Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 8, pp. 1787–1795
DOI : 10.47974/JIM-2042

Fixed points theorems for generalized C_{β}^{ψ}-rational contraction mapping in relational metric space

Anjana *
Naveen Mani †
Department of Mathematics
Chandigarh University
Gharuan 140301
Mohali, Punjab
India

Amit Sharma §
Department of Mathematics
Amity University
Gurugram 122001
Haryana
India

Megha Pingale ‡
Department of Mathematics
Guru Gobind Singh College of Engineering and Research Centre
Nashik 422001
Maharashtra
India

Abstract

In this article, by employing the notion of locally f – transitivity binary relation in relational theoretic metric space, to find the existence and uniqueness of fixed point for single self-map with satisfying C_{β}^{ψ} – rational type contractive mapping. Moreover, some examples

* *E-mail:* anjanabhardwaj1999@gmail.com (Corresponding Author)
† *E-mail:* naveenmani81@gmail.com
§ *E-mail:* dba.amitsharma@gmail.com
‡ *E-mail:* meghakothawade17@gmail.com

are also presented to justify the importance of our proved results. Our main outcome extends and generalize in several existing results of literature.

Subject Classification: *47H10, 54H25.*

Keywords: *Fixed point, Binary relation, C_β^ψ - rational type contractive mappings, $\Re^s$- connected, metric space.*

1. Introduction

Numerous extensions of pivotal result of the contraction principle based on metric fixed point theory can be found in the existing literature. In the realm of Mathematics, this principle assures the "existence and uniqueness of fixed points" for self-maps with contraction characteristics in complete metric spaces. Moreover, it presents a useful approach for determining the fixed point of the underlying mapping. In recent times, this theorem improved by number of authors by utilizing further inclusive contractive mappings on many different kinds of spaces. Few of them are rational-type contractions. Many results in existing literature deal with self-maps and pairs of maps verifying rational expressions in various spaces, can be found in [7,8]. The weaker C - contractive condition and established certain theorems concerning fixed points introduced by Suzuki [9] in 2007. A self – map $f: E \to E$ satisfies the C – condition if, $\forall\, \zeta, \eta \in E$,

$$\frac{1}{2}\rho(\zeta, f\zeta) \le \rho(\zeta, \eta) \Rightarrow (\rho(f\zeta, f\eta) \le (\rho(\zeta, \eta)).$$

Motivated by work of the Suzuki [10], Gupta and Mani [10] introduced the concept of "generalized C_β^ψ- contraction condition" and proved certain fixed point theorems in metric spaces with and without partial order relation.

Theorem 1.1: [11] *Let a metric space (E, ρ) and self – map f on E satisfy C_β^ψ- condition if*

$$\frac{1}{2}\rho(\zeta, f\zeta) \le \rho(\zeta, \eta) \Rightarrow \psi\big(\rho(f\zeta, f\eta)\big) \le \beta\big(\rho(\zeta, \eta)\big), \forall\, \zeta, \eta \in E,$$

where $\psi \in \Psi$ and function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Some more relevant literature on study of fixed points defined on above for maps are examined in [3,4,13,14]. In recent times, by utilizing an arbitrary binary relation, Alam and Imdad [2] provided a major extensions of the "Banach contraction principle" introduced the interesting concept which generalized by several authors.

In this paper our main motive is to find some fixed point theorem with "C_β^ψ- rational contraction" condition in relational metric spaces by employing locally f – transitivity property. Our findings broaden and improve the results of

Alam [2] and Mani [11]. Additionally, to get the uniqueness of the fixed point, we've assumed range space to be $\mathfrak{R}^s$ – connected.

2. Foundational concepts:

Here the following section, using notations $E, \mathfrak{R}, \mathbb{N}$ and $\mathbb{N}_0$ stands for a non-void set, nonempty binary relation, set of non-zero positive integers and set of whole numbers as our abbreviations for the various concepts throughout the papers. The following notations along with relevant findings should be studied.

Definition 2.1: [1] Define a binary relation $\mathfrak{R}$ on E, where $E \neq \emptyset$ and $\zeta, \eta \in E$. If either $(\zeta, \eta) \in \mathfrak{R}$ or $(\eta, \zeta) \in \mathfrak{R}$, then we state that ζ and η are $\mathfrak{R}$ - comparative . We written it as $[\zeta, \eta] \in \mathfrak{R}$.

Definition 2.2: [2] Define a binary relation $\mathfrak{R}$ on E, where $E \neq \emptyset$,

(i) $\mathfrak{R}^{-1} = \{(\zeta, \eta) \in E^2 : (\eta, \zeta) \in \mathfrak{R}\}$ is said to be inverse relation of $\mathfrak{R}$.

(ii) $\mathfrak{R}^s = \mathfrak{R} \cup \mathfrak{R}^{-1}$, formally known as symmetric closure.

Definition 2.3: [2] Define a binary relation $\mathfrak{R}$ and self - mapping f on E, where $E \neq \emptyset$, then

(i) $(\zeta, \eta) \in \mathfrak{R}^s \Leftrightarrow [\zeta, \eta] \in \mathfrak{R}$.

(ii) If for any $\zeta, \eta \in E, (\zeta, \eta) \in \mathfrak{R} \Leftrightarrow (f\zeta, f\eta) \in \mathfrak{R}$. Then $\mathfrak{R}$ is called f- closed.

(iii) $\mathfrak{R}^s$ is also f- closed, if $\mathfrak{R}$ is f- closed.

(iv) $\forall\, n \in \mathbb{N}_0$,if $\mathfrak{R}$ is f- closed, say $\mathfrak{R}$ is also f^n- closed, where f^n denotes nth iterate.

(v) If $(\zeta_n, \zeta_{n+1}) \in \mathfrak{R}, n \in \mathbb{N}_0$, then the sequence $\{\zeta_n\}$ contained in E is called $\mathfrak{R}$- preserving.

(vi) For all $k \in \mathbb{N}$, $\exists$ a subsequences $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ with $[\zeta_{n_k}, \zeta] \in \mathfrak{R}$, if whenever $\{\zeta_n\}$ is $\mathfrak{R}$ - preserving sequence and $\zeta_n \overset{\rho}{\rightarrow} \zeta$ as $n \rightarrow \infty$, then $\mathfrak{R}$ is called ρ – self closed.

Definition 2.4: [1] A subset $D \subseteq E$ is described by $\mathfrak{R}$ - connected if for every pair $\zeta, \eta \in D$, $\exists$ a path in $\mathfrak{R}$ from ζ to η.

Definition 2.5: [2] Define a binary relation $\mathfrak{R}$ on a metric space (E, ρ) and a pair of points ζ, η in E. Then, $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k\} \subset E$ be a finite sequence satisfying the conditions are as follows: for every j $(0 \leq j \leq k - 1)$

(i) $\lambda_0 = \zeta$ and $\lambda_k = \eta$,

(ii) $(\lambda_j, \lambda_{j+1}) \in \mathfrak{R}$.

Definition 2.6: [5] A function $\psi \in \Psi$ (collection of all functions), $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous, monotone increasing and $\psi(\tau) = 0 \Leftrightarrow \tau = 0$.

Now, Alam and Imdad [1] introduce by localizing the "notion of transitivity" employed in the framework of binary relation modified by Turinici [6]. Also, for a detailed understanding of $\mathfrak{R}$- completeness and $\mathfrak{R}$- continuity, readers can turn to the insights provided in [1].

3. Fixed Point Results with Binary Relation

Lets begin this section to state and prove our main findings as follows:

Theorem 3.1: Consider a metric space (E, ρ) and arbitrary binary relation $\mathfrak{R}$ on E. A mapping $f : E \to E$ satisfying generalized C_β^ψ- rational contraction and remaining conditions are follows as:

a) $\exists\, \Upsilon \subseteq E, f(E) \subseteq \Upsilon \subset E$ such that (Υ, ρ) is $\mathfrak{R}$ - complete,

b) $E(f, \mathfrak{R})$ is non -void,

c) $\mathfrak{R}$ is locally f - transitivity and f - closed,

d) either f is $\mathfrak{R}$ - continuous or $\mathfrak{R}|_\Upsilon$ is ρ - self closed,

e) for all $\zeta, \eta \in E$ with $(\zeta, \eta) \in \mathfrak{R}$,

$\frac{1}{2}\rho(\zeta, f\zeta) \leq \rho(\zeta, \eta)$ implies $\psi(\rho(f\zeta, f\eta)) \leq \beta(M_f(\zeta, \eta))$,

where,

$$M_f(\zeta, \eta) = \max\left\{\rho(\zeta, \eta), \frac{\rho(\zeta, f\zeta)\rho(\eta, f\eta)}{[1 + \rho(\zeta, \eta)]}, \frac{\rho(\eta, f\eta)[1 + \rho(\zeta, f\zeta)]}{[1 + \rho(\zeta, \eta)]}\right\},$$

$\psi \in \Psi$ and function $\beta : [0, \infty) \to [0, \infty)$ remains continuous satisfying $0 < \beta(\tau) < \psi(\tau), \tau > 0$. Then f admits a fixed point.

Proof: Here three steps, we finish the proof of theorem.

Step 1: We have $E(f, \mathfrak{R})$ is non -void. Let $\zeta_0 \in E(f, \mathfrak{R})$ then $(\zeta_0, f\zeta_0) \in \mathfrak{R}$. Construct a Picard sequence $\{\zeta_n\}$ based at initial point ζ_0 with iteration i.e., $\zeta_n = f^n(\zeta_0)$, for all $n \in \mathbb{N}_0$.

Since $(\zeta_0, f\zeta_0) \in \mathfrak{R}$ and $\mathfrak{R}$ is f - closed therefore

$$(f\zeta_0, f^2\zeta_0), (f^2\zeta_0, f^3\zeta_0), \ldots, (f^n\zeta_0, f^{n+1}\zeta_0), \ldots, \in \mathfrak{R}$$

and

$$(\zeta_n, \zeta_{n+1}) \in \mathfrak{R}, \forall\, n \in \mathbb{N}_0 \tag{1}$$

which yields $\{\zeta_n\}$ is $\mathfrak{R}$ - preserving sequence. Now, from Theorem $3.1(e)$ we obtain, $\forall\, n \in \mathbb{N}_0$, that

$$\frac{1}{2}\rho(\zeta_n, f\zeta_n) = \frac{1}{2}\rho(\zeta_n, \zeta_{n+1}) \leq \rho(\zeta_n, \zeta_{n+1})$$

implies

$$\psi(\rho(f\zeta_n, f\zeta_{n+1})) \leq \beta(M_f(\zeta_n, \zeta_{n+1})), \tag{2}$$

where on solving , we get,

$$M_f(\zeta_n, \zeta_{n+1}) \leq \max\{\rho(\zeta_n, \zeta_{n+1}), \rho(\zeta_{n+1}, \zeta_{n+2})\}.$$

Using (2), we get

$$\psi\big(\rho(f\zeta_n, f\zeta_{n+1})\big) \le \beta(\max\{\rho(\zeta_n,\zeta_{n+1}), \rho(\zeta_{n+1},\zeta_{n+2})\}) \tag{3}$$

If $\psi(\rho(\zeta_{n+1},\zeta_{n+2})) \le \beta(\rho(\zeta_{n+1},\zeta_{n+2})$, then it contradicts to our assumption and hence, from (3), we have

$$\psi\big(\rho(\zeta_{n+1},\zeta_{n+2})\big) \le \beta\big(\rho(\zeta_n,\zeta_{n+1})\big) \tag{4}$$

We get, $\rho(\zeta_{n+1},\zeta_{n+2}) \le \rho(\zeta_n,\zeta_{n+1})$ [by the property of ψ and β]

Thus, the sequence $\{\rho(\zeta_n,\zeta_{n+1})\}$ is bounded and monotone non-increasing. Therefore, $\exists\ \ell \ge 0$ such that $\lim_{n\to\infty} \rho(\zeta_n,\zeta_{n+1}) = \ell \ge 0$

Letting the limit as $n \to \infty$ in inequality (4), we get $\psi(\ell) \le \beta(\ell)$, which contradicts to our assumption. Thus, we conclude that $\ell = 0$, therefore,

$$\lim_{n\to\infty} \rho(\zeta_n,\zeta_{n+1}) = \ell = 0 \tag{5}$$

Step 2: On contrary, suppose that $\{\zeta_n\}$ is not Cauchy sequence. Then for an $\epsilon > 0$, $\exists\ \mathbb{N}_0 \in \mathbb{N}$ and $\mathbb{N}_1 = \max\{m_i, \mathbb{N}_0\}$ such that $\forall\ m_k > n_k \ge \mathbb{N}_1$, we obtain

$$\rho\big(\zeta_{n_k}, \zeta_{n_{k+1}}\big) < \epsilon \le \rho\big(\zeta_{n_k}, \zeta_{m_k}\big),$$

Applying the assumption of locally f - transitivity of binary relation $\mathfrak{R}$ yields $\big(\zeta_{n_k}, \zeta_{m_k}\big) \in \mathfrak{R}$ and from assumption (e), we have

$$\tfrac{1}{2}\rho\big(\zeta_{n_k}, \zeta_{n_{k+1}}\big) \le \rho\big(\zeta_{n_k}, \zeta_{m_k}\big)$$

implies $$\psi(\rho(f\zeta_{n_k}, f\zeta_{m_k})) \le \beta(M_f(\zeta_{n_k}, \zeta_{m_k})). \tag{6}$$

On solving and letting the limit as $k \to \infty$ in (6) and by using Lemma,(Lemma 2.9, [12]), we obtain $\psi(\epsilon) \le \beta(\epsilon)$ which contradicts to our assumption. Since (Υ,ρ) is $\mathfrak{R}$ - complete, $\exists\ \lim_{n\to\infty} \zeta_n = \upsilon$. Therefore, the sequence $\{\zeta_n\}$ is a $\mathfrak{R}$- preserving Cauchy in Υ. **Step 3.** Now, we claim that υ is a fixed point of f.

Case (i): Firstly, if f is continuous mapping and $\{\zeta_n\}$ is $\mathfrak{R}$ - preserving with $\zeta_n \xrightarrow{\rho} \upsilon$, then $\mathfrak{R}$ - continuity of f implies $\zeta_{n+1} = f\zeta_n \xrightarrow{\rho} f\upsilon$ and by uniqueness of the limit, i.e., $f\upsilon = \upsilon$ that is, υ is a fixed point of f.

Case (ii): Alternatively, $\exists$ a subsequences $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ with $[\zeta_{n_k}, z] \in \mathfrak{R}$ if $\mathfrak{R}|_\Upsilon$ is ρ - self closed. So, using condition Theorem $3(e)$, we have

$$\tfrac{1}{2}\rho(\zeta_{n_k}, f\zeta_{n_k}) = \tfrac{1}{2}\rho(\zeta_{n_k}, \zeta_{n_{k+1}}) \le \rho(\zeta_{n_k}, \upsilon)\ \forall\ k \in \mathbb{N},$$

this implies

$$\psi(\rho(f\zeta_{n_k}, f\upsilon)) \le \beta(M_f(\zeta_{n_k}, \upsilon)), \tag{7}$$

where, on solving, we get, $M_f\big(\zeta_{n_k}, \upsilon\big) \le \max\{\rho(\upsilon, f\upsilon), \rho(\zeta_{n_k}, \upsilon)\}$.

If $M_f(\zeta_{n_k}, \upsilon) = \rho(\upsilon, f\upsilon)$, then (7) give that

$$\psi(\rho(f\zeta_{n_k}, f\upsilon)) \le \beta(\rho(\upsilon, f\upsilon)). \tag{8}$$

On taking the limit as $k \to \infty$ and using the fact that $\rho(\upsilon, f\upsilon) > 0$, we have $\psi(\rho(\upsilon, f\upsilon)) \leq \beta(\rho(\upsilon, f\upsilon))$. This contradicts our assumption. Thus, our findings that $\rho(\upsilon, f\upsilon) = 0 \Rightarrow f\upsilon = \upsilon$.

Secondly, if $M_f\ (\zeta_{n_k}, \upsilon) = \rho(\zeta_{n_k}, \upsilon)$, then on using (7), we get

$$\psi\left(\rho\left(\zeta_{n_{k+1}}, f\upsilon\right)\right) = \psi\left(\rho\left(f\zeta_{n_k}, f\upsilon\right)\right) \leq \beta\left(\rho\left(\zeta_{n_k}, \upsilon\right)\right), \forall\, k \in \mathbb{N}_0. \tag{9}$$

Let us consider a partition $\{\mathbb{N}_0, \mathbb{N}^+\}$ of N due to the two distinct possibilities that find here , i.e., $\mathbb{N}_0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}_0 \cap \mathbb{N}^+ = \emptyset$ satisfying that

(i) $\rho(\zeta_{n_k}, \upsilon) = 0\ \forall\, k \in \mathbb{N}_0$. (ii) $\rho(\zeta_{n_k}, \upsilon) > 0\ \forall\, k \in \mathbb{N}_0$.

In case (i), if $\rho(\zeta_{n_k}, \upsilon) = 0$ for all $\in \mathbb{N}_0$, that is $\psi(\rho(f\zeta_{n_k}, f\upsilon)) = 0 \Leftrightarrow \rho(f\zeta_{n_k}, f\upsilon) = 0$, for all $k \in \mathbb{N}_0$, then condition (9) holds for all $k \in \mathbb{N}_0$.

In case (ii), if $\rho(\zeta_{n_k}, \upsilon) > 0\ \forall\, k \in \mathbb{N}_0$, then we obtain $\psi(\rho(f\zeta_{n_k}, f\upsilon)) \leq \beta(\rho(\zeta_{n_k}, \upsilon))$ for all $k \in \mathbb{N}^+$. Letting limit as $k \to \infty$ and by the continuity of ψ and β, we get $\psi(\rho(\upsilon, f\upsilon)) \leq \beta(\rho(\upsilon, \upsilon)) = 0$. By using the property of ψ, we get $\rho(\upsilon, f\upsilon) = 0 \Rightarrow f\upsilon = \upsilon$, which concludes condition also holds for all $k \in \mathbb{N}^+$. Thus, the condition (9) holds for all $n \in \mathbb{N}$. Letting limit as $k \to \infty$ in (9) and using $\zeta_{n_k} \overset{\rho}{\to} \upsilon$, we get $\zeta_{n_{k+1}} \overset{\rho}{\to} f\upsilon$. Since the limit is unique, we get $f\upsilon = \upsilon$ that is, υ is a fixed point of f. Hence, the result is proved.

Remark 3.2: If the assumption of Theorem 3.1 that locally f- transitivity of $\mathfrak{R}$ remains true is replaced by the assumptions are follows as: (a) $\mathfrak{R}$-transitivity, (b) f-transitivity of $\mathfrak{R}$ and (c) locally transitivity of $\mathfrak{R}$.

Theorem 3.3: *In addition to the assumptions of Theorem* 3.1*, if $f(E)$ is $\mathfrak{R}^s$-connected, then f admits a unique fixed point.*

Proof: Let two fixed points ζ and η of f in relation metric space (E, ρ). Then,

$$f^n\zeta = \zeta \text{ and } f^n\eta = \eta, \text{ for all } n \in \mathbb{N}.$$

Clearly, $\zeta, \eta \in f(E)$ and $f(E)$ is $\mathfrak{R}^s$- connected, $\exists$ a path, say $(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_k)$ of some finite length k in $\mathfrak{R}^s$, from ζ to η such that $\lambda_0 = \zeta$, $\lambda_k = \eta$, and $[\lambda_j, \lambda_{j+1}] \in \mathfrak{R}$. As $\mathfrak{R}$ is f - closed, we get $[f^n\lambda_j, f^n\lambda_{j+1}] \in \mathfrak{R}$, $\forall\, n \in \mathbb{N}_0$ and $j\ (0 \leq j \leq k-1)$. If $(\zeta, \eta) \in \mathfrak{R}^s$ then either $(f^n\zeta, f^n\eta) \in \mathfrak{R}$ or $(f^n\eta, f^n\zeta) \in \mathfrak{R}$ and by f - closedness of $\mathfrak{R}$, we have $(f^n\zeta, f^n\eta) \in \mathfrak{R}$, for $n = 0, 1, 2, \ldots$.

From Theorem 3.1. (e), we get,

$$\tfrac{1}{2}\rho(\zeta, f\zeta) \leq \rho(\zeta, \eta)$$

implies, on solving

$$\psi(\rho(\zeta, \eta)) = \psi(\rho(f^n\zeta, f^n\eta)) \leq \beta\left(M_f\ (f^{n-1}\zeta, f^{n-1}\eta)\right) \leq \beta(\rho(\zeta, \eta)) \tag{10}$$

This leads to a contradiction. Thus, we obtain that $\psi\big(\rho(\zeta,\eta)\big) = 0$ implies $\rho(\zeta,\eta) = 0$, that is $\zeta = \eta$. Hence f admits a unique fixed point.

Instead, $\exists$ a path of length $k > 1$ in $\mathfrak{R}^s$, if $(\zeta,\eta) \neq \mathfrak{R}^s$. We define $t_n^j = \rho(f^n\lambda_j, f^n\lambda_{j+1}) \in \mathfrak{R}^s$, for $j\ (0 \leq j \leq (k-1))$ and $n \in \mathbb{N}_0$. Moreover, from the Theorem 3.1 and for any fix j, we have

$$\psi(t_n^j) = \psi\left(\rho(f^n\lambda_j, f^n\lambda_{j+1})\right) \leq \beta\left(M_f\left(f^{n-1}\lambda_j, f^{n-1}\lambda_{j+1}\right)\right)$$

where $M_f\left(f^{n-1}\lambda_j, f^{n-1}\lambda_{j+1}\right) = \rho\left(f^{n-1}\lambda_j, f^{n-1}\lambda_{j+1}\right)$ for all $n \in \mathbb{N}$

By using (10), we get

$$\psi\big(t_n^j\big) = \psi\left(\rho\big(f^n\lambda_j, f^n\lambda_{j+1}\big)\right) \leq \beta\left(\rho\big(f^{n-1}\lambda_j, f^{n-1}\lambda_{j+1}\big)\right) \leq \beta(t_{n-1}^j) \quad (11)$$

Consequently, $\{\psi(t_n^j)\} = \{\psi\big(\rho\big(f^{n-1}\lambda_j, f^{n-1}\lambda_{j+1}\big)\big\}$ is a positive decreasing sequence. Monotonicity of ψ gives us the sequence $\{t_n^j\}$ is also a decreasing and consequently, $\exists\, t \geq 0$ such that $\lim_{n\to\infty} t_n^j = t$. Taking the limit a $n \to \infty$ in (11) and utilizing the monotonicity of functions ψ and β, we get, $\psi(t) \leq \beta(t)$

This implies $\psi(t) = 0$ and consequently $t = 0$, for every $j\ (0 \leq j \leq k-1)$. Lastly, from above conclusion and using triangular inequality, we obtain

$$\rho(\zeta,\eta) = \rho(f^n\lambda_0, f^n\lambda_k) \leq t_n^0 + t_n^1 + \ldots.. + t_n^{k-1} \to 0, \text{as } n \to \infty.$$

Hence f admits a unique fixed point.

Remark 3.4: If the assumption of Theorem 3.3 replace by the assumption that either $R|_{f(E)}$ is complete or $f(E)$ is $\mathfrak{R}^s$- directed [2]. Then the Theorem 3.1 remains true.

Example 3.5: Define binary relation $(\zeta,\eta) \in \mathfrak{R}$ implies $\zeta < \eta$ on E and a self mapping f on E , where $E = [0,\infty)$ with usual metric ρ such that

$$f\zeta = \frac{\zeta}{\zeta+1}, \forall\, \zeta \in E.$$

Then, it is verify that $\mathfrak{R}$ is f-closed and mapping f is $\mathfrak{R}$ is continuous. Again, $\forall\, \zeta,\eta \in E$ with $(\zeta,\eta) \in \mathfrak{R}$, define ψ and $\beta : [0,\infty) \to [0,\infty)$ be an altering distance function and continuous function respectively by $\psi(\tau) = \tau$; $\beta(\tau) = \frac{2}{3}\tau, \forall\, \tau > 0$, we have

$$\psi(\rho(f\zeta, f\eta)) < \beta(M_f(\zeta,\eta)).$$

Thus, Theorem 3.1 are verified all the assumptions and f has a fixed points in E (i.e., at $\zeta = 0$).

Example 3.6: Define binary relation $(\zeta,\eta) \in \mathfrak{R}$ implies $\zeta < \eta$ on E and a self mapping f on E , where $E = [0,\infty)$ with usual metric ρ such that

$$f\zeta = \frac{\zeta}{2}, \forall\, \zeta \in E,$$

Then, it is verify that $\mathfrak{R}$ is f-closed and mapping f is $\mathfrak{R}$ is continuous. Again, $\forall\, \zeta, \eta \in E$ with $(\zeta, \eta) \in \mathfrak{R}$, define ψ and $\beta : [0, \infty) \to [0, \infty)$ be an altering distance function and continuous function respectively by $\psi(\tau) = \tau$; $\beta(\tau) = \frac{4}{5}\tau, \forall\, t > 0$. we have

$$\psi(\rho(f\zeta, f\eta)) < \beta(M_f(\zeta, \eta)).$$

Thus, Theorem 3.1 are verified all the assumptions and f has a fixed points in E (i.e., at $\zeta = 0$).

4. Conclusion

In the present study, we've demonstrated the "existence of a fixed point" that satisfies "generalized C_β^ψ- rational contraction" mapping in "Relational Theoretic Metric Spaces". To ensure the "uniqueness of the fixed point", we've assumed that the range space is $\mathfrak{R}^s$ – connected. This proof relies in the utilization of the concept of locally f - transitivity within relational theoretic metric spaces. Furthermore, to support the validity of our findings, examples 3.5 and 3.6 are provided.

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Received July, 2024