0}^\gamma g_2(\eta_p, \mu_\beta)g_2(\eta_\gamma, \mu_\gamma)\alpha^\gamma$$

Hence, we have

$$g_1(\eta_\delta, \mu_\beta) \leq k^{n+1}[g_1(\eta_0, \mu_1)[g_2(\eta_\delta, \mu_{\delta+1})\alpha^\delta + (G_\beta - G_\delta)] + g_1(\eta_0, \mu_0)$$

$$[g_2(\eta_{\delta+1}, \mu_{\delta+1})\alpha^{\delta+1} + (Q_{\beta-1} - Q_\beta)] \quad (3)$$

Now by condition (1) and (2) and the using ratio test we get $\lim_{\delta \rightarrow \infty} G_\delta$ and exists.

Therefore, $\{G_\delta\}$ and $\{Q_\delta\}$ are Cauchy Sequences.

As $\delta, \beta \rightarrow \infty$ we get

$$\lim_{\delta, \beta \rightarrow \infty} g_1(\eta_\delta, \mu_\beta) = 0 \quad (4)$$

Similarly, we can derive

$$\lim_{\delta, \beta \rightarrow \infty} g_2(\eta_\beta, \mu_\delta) = 0 \quad (5)$$

Hence $(\{\eta_\delta\}, \{\mu_\delta\})$ is a Cauchy Bisequence.

By completeness property $\{\eta_\delta\} \rightarrow e$, $\{\mu_\delta\} \rightarrow e$ where $e \in R \cap S$ and $\{T(\eta_\delta)\} = \{\mu_{\delta+1}\} \rightarrow e \in R \cap S$.

Since T is continuous $T(\mu_\delta) \rightarrow T(e)$ so $T(e) = e$. Hence T has a fixed point e . If ∂ is additional fixed point of T , then $T(\partial) = \partial$ $\partial \in R \cap S$ and we have

$$g_1(e, \partial) = g_1(T(e), T(\partial)) \leq \alpha g_1(e, \partial)$$

where $0 < \alpha < 1$, $g_1(e, \partial) = 0$ and so $e = \partial$.

Example 3.1: Consider $R = [0, 1]$ and $S = [1, 2]$ be spaces with $g_1(\eta, \mu) = |\eta - \mu|^2$ and $g_2(\eta, \mu) = \eta\mu + 1$ $\forall \eta \in R$ and $\mu \in S$. Then (R, S, g_1) is a complete BCBDS.

Solution: Let us consider $T: R \cup S \rightarrow R \cup S$ by

$$T(\eta) = \begin{cases} \frac{\eta}{4}, & \text{if } \eta \in (0, 1) \\ 1, & \text{if } \eta \in [1, 2] \end{cases}, \quad \forall \eta \in R \cup S \text{ and } k = 1,$$

Let $\alpha = \frac{1}{2} \in (0, 1)$, $g_0 = \frac{1}{2} \in R$, $g_0 = 2 \in S$ then

$$\eta_\delta = T(\eta_{\delta-1}) = \frac{1}{4S}, \eta_\delta = T(\eta_{\delta-1}) = 1 \forall \delta \in S$$

We get

$$\begin{aligned} \sup_{\beta \geq 1} \lim_{\delta \rightarrow \infty} \frac{T(\eta_{\delta+1}, \mu_{\delta+2})}{T(\eta_\delta, \mu_{\delta+1})} T(\eta_{\delta+1}, \mu_\beta) &\leq \frac{1}{\alpha} \\ \sup_{\beta \geq 1} \lim_{\delta \rightarrow \infty} \frac{T(\eta_{\delta+1}, \mu_{\delta+1})}{T(\eta_\delta, \mu_\delta)} T(\eta_{\delta+1}, \mu_\beta) &\leq \frac{1}{\alpha} \end{aligned}$$

Let $\eta \in R$ and $\mu \in S$ then

$$R(T_\eta, T_\mu) = \left| \frac{\eta}{4} - 1 \right|^2 \leq \frac{1}{2} |\eta - \mu|^2$$

Therefore, T has a unique fixed point, and the theorem holds at $\eta = 1$.

Example 3.2: Let $R = Z_\delta(R)$ and $S = S_\delta(R)$ be the collection of all upper- and lower-triangular matrices with order-dependent properties. δ consequently. Introduce $g_1: Z(R) \times S(R) \rightarrow [0, \infty)$ be enumerated by $g_1(\Sigma, \Gamma) = \sum_{\beta, s=1}^{\delta} |\eta_{\beta s} -$

$|\mu_{\beta s}|^2$ for all $\Sigma = (\eta_{\beta s})\delta \times \delta \in R$ and $\Gamma = (\mu_{\beta s})_{\delta \times \delta} \in R$ and $g_2(\eta, \mu) = \eta\mu + 1 \forall \eta \in R$ and $\mu \in S$. Then (R, S, g_1) is a complete BCBDMs.

Solution: Let us consider $T: R \cup S \rightarrow R \cup S$ by $T(\eta_{\beta s})\delta \times \delta = \left(\frac{\eta_{\beta s}}{4}\right)\delta \times \delta \forall (\eta_{\beta s})_{\delta \times \delta} \in R \cup S$ and $k = 1$ therefore,

Let $\alpha = \frac{1}{2} \in (0, 1)$ and $\eta_0 = \left(\frac{\eta_{\beta s}}{2}\right)_{\delta \times \delta} \in R \times S$ then

$$(\eta_{\beta s})\delta \times \delta = T(\eta_{\beta s} - 1)\delta \times \delta = \left(\frac{\eta_{\beta s}}{2(4\delta)}\right)_{\delta \times \delta} \text{ for all } \delta \in N.$$

We get

$$\sup_{\beta \geq 1} \lim_{\delta \rightarrow \infty} \frac{T(\eta_{\delta+1}, \mu_{\delta+2})}{T(\eta_{\delta}, \mu_{\delta+1})} T(\eta_{\delta+1}, \mu_{\beta}) \leq \frac{1}{\alpha}$$

$$\sup_{\beta \geq 1} \lim_{\delta \rightarrow \infty} \frac{T(\eta_{\delta+1}, \mu_{\delta+1})}{T(\eta_{\delta}, \mu_{\delta})} T(\eta_{\delta+1}, \mu_{\beta}) \leq \frac{1}{\alpha}$$

Let $\Sigma \in R$ and $\Gamma \in S$ then

$$g_1(T(\Sigma), T(\Gamma)) = \frac{1}{4} \sum_{\beta, s=1}^{\delta} |\eta_{\beta s} - \mu_{\beta s}|^2 \leq \frac{1}{2} \sum_{\beta, s=1}^{\delta} |\eta_{\beta s} - \mu_{\beta s}|^2 = \frac{1}{2} g_1(\Sigma, \Gamma)$$

As a result, the theorem is true and S is an original fixed point $R_{\delta \times \delta}$.

4. Applications

Theorem 4.1: Consider the below fractional differential equation (FDE). Assume the following is true:

- (1) $\exists \alpha \in [0, 1], \beta \in (0, 1)$ and $(\eta, \mu) \rightarrow R \times S$ such that $|f(\alpha, \eta) - (\alpha, \mu)| \leq \sqrt{\beta} |\eta(\alpha) - \mu(\alpha)|$
- (2) $\sup_{\alpha \in [0, 1]} \int_0^1 |G(\alpha, e)|^2 dq \leq 1$

The FDE follows

$${}^e D^\gamma \eta(\alpha) + f(\alpha, \eta(\alpha)) = 0 \quad (6)$$

$$1 \leq \alpha \leq 0, \alpha \leq \gamma > 1, \eta(0) = \eta(1) = 0.$$

where ${}^e D^\gamma$ indicates the Caputo Fractional Derivative of order γ and is determined by

$${}^e D^\gamma = \frac{1}{\Gamma \xi - \gamma} \int_0^\sigma \frac{\eta^\xi(e) de}{(\alpha - e)^{\gamma - \xi + 1}} \quad (7)$$

provides a unique solution in $R \times S$.

Proof: As a result of the above equation, $\eta(\alpha) = \int_0^1 G(\alpha, e) f(q, \eta(e)) de$ where

$$G(\alpha, e) = \begin{cases} \frac{[\alpha(1-e)]^{\gamma-1} - (\alpha-e)^{\gamma-1}}{\Gamma \gamma}, & 0 \leq e \leq \alpha \leq 1 \\ \frac{[\alpha(1-e)]^{\gamma-1}}{\Gamma \gamma}, & 0 \leq \alpha \leq e \leq 1 \end{cases}$$

Define the covariant mapping $T: R \cup S \rightarrow R \cup S$ defined by

$$\begin{aligned}
T_\eta(\alpha) &= \int_0^1 G(\alpha, e)f(q, \eta(e))de \\
\Rightarrow |T_\eta(\alpha) - T_\mu(\alpha)|^2 &= \left| \int_0^1 G(\alpha, e)f(q, \eta(e))de - \int_0^1 G(\alpha, e)f(q, \mu(e))de \right|^2 \\
&\leq k \int_0^1 |G(\alpha, e)|^2 de \int_0^1 |f(q, \eta(e)) - f(q, \mu(e))|^2 de \leq k\beta |\eta(\alpha) - \mu(\alpha)|^2
\end{aligned}$$

Taking sup on both the sides we get $\hat{d}_1(T_x, T_y) \leq k\beta \hat{d}_1(x, y)$

Similarly, we can say that $g_2(T_\eta, T_\mu) \leq k\beta g_2(\eta, \mu)$

Thus, the requirements of theorem 3.1 are fulfilled.

5. Conclusion

This article presented the idea of bipolar controlled b-dislocated metric space and addressed some of its fundamental topological characteristics. With appropriate examples that illustrate our primary findings, we brought out the original fixed point outcomes, on circumstances established on BCBDMs here. To illustrate the primary finding, one application is also given.

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