

Integral transform of product of generalized k-Bessel function and generalized Hurwitz-Lerch Zeta function

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Abstract

The paper presents new integral formulae involving product of generalized K-Bessel function of first kind $\omega_{k,q,b,c}^{\gamma,\alpha}(z)$ and Hurwitz-Lerch Zeta (HLZ function) function $\phi_{\epsilon,\eta}^{\rho,\sigma}(z, s, d)$ are obtained and presented in terms of the generalized (Wright) hypergeometric function. Some useful and interesting cases of main results are also deliberated. The results are obtained using a fascinating integral known as Oberhettinger integral.

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1. Introduction

In the last few years, so many valuable integral transform formulae linked with the generalized special functions have been explored by many researchers Menaria *et al.* [1-3] and Agrawal *et al* [4]. Numerous interesting results are also given by several researchers on zeta function and families of HLZ functions in recent years like Lin and Srivastava [5], Pathan *et al.*[6] and Jamini and Gupta[7]. Generalized k-Bessel $\omega_{k,q,b,c}^{\gamma,\alpha}(z)$ introduced by Nisar and Mondal [8] defined for $k \in \mathbb{R}$, $\alpha, \gamma, q, b, c \in \mathbb{C}$, $\Re(q) > 0$ is given by

$$\omega_{k,q,b,c}^{\gamma,\alpha}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n ((\gamma)_{n,k}) \left(\frac{z}{2}\right)^{q+2n}}{r_k(\alpha n + q + \frac{b+1}{2})(n!)^2} \quad (1)$$

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Recently, Diaz and Parigaan[9] introduced k-gamma function and k-pochhammer symbol, given by equation (2) and also established a valuable connection with gamma function see[10-11] given by equation (3)

$$(\gamma)_{n,k} = \begin{cases} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)} & (k \in \mathbb{R}; \gamma \in \mathbb{C} \setminus \{0, -k, -2k, \dots\}) \\ \gamma(\gamma+k) \dots (\gamma+(n-1)k) & n \in \mathbb{N}; \gamma \in \mathbb{C} \end{cases} \quad (2)$$

$$\Gamma_k(z) = k^{\frac{z}{k}-1} \Gamma\left(\frac{z}{k}\right). \quad (3)$$

Equation (2) and (3) transform to the simple form of pochhammer symbol and Euler's Gamma function for $k=1$, [12].

A generalization of the series ${}_p\Psi_q$ presented by [13]-[15]

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, E_1), \dots, (\alpha_p, E_p) \\ (\beta_1, L_1), \dots, (\beta_q, L_q) \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(\alpha_j + E_j k)}{\prod_{j=1}^q \Gamma(\beta_j + L_j k)} \frac{z^k}{k!} \quad (4)$$

Where the coefficients $E_1, \dots, E_q$ and $L_1, \dots, L_q$ are real positive numbers such that $1 + \sum_{j=1}^q L_j - \sum_{j=1}^p E_j \geq 0$

Another form of (4) is

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, 1), \dots, (\alpha_p, 1) \\ (\beta_1, 1), \dots, (\beta_q, 1) \end{matrix} ; z \right] = \frac{\prod_{j=1}^p \Gamma(\alpha_j)}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix} ; z \right], \quad (5)$$

${}_pF_q$ is the generalized hypergeometric series.

Lin and Srivastava [5] introduced generalized family of HLZ functions $\phi_{\xi, \eta}^{\rho, \sigma}(T, s, d)$ defined for $\xi \in \mathbb{C}, d, \eta \neq 0, -1, -2, \dots; \rho, \sigma \in \mathbb{R}^+; \rho < \sigma$ when $T \in \mathbb{C}; \rho = \sigma$ and $s \in \mathbb{C}$ when $|T| < 1; \rho = \sigma$ and $\Re(s - \xi + \eta) > 1$ when $|T| = 1$ is

$$\phi_{\xi, \eta}^{\rho, \sigma}(T, s, d) = \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{T^m}{(d+m)^s} \quad (6)$$

The following Oberhttinger integral formula [16] is needed for the present investigation

$\int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} dx = 2\partial a^{-\partial} \left(\frac{a}{2}\right)^{\mu} \frac{\Gamma(2\mu)\Gamma(\partial-\mu)}{\Gamma(1+\partial+\mu)}$, on condition that

$$0 < \Re(\mu) < \Re(\partial). \quad (7)$$

2. Main Results

Here integral formulae containing product of two generalized special functions are established and the outcomes are stated in connection with (Wright) hypergeometric function, as in theorem 2.1 and theorem 2.2.

Theorem 2.1: Let $k \in \mathbb{R}, \alpha, \gamma, \varrho, b, c \in \mathbb{C}, \Re(\varrho) > 0$ and for $\mu \in \mathbb{C}, a, \eta \neq 0, -1, -2, \dots; \rho, \sigma \in \mathbb{R}^+; \rho < \sigma$ when $z \in \mathbb{C}; \rho = \sigma$ and $s \in \mathbb{C}$ when $|z| < 1; \rho = \sigma$ and $\Re(s - \xi + \eta) > 1$ when $|z| = 1$ then

$$\int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} \omega_{k, \varrho, b, c}^{\gamma, \alpha}(z) \phi_{\xi, \eta}^{\rho, \sigma}(z, s, d) dx$$

$$\begin{aligned}
 &= \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-q} a^{\mu-\theta-m-q} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{b+1}{2k})^{-1}} \\
 & \cdot {}_3\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (\partial + q + m - \mu, 2), (\partial + q + m + 1, 2) \\ (1, 1), \left(\frac{q}{k} + \frac{b+1}{2k}, \frac{\alpha}{k}\right), (\partial + q + m, 2), (1 + \partial + q + \mu + m, 2) \end{matrix} ; \frac{-cy^2}{4a^2 k\bar{k}^{-1}} \right]
 \end{aligned} \quad (8)$$

Proof: Let $\mathfrak{l}$ represent the left hand side of equation (8).

$$\begin{aligned}
 \mathfrak{l} &= \int_0^{\infty} x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\partial} \omega_{k, \varrho, b, c}^{\gamma, \alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) \\
 & \quad \Phi_{\xi, \eta}^{\rho, \sigma} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx
 \end{aligned}$$

By suitable arrangements, we get

$$\begin{aligned}
 \mathfrak{l} &= \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} 2^{1-\mu-q} a^{\mu-\theta-m-q} \\
 & \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k} + n\right)}{\Gamma\left(\frac{\gamma}{k}\right)} \frac{\Gamma(2\mu)}{n! \Gamma(n+1)} \frac{\Gamma(1+\partial+q+m+2n)}{\Gamma(\partial+m+q+2n)} \frac{\Gamma(\partial+q+m-\mu+2n)}{\Gamma(\partial+q+m+\mu+2n+1)} \\
 & \quad \cdot \frac{1}{(k\bar{k} + \frac{b+1}{2k})^{-1} \Gamma\left(\frac{q}{k} + \frac{b+1}{2k} + \frac{\alpha n}{k}\right)} \cdot \left(\frac{-cy^2}{4a^2 k\bar{k}^{-1}} \right)^n \\
 &= \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-q} a^{\mu-\theta-m-q} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{b+1}{2k})^{-1}} {}_3\Psi_4 \\
 & \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (\partial + m + q - \mu, 2), (\partial + m + q + 1, 2) \\ (1, 1), \left(\frac{q}{k} + \frac{b+1}{2k}, \frac{\alpha}{k}\right), (q + m + \partial, 2), (q + m + \partial + \mu + 1, 2) \end{matrix} ; \frac{-cy^2}{4a^2 k\bar{k}^{-1}} \right]
 \end{aligned}$$

which is the main result.

Theorem 2.2: Let $k \in \mathbb{R}$, $\alpha, \gamma, \varrho, b, c \in \mathbb{C}$, $\Re(\varrho) > 0$ and for $\mu \in \mathbb{C}$, $\alpha, \varrho \neq 0, -1, -2, \dots$; $\rho, \sigma \in \mathbb{R}^+$; $\rho < \sigma$ when $s \in \mathbb{C}$; $\rho = \sigma$ and $s \in \mathbb{C}$ when $|z| < 1$; $\rho = \sigma$ and $\Re(s - \mu + q) > 1$ when $|z| = 1$

Then

$$\begin{aligned}
 & \int_0^{\infty} x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\partial} \omega_{k, \varrho, b, c}^{\gamma, \alpha} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})} \right) \\
 & \quad \Phi_{\xi, \eta}^{\rho, \sigma} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx = \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-q} a^{\mu-\theta-m-q} \Gamma(\partial-\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{b+1}{2k})^{-1}} {}_3\Psi_4 \\
 & \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (m + \partial + q + 1, 2), (2q + 2m + 2\mu, 4) \\ (1, 1), \left(\frac{q}{k} + \frac{b+1}{2k}, \frac{\alpha}{k}\right), (m + \partial + q, 2), (1 + \partial + 2q + 2m + \mu, 4) \end{matrix} ; \frac{-cy^2}{16k\bar{k}^{-1}} \right]
 \end{aligned} \quad (9)$$

Proof: Let $\mathfrak{f}$ represent the left hand side of (9).

$$\mathcal{E} = \int_0^\infty x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\partial} \omega_{k,\varrho,b,c}^{\gamma,\alpha} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})} \right) \\ \Phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx$$

By suitable arrangements, we get

$$\mathcal{E} = \sum_{m=0}^\infty \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+\varrho}}{(d+m)^s} 2^{1-\mu-\varrho-m} a^{\mu-\partial} \\ \sum_{n=0}^\infty \frac{\Gamma(\frac{\gamma}{k}+n)}{\Gamma(\frac{\gamma}{k})} \frac{\Gamma(\partial-\mu)}{n!\Gamma(n+1)} \frac{\Gamma(\varrho+\partial+m+2n+1)}{\Gamma(\varrho+\partial+m+2n)} \frac{\Gamma(2\varrho+2m+2\mu+4n)}{\Gamma(1+\partial+2\varrho+2m+\mu+4n)} \\ \cdot \frac{1}{(k\frac{\varrho}{k}+\frac{b+1}{2k})^{-1}\Gamma(\frac{\varrho}{k}+\frac{b+1}{2k}+\frac{\alpha n}{k})} \cdot \left(\frac{-cy^2}{16k\frac{\alpha}{k}-1} \right)^n \\ = \sum_{m=0}^\infty \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+\varrho}}{(d+m)^s} \frac{2^{1-\mu-\varrho-m} a^{\mu-\partial} \Gamma(\partial-\mu)}{\left(k\frac{\varrho}{k}+\frac{b+1}{2k}-1 \right) \Gamma(\frac{\gamma}{k})} \\ {}_3\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1 \right), (1+\partial+\varrho+m, 2), (2\varrho+2m+2\mu, 4) \\ (1, 1), \left(\frac{\varrho}{k} + \frac{b+1}{2k}, \frac{\alpha}{k} \right), (\partial+\varrho+m, 2), (1+\partial+2\varrho+2m+\mu, 4) \end{matrix} ; \frac{-cy^2}{16k\frac{\alpha}{k}-1} \right]$$

This is the main result.

3. Special cases and Corollaries

In this part, Relation of generalized k Bessel function $\omega_{k,\varrho,b,c}^{\gamma,\alpha}(z)$ with some generalized special functions is discussed and interesting corollaries have been given

Corollary 3.1: For $b = 1$ and $c = 1$, theorem 2.1 reduces as follows, where $J_{k,\varrho}^{\gamma,\alpha}(z)$ is another form of k -Bessel function.

$$\int_0^\infty x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\partial} J_{k,\varrho}^{\gamma,\alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) \Phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx \\ = \sum_{m=0}^\infty \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+\varrho}}{(d+m)^s} \frac{2^{1-\mu-\varrho} a^{\mu-\partial-m-\varrho} \Gamma(2\mu)}{\Gamma(\frac{\gamma}{k}) (k\frac{\varrho}{k}+\frac{1}{k}-1)} {}_3\Psi_4 \\ \left[\begin{matrix} \left(\frac{\gamma}{k}, 1 \right), (\partial+\varrho+m-\mu, 2), (\partial+\varrho+m+1, 2) \\ (1, 1), \left(\frac{\varrho}{k} + \frac{1}{k}, \frac{\alpha}{k} \right), (\partial+\varrho+m, 2), (1+\partial+\varrho+m+\mu, 2) \end{matrix} ; \frac{-y^2}{4a^2k\frac{\alpha}{k}-1} \right] \quad (10)$$

Corollary 3.2: With $c = -1$ and $b = 1$, theorem 2.1 reduces as follows, where $I_{k,\varrho}^{\gamma,\alpha}(z)$ is k -modified Bessel function

$$\int_0^\infty x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\partial} I_{k,\varrho}^{\gamma,\alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) \Phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx$$

$$\begin{aligned}
 &= \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-\varrho} a^{\mu-\partial-m-\varrho} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k}^{\frac{1}{k}-1})} \\
 &{}_3\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (m+\partial+\varrho-\mu, 2), (m+1+\varrho+\partial, 2) \\ (1, 1), \left(\frac{\varrho}{k} + \frac{1}{k}, \frac{\alpha}{k}\right), (m+\varrho+\partial, 2), (m+\partial+\varrho+\mu+1, 2) \end{matrix} ; \frac{y^2}{4a^2 k\bar{k}^{-1}} \right] \quad (11)
 \end{aligned}$$

Corollary 3.3: For $b = 2$ and $c = 1$ theorem 2.1 reduces as follows, where $K_{k,\varrho}^{\gamma,\alpha}(z)$ is k -spherical Bessel function

$$\begin{aligned}
 &\int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} K_{k,\varrho}^{\gamma,\alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) \phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx \\
 &= \sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-\varrho} a^{\mu-\partial-m-\varrho} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k}^{\frac{1}{k}-1})} \\
 &{}_3\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (m+\varrho+\partial-\mu, 2), (m+1+\partial+\varrho, 2) \\ (1, 1), \left(\frac{\varrho}{k} + \frac{3}{2k}, \frac{\alpha}{k}\right), (m+\varrho+\partial, 2), (m+\partial+\varrho+\mu+1, 2) \end{matrix} ; \frac{-y^2}{4a^2 k\bar{k}^{-1}} \right] \quad (12)
 \end{aligned}$$

Similarly, following corollaries (from 3.4 to 3.6) are given with the help of theorem 2.2

Corollary 3.4: For $b = c = 1$ theorem 2.2 reduces in the following form where $I_{k,\varrho}^{\gamma,\alpha}(z)$ is another form of k -Bessel function.

$$\begin{aligned}
 &\int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} I_{k,\varrho}^{\gamma,\alpha} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})} \right) \phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx = \\
 &\sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-\varrho-m} a^{\mu-\partial} \Gamma(\partial-\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k}^{\frac{1}{k}-1})} {}_3\Psi_4 \\
 &\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (1+\partial+\varrho+m, 2), (2\varrho+2m+2\mu, 4) \\ (1, 1), \left(\frac{1}{k} + \frac{\varrho}{k}, \frac{\alpha}{k}\right), (m+\partial+\varrho, 2), (2\varrho+2m+\partial+1+\mu, 4) \end{matrix} ; \frac{-y^2}{16k\bar{k}^{-1}} \right] \quad (13)
 \end{aligned}$$

Corollary 3.5: With $c = -1$ and $b = 1$, theorem 2.2 transform as follows where $I_{k,\varrho}^{\gamma,\alpha}(z)$ is k -modified Bessel function

$$\begin{aligned}
 &\int_0^{\infty} x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} I_{k,\varrho}^{\gamma,\alpha} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})} \right) \phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx = \\
 &\sum_{m=0}^{\infty} \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-\varrho-m} a^{\mu-\partial} \Gamma(\partial-\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k}^{\frac{1}{k}-1})} {}_3\Psi_4 \\
 &\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (1+\partial+\varrho+m, 2), (2\varrho+2m+2\mu, 4) \\ (1, 1), \left(\frac{1}{k} + \frac{\varrho}{k}, \frac{\alpha}{k}\right), (\partial+\varrho+m, 2), (\partial+2\varrho+2m+1+\mu, 4) \end{matrix} ; \frac{y^2}{16k\bar{k}^{-1}} \right] \quad (14)
 \end{aligned}$$

Corollary 3.6: For $b = 2$ and $c = 1$, theorem 2.2 reduces as follows, where $K_{k,q}^{\gamma,\alpha}(z)$ is k -spherical Bessel function

$$\int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} K_{k,q}^{\gamma,\alpha} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})} \right) \phi_{\xi,\eta}^{\rho,\sigma} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})}, s, d \right) dx =$$

$$\sum_{m=0}^\infty \frac{(\xi)_{\rho m}}{(\eta)_{\sigma m}} \frac{y^{m+q}}{(d+m)^s} \frac{2^{1-\mu-q-m} a^{\mu-\partial} \Gamma(\partial-\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{3}{2k})} {}_3\Psi_4$$

$$\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (1+\partial+q+m, 2), (2q+2m+2\mu, 4) \\ (1,1), \left(\frac{q}{k} + \frac{3}{2k}, \frac{\alpha}{k}\right), (\partial+q+m, 2), (1+\partial+2q+2m+\mu, 4) \end{matrix} ; \frac{-y^2}{16k\bar{k}^{-1}} \right] \quad (15)$$

Corollary 3.7: Let all the situations of theorem 2.1 be fulfilled and Hurwitz-Lerch Zeta function become unity $\phi_{\xi,\eta}^{\rho,\sigma}(T, s, d) \rightarrow 1$ then following result holds true due to Menaria et al. [2]

$$\int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} \omega_{k,q,b,c}^{\gamma,\alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) dx$$

$$= y^q \frac{2^{1-\mu-q} a^{\mu-\partial-q} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{q+b+1}{2k})} {}_3\Psi_4$$

$$\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (q+\partial-\mu, 2), (1+\partial+q, 2) \\ (1,1), \left(\frac{q}{k} + \frac{b+1}{2k}, \frac{\alpha}{k}\right), (\partial+q, 2), (1+\partial+q+\mu, 2) \end{matrix} ; \frac{-cy^2}{4a^2 k\bar{k}^{-1}} \right] \quad (16)$$

Corollary 3.8: Let all the situations of theorem 2.2 be satisfied and Hurwitz-Lerch Zeta function become unity $\phi_{\xi,\eta}^{\rho,\sigma}(T, s, d) \rightarrow 1$ then following result holds true due to Menaria et al. [2]

$$\int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} \omega_{k,q,b,c}^{\gamma,\alpha} \left(\frac{xy}{(x+a+\sqrt{x^2+2ax})} \right) dx = \frac{y^q 2^{1-\mu-2q} a^{\mu-\partial} \Gamma(\partial-\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{q+b+1}{2k})} {}_3\Psi_4$$

$$\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (1+\partial+q, 2), (2q+2\mu, 4) \\ (1,1), \left(\frac{q}{k} + \frac{b+1}{2k}, \frac{\alpha}{k}\right), (\partial+q, 2), (1+\partial+2q+\mu, 4) \end{matrix} ; \frac{-cy^2}{16k\bar{k}^{-1}} \right] \quad (17)$$

Corollary 3.9: Assuming that all the conditions of Corollary (3.7) fulfilled then following result holds true due to Menaria et al. [2]

$$\int_0^\infty x^{\mu-1} (x+a+\sqrt{x^2+2ax})^{-\partial} J_{k,q}^{\gamma,\alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) dx = \frac{y^q 2^{1-\mu-q} a^{\mu-\partial-q} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k\bar{k} + \frac{q+1}{k})} {}_3\Psi_4$$

$$\left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (\partial+q-\mu, 2), (1+\partial+q, 2) \\ (1,1), \left(\frac{q}{k} + \frac{1}{k}, \frac{\alpha}{k}\right), (q+\partial, 2), (1+\partial+q+\mu, 2) \end{matrix} ; \frac{-y^2}{4a^2 k\bar{k}^{-1}} \right] \quad (18)$$

Corollary 3.10: Assuming that all the conditions of corollary (3.7) fulfilled then following result holds true due to Menaria et al. [2]

$$\int_0^\infty x^{\mu-1} (x + a + \sqrt{x^2 + 2ax})^{-\partial} I_{k,\varrho}^{\gamma,\alpha} \left(\frac{y}{(x+a+\sqrt{x^2+2ax})} \right) dx = \frac{y^\varrho 2^{1-\mu-\varrho} a^{\mu-\partial-\varrho} \Gamma(2\mu)}{\Gamma\left(\frac{\gamma}{k}\right) (k^{\frac{\varrho}{k}+1} k^{-1})} \cdot {}_3\Psi_4 \left[\begin{matrix} \left(\frac{\gamma}{k}, 1\right), (\partial + \varrho - \mu, 2), (1 + \partial + \varrho, 2) \\ (1, 1), \left(\frac{1}{k} + \frac{\varrho}{k}, \frac{\alpha}{k}\right), (\partial + \varrho, 2), (1 + \mu + \partial + \varrho, 2) \end{matrix} ; \frac{y^2}{4a^2 k^{\frac{\alpha}{k}-1}} \right] \quad (19)$$

4. Conclusion

In present paper, new generalized integral formulae including the product of generalized k-Bessel function and Hurwitz-Lerch Zeta (HLZ) function are obtained and presented in form of the generalized hypergeometric function. Useful and interesting special cases of the main results are also deliberated. In future more integral transform formulae will be used on above functions to find out more interesting results.

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