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Kannan contraction mapping in strong controlled metric-type space

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Abstract

In this present paper, one fixed point theorem is proved using Kannan Contraction in strong controlled metric-type space (SCMTS). This result extends the existing theorems on metric space of two types: strong b and strong controlled.

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Keywords: *Fixed point, Kannan contraction mapping, SCMTS.*

1. Introduction

Fixed point (FP) theorems are useful tools for studying nonlinear problems. In various branches of science and technology, FP theory is used to find the unique solution's existence for the given problem and also contributes to solving real-world challenges. S. Banach introduced a very famous result, defined as Banach Contraction Principle [9], which claims that if space is complete and given mapping satisfies contraction condition, then solution of the problem exists and is unique. After this result is being modified by many publishers see [1–8, 10, 11]. This paper is centred on b-metric space (b-MSs) which is introduced by Czerwik [12]. By altering the triangle inequality of a MSs, a b-MSs can be obtained. So, every MSs is a b-MSs. Some main generalizations of b-MSs are strong b-MSs [13], extended b-MSs [14], controlled MTSs and double-controlled MTSs [15], $\boldsymbol{S_b}$ −MSs [16], fuzzy b-MSs [17]. Various FP results haved proved in these various b-metric space. Likewise,

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the objective of this paper is to prove new FP results in strong controlled MTSs by using Kannan Contraction mapping.

There are three sections to this study. Section 1, consist of abstract and introduction. Section 2, present the definition of (SbMS), (CSbMS), Kannan Contraction mapping and extended b- metric space. And in section 3, one new FP theorem using Kannan Contractions in strong controlled metric-type space is proved.

2. Preliminaries

Definition 2.1: [17] Let $\boldsymbol{D' \neq \emptyset}$ and $\mathfrak{g} \geq \mathbf{1}$, define a real-valued function $\mathfrak{h}: \boldsymbol{D' \times D' \to [0, \infty)}$ is known as b-metric if:

(a) $(\mathfrak{c}^o, \mathfrak{e}^o) = 0 \; iff \; \mathfrak{c}^o = e^o$;

(b) $\mathfrak{h}(\mathfrak{c}^o, \mathfrak{e}^o) = \mathfrak{h}(\mathfrak{e}^o, \mathfrak{c}^o)$;

(c) $\boldsymbol{\mathfrak{h}(\mathfrak{c}^o, \mathfrak{e}^o) \leq k\big(\mathfrak{h}(\mathfrak{c}^o, \gamma^o) + \mathfrak{h}(\gamma^o, \mathfrak{e}^o)\big)}$ for all $\boldsymbol{\mathfrak{c}^o, \mathfrak{e}^o, \gamma^o \in D'}$.

Then $(\boldsymbol{D', \mathfrak{h}})$ is called b-MSs.

Definition 2.1: [18] Let $\boldsymbol{D' \neq \emptyset}$ as a set and $\boldsymbol{k \geq 1}$. Then, $\mathfrak{h}: \boldsymbol{D' \times D' \to [0, \infty)}$ be a function is known as strong b-metric if

(a) $(\mathfrak{c}^o, \mathfrak{e}^o) = 0 \; iff \; \mathfrak{c}^o = e^o$;

(b) $\mathfrak{h}(\mathfrak{c}^o, \mathfrak{e}^o) = \mathfrak{h}(\mathfrak{e}^o, \mathfrak{c}^o)$;

(c) $\mathfrak{h}(\mathfrak{c}^o, \mathfrak{e}^o) \leq \mathfrak{h}(\mathfrak{c}^o, \gamma^o) + k\mathfrak{h}(\gamma^o, \mathfrak{e}^o)$,

for all $\mathfrak{c}^o, \mathfrak{e}^o, \gamma^o \in D'$. The pair $(D', \mathfrak{h})$ is the strong b-MSs.

Definition 2.2: [18] Assume $D' \neq \emptyset$ as a set and $g: D' \times D' \to [1, \infty)$. Then $\rho: D' \times D' \to [0, \infty)$ be a function is known as strong controlled b-metric if

(a) $\rho(o, \upsilon) = 0 \iff o = \upsilon$;

(b) $\rho(o, \upsilon) = \rho(\upsilon, o,)$;

(c) $\rho(o, \upsilon) \leq \rho(o, \gamma) + g(\gamma, \upsilon)\rho(\gamma, \upsilon)$,

for all $o, \upsilon, \gamma \in D'$. The pair (D', ρ) is the strong controlled b-metric space (SCbMS).

Thus, every Sb-MS can be an SCb-MS if we take $g(\gamma, \upsilon) = k$; but the converse is not necessarily true.

Definition 2.3: [18] Consider a SCbMS (D', g) of one and two functions and $\{v_s\}_{s \geq 0}$ is a sequence in D'.

(a) Thus, $\{v_s\}$ converges at an arbitrary $\upsilon \in D'$, if $\forall \, \zeta > 0, \exists \, \mathfrak{M} = \mathfrak{M}(\zeta) \in N$ such that $g(v_s, \upsilon) < \zeta \; \forall \; s \geq \mathfrak{M}$. Here, the $\lim_{n \to \infty} v_s = \upsilon$.

(b) Suppose consider a Cauchy sequence $\{v_s\}$ if $\forall \, \zeta > 0, \exists \, \mathfrak{M} = \mathfrak{M}(\zeta) \in N$ such that $g(v_r, v_s) < \zeta \; \forall \, r, s \geq \mathfrak{M}$.

(c) The SCbMS (D', g) assume to be completed if it is convergent for all Cauchy sequences.

Theorem 2.4: [19] *Consider* (D', d) *as a complete MSs and* $\Upsilon : D' \to D'$ *is a mapping which follows that*

$$d(\Upsilon m, \Upsilon n) \leq sd(m, \Upsilon m) + sd(n, \Upsilon n),$$

$\forall\, m, n \in D'$ and $s \in \left[0, \frac{1}{2}\right)$. Then, $\bar{m} \in D'$ is the FP of Υ which is unique and for any $m \in D'$, $\{\Upsilon^r m\}$ converges to $\bar{m}$.

This contraction conditions satisfied mapping is known as Kannan mapping which need not be continuous. The complete space in ways of the FP property of the mapping can be expressed by this mapping.

3. Main result

At this section, by using Kannan Contraction Mapping in strong controlled b-metric space (SCbMS), we have:

Theorem 3.1: *Assume* $(D', \mathcal{K})$ *be a strong controlled b-MSs that is complete and define by the function* ϕ: $D' \times D' \to [1, \infty)$. *And let* Υ: $D' \to D'$ *holds*

$$\mathcal{K}(\Upsilon\bar{\mathfrak{x}}, \Upsilon\bar{\mathfrak{w}}) \leq l\{\mathcal{K}(\bar{\mathfrak{x}}, \Upsilon\bar{\mathfrak{x}}) + \mathcal{K}(\bar{\mathfrak{w}}, \Upsilon\bar{\mathfrak{w}})\} \tag{1}$$

$\forall\, \bar{\mathfrak{x}}, \bar{\mathfrak{w}} \in D'$, where $0 < l < 1$. For $\bar{\mathfrak{x}}_0 \in D'$ choose $\bar{\mathfrak{x}}_{\dot{z}} = \Upsilon^{\dot{z}}\bar{\mathfrak{x}}_0$. Assume that

$$\phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s) < \frac{1}{l}. \tag{2}$$

Also, $\forall \bar{\mathfrak{x}} \in D'$, consider that

$$\lim_{\dot{z}\to\infty} \phi(\bar{\mathfrak{x}}, \bar{\mathfrak{x}}_{\dot{z}}) \text{ is exists and finite as well.} \tag{3}$$

Therefore, $\exists$ a point of Υ which is fixed and unique.

Proof: We consider the sequence $\{\bar{\mathfrak{x}}_{\dot{z}} = \Upsilon^{\dot{z}}\bar{\mathfrak{x}}_0\}$ in D'. From (1), we obtain

$$\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) \leq l^r\{\mathfrak{S}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\mathfrak{w}}_0, \bar{\mathfrak{w}}_1)\} \forall\, \dot{z} \in [0, \infty). \tag{4}$$

For $\dot{z} < s$ where $\mathfrak{q}, s \in \mathbb{Z}$, we get

$$\begin{aligned}
\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_s) &\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s), \\
&\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_{\dot{z}+2}) \\
&\quad + \phi(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_s)\phi(\bar{\mathfrak{x}}_{\dot{z}+2}, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+2}, \bar{\mathfrak{x}}_s) \\
&\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_{\dot{z}+2}) \\
&\quad + \phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s)\phi(\bar{\mathfrak{x}}_{\dot{z}+2}, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+2}, \bar{\mathfrak{x}}_{\dot{z}+3}) \\
&\quad + \phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \bar{\mathfrak{x}}_s)\phi(\bar{\mathfrak{x}}_{\dot{z}+2}, \bar{\mathfrak{x}}_s)\phi(\bar{\mathfrak{x}}_{\dot{z}+3}, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+3}, \bar{\mathfrak{x}}_s) \\
&\leq \cdots \\
&\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \textstyle\sum_{m=\dot{z}+1}^{s-2}\left(\prod_{n=\dot{z}+1}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s)\right)\mathcal{K}(\bar{\mathfrak{x}}_m, \bar{\mathfrak{x}}_{m+1}) \\
&\quad + \textstyle\prod_{t=\mathfrak{q}+1}^{s-1} \phi(\bar{\mathfrak{x}}_t, \bar{\mathfrak{x}}_s)\mathcal{K}(\bar{\mathfrak{x}}_{s-1}, \bar{\mathfrak{x}}_s) \\
&\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \textstyle\sum_{m=\dot{z}+1}^{s-1}\left(\prod_{n=\dot{z}+1}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s)\right)\mathcal{K}(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_{m+1}) \\
&\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \textstyle\sum_{m=\dot{z}+1}^{s-1}\left(\prod_{n=\dot{z}+1}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s)\right) l^m\{(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\mathfrak{w}}_0, \bar{\mathfrak{w}}_1)\}
\end{aligned}$$

$$\leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + \sum_{m=0}^{s-1}(\prod_{n=0}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s))\, l^m\{\mathcal{K}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\varpi}_0, \bar{\varpi}_1)\}$$
$$- \sum_{m=0}^{\dot{z}}(\prod_{n=0}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s))\, l^m\{\mathcal{K}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\varpi}_0, \bar{\varpi}_1)\}$$

Consider $\phi(\bar{\mathfrak{x}}, \bar{\varpi}) \geq 1$. Let

$$S_{\bar{\mathfrak{x}}} = \sum_{m=0}^{\bar{\mathfrak{x}}}(\prod_{n=0}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s))\, l^m\{\mathcal{K}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\varpi}_0, \bar{\varpi}_1)\}$$

Thus, we have

$$\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_s) \leq \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1}) + (S_{s-1} - S_{\dot{z}}) \tag{5}$$

Applying the term's ratio test, we get

$$U_m = (\prod_{n=0}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s)) l^m\{\mathcal{K}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\varpi}_0, \bar{\varpi}_1)\}$$
$$\frac{U_{m+1}}{U_m} = \frac{\prod_{n=0}^{m+1} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s) l^{m+1}\{\mathcal{K}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\varpi}_0, \bar{\varpi}_1)\}}{\prod_{n=0}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s) l^m\{\mathcal{K}(\bar{\mathfrak{x}}_0, \bar{\mathfrak{x}}_1) + \mathcal{K}(\bar{\varpi}_0, \bar{\varpi}_1)\}}$$

But

$$\frac{\prod_{n=0}^{m+1} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s)}{\prod_{n=0}^{m} \phi(\bar{\mathfrak{x}}_n, \bar{\mathfrak{x}}_s)} = \phi(\bar{\mathfrak{x}}_{m+1}, \bar{\mathfrak{x}}_s) l$$

then $\frac{U_{m+1}}{U_m} = \phi(\bar{\mathfrak{x}}_{m+1}, \bar{\mathfrak{x}}_s) l$

using (2), we conclude that

$$\frac{U_{m+1}}{U_m} < \frac{1}{l} \times l < 1$$

demonstrate the convergence of this term. It concludes $\{S_{\mathfrak{q}}\}$ exists by using this result and Equation (2) together. Therefore, $S_{\dot{z}}$ is Cauchy. On the assumption that (5) uses $\dot{z}, s$ approach to ∞ it implies that

$$\lim_{\dot{z},s\to\infty} \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_s) = 0 \tag{6}$$

Thus, it follows that the stated sequence $\{\bar{\mathfrak{x}}_{\mathfrak{q}}\}$ assume to be Cauchy. And there exists some $a' \in D'$ as $(D', \mathcal{K})$ is a complete SCbMS. So, we have

$$\lim_{\dot{z}\to\infty} \mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, a') = 0$$

According to our statement $\Upsilon a' = a'$. Utilizing the triangular inequality and the strong controlled metric-type space (SCMTS) definition, it results

$$\mathcal{K}(a', \bar{\mathfrak{x}}_{\dot{z}+1}) \leq \mathcal{K}(a', \bar{\mathfrak{x}}_{\dot{z}}) + \phi(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1})\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, \bar{\mathfrak{x}}_{\dot{z}+1})$$

As a result of (3) and (6), we get:

$$\lim_{\dot{z}\to\infty} \mathcal{K}(a', \bar{\mathfrak{x}}_{\dot{z}+1}) = 0 \tag{7}$$

Utilizing (1), we get

$$\mathcal{K}(a', \Upsilon a') \leq \mathcal{K}(a', \bar{\mathfrak{x}}_{\dot{z}+1}) + \phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \Upsilon a')\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}+1}, \Upsilon a')$$
$$\leq \mathcal{K}(a', \bar{\mathfrak{x}}_{\mathfrak{q}+1}) + l\phi(\bar{\mathfrak{x}}_{\dot{z}+1}, \Upsilon a')\mathcal{K}(\bar{\mathfrak{x}}_{\dot{z}}, a')$$

With the help of (3) and (7), we are able to reach the limit $\mathfrak{J}(a', \Upsilon a') = 0$, which means that $\Upsilon a' = a'$. Consider that $c' \in D'$, such that $\Upsilon c' = c'$ and a', c' are identical that is $a' \neq c'$. As a result,

$$0 < \mathcal{K}(a', c') = \mathcal{K}(\Upsilon a', \Upsilon c') \leq l\{\mathcal{K}(a', \Upsilon a') + \mathcal{K}(c', \Upsilon c')\}.$$

Therefore, the inequality's extremes contradict that $a' = c'$. As a result, Υ contains a fixed and unique point a'.

4. Conclusion

Firstly, we discuss the general background of strong controlled metric type space. Then, a new type of FP theorem using Kannan Contraction have been proved in this paper by using a constant $l > 0$ in strong controlled MTSs. Our result is an extension version of classical result in this literature.

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