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Fixed point theorems for $(\dot{\zeta}, \dot{\alpha}, \dot{\beta})$-contraction

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Abstract

This project aims to introduce $(\dot{\zeta}, \dot{\alpha}, \dot{\beta})$-contractions of type-A in b-metric-like spaces (BMLS) utilizing $(\dot{\alpha}, \dot{\beta})$-Geraghty contractions [9] introduced in 2019. In the first section of this work, a new class $\dot{\Upsilon}_p$ is defined as a collection of maps with certain appropriate constraints. Then, using this class $\dot{\zeta} : [0,\infty)^p \to [0,\infty)$, some fixed point findings are presented with an illustration for efficacy check. Many results in the literature are supported by our findings.

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1. Introduction

In 1922, Banach [12] started researching constructive theory in metric space. One of the most valuable and significant theorems in traditional functional analysis is the Banach Contraction Theorem (BCT). Numerous results have been proved to expand this renowned finding. In 1969, Nadler [4] established some related findings for multi-value maps. This idea has

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several more expansions, which one may find in ([1], [3], [5], [6], [10]). In order to generalize BCT, Samet *et al.* [2] created $\alpha-\psi$-contractive map and investigated some results using this class. In 2015, S.Chandok [13] developed the idea of $(\dot{\alpha},\dot{\beta})$-admissible mappings. Later on, Pant and Panicker [11] worked on $(\dot{\alpha},\dot{\beta})$-Geraghty contractions in 2016 and then Mlaiki *et al.* [9] extended the research of [11] in 2019 and obtained some findings for this family of maps.

Czerwik [14] defined the concept of b-metric space as follows:

Definition 1.1: [14] Assume that $K\geq 1$ is a real. A map $f_b : E\times E \to [0,\infty]$ if satisfies the axioms for $l,w,p\in E,$ is referred to as a b-metric space.

$$(f_b1)\ f_b(l,w)=0 \Leftrightarrow l=w;$$

$$(f_b2)\, f_b(l,w)=f_b(w,l);$$

$$(f_b3)\, f_b(l,p)\leq K[f_b(l,w)+f_b(w,p)].$$

A bMS is defined as a pair (E,f_b).

We require the ideas and literary findings listed below to support our findings.

Definition 1.2: [13] For a bMS (E,f_b), let $\dot{G}:E\to E$ and $\dot{\alpha},\ \dot{\beta}:E\times E\to [0,\infty)$. Then map $\dot{G}$ is $(\dot{\alpha},\dot{\beta})$-admissible, if $\dot{\alpha}(l,w)\geq 1$ and $\dot{\beta}(l,w)\geq 1 \Rightarrow \dot{\alpha}(\dot{G}l,\ \dot{G}w)\geq 1$ and $\dot{\beta}(\dot{G}l,\ \dot{G}w)\geq 1$ for all $l,w\in E$.

Definition 1.3: [13] For $\dot{G}:E\to E$ and $\dot{\alpha},\dot{\beta}\ :\ E\times E\to[0,\infty)$, set E is called $(\dot{\alpha},\dot{\beta})$-regular, if for a sequence $\{l_r\}\in E$ such that $\{l_r\}\to l;$ $\dot{\alpha}(l_r,l_{r+1})\geq 1$ and $\dot{\beta}(l_r,l_{r+1})\geq 1$ for all r, there is a corresponding subsequence $\{l_{r_k}\}$ of $\{l_r\}$ such that $\dot{\alpha}(l_{r_k},\ l_{r_{k+1}})\geq 1$ and $\dot{\beta}(l_{r_k},\ l_{r_{k+1}})\geq 1$ for all $k,$ $\dot{\alpha}(l,\ \dot{G}l)\geq 1$ and $\dot{\beta}(l,\ \dot{G}l)\geq 1$.

Definition 1.4: [13] Let $\dot{\Psi}$ be the collection of maps $\dot{\psi}:[0,\infty)\to[0,\infty)$ which are strictly increasing and continuous with $\dot{\psi}(0)=0$.

Definition 1.5: [13] Let $\dot{\Theta}$ be the collection of maps $\dot{\theta}:[0,\infty)\to[0,1)$ such that $\dot{\theta}(s_r)\to 1\Rightarrow s_r\to 0$ for a bounded sequence $\{s_r\}$ of +ve reals.

The following contractive mapping was introduced by Pant and Panicker [11] in 2016.

Definition 1.6: [11] Let $(E, f_b, K \geq 1)$ be a bMS. Then a map $\dot{G}: E \to E$ is called $(\dot{\alpha}, \dot{\beta})$-Geraghty contraction map, if there are $\dot{\alpha}, \dot{\beta} : E \times E \to [0, \infty)$, $\dot{\psi} \in \dot{\Psi}$, $\dot{\theta} \in \dot{\Theta}$ such that

$$\dot{\alpha}(l, \dot{G}l)\dot{\beta}(w, \dot{G}w)\dot{\psi}(K^3 f_b(\dot{G}l, \dot{G}w)) \leq \dot{\theta}(\dot{\psi}(N(l, w)))\dot{\psi}(N(l, w)); \, l, \, w \in E,$$

where $N(l, w) = \max\left\{ f_b(l, w), f_b(l, \dot{G}l), f_b(w, \dot{G}w), \frac{f_b(l, \dot{G}w) + f_b(\dot{G}l, w)}{2K} \right\}$

The following contraction was established by Mlaiki *et al.* [9] in 2019 by giving extension to the work of Pant and Panicker [11].

Definition 1.7: [9] Let $(E, f_b, K \geq 1)$ be a bMS. Then map $\dot{G}: E \to E$ is called an $(\dot{\alpha}, \dot{\beta})$-contraction, if there are $\dot{\alpha}, \dot{\beta} : E \times E \to [0, \infty)$, $\dot{\psi} \in \dot{\Psi}$, $\varepsilon > 1$ such that $\dot{\alpha}(l, \dot{G}l)\ \dot{\beta}(w, \dot{G}w)\ \dot{\psi}(K^{\varepsilon}\ f_b(\dot{G}l,\ \dot{G}w)) \leq \dot{\psi}(N(l,\ w)); \, l, w \in E,$ where

$$N(l, w) = \max\left\{ f_b(l, w), f_b(l, \dot{G}l), f_b(w, \dot{G}w), \frac{f_b(l, \dot{G}w) + f_b(\dot{G}l, w)}{2K} \right\}.$$

Definition 1.8: [9] Let $(E, f_b, K \geq 1)$ be a complete bMS and $\dot{G}: E \to E$ be a map holding following claims:

1. $\dot{G}$ is $(\dot{\alpha}, \dot{\beta})$-contraction;
2. $\dot{G}$ is $(\dot{\alpha}, \dot{\beta})$-admissible;
3. there is $l_0 \in E$ with $\dot{\alpha}(l_0, \dot{G}l_0) \geq 1$ and $\dot{\beta}(l_0, \dot{G}l_0) \geq 1$;
4. E is $(\dot{\alpha}, \dot{\beta})$-regular or $\dot{G}$ is continuous.

Then $\dot{G}$ has a single fixed point.

Mohammed Ali Alghamdi [7] presented an extension to bMS in 2013 that took non-zero self distance into account and went by the name of b-metric-like space (bMLS).

Definition 1.9: [7] A b-metric-like on the set E is a map $f_{bl} : E \times E \to [0, +\infty)$ if the following claims hold for $l, w, z \in E$:

$$(f_{bl}1)\ f_{bl}(l, w) = 0 \Rightarrow l = w;$$

$$(f_{bl}2)\ f_{bl}(l, w) = f_{bl}(w, l);$$

$(f_{bl}3)$ $f_{bl}(l,w) \le K(f_{bl}(l,z)+f_{bl}(z,w))$ where $K \ge 1$.

The triplet $(E, f_{bl}, K \ge 1)$ is called a bMLS.

Definition 1.10: [7] Let be a bMLS. For a sequence $\{l_r\} \in E$ and $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ with $\lim_{r\to\infty} f_{bl}(l,l_r) = f_{bl}(l,l)$ for $l \in E$, then l is called the limit of this sequence.

Definition 1.11: [7] In a bMLS:

(a) A sequence $\{l_r\} \in E$ is called Cauchy if $\lim\limits_{r,p\to\infty} f_{bl}(l_r,l_p)$ finitely exists.
(b) (E, f_{bl}) is called a complete space iff every Cauchy sequence $\{l_r\}$ converges in it and $\lim\limits_{r,p\to\infty} f_{bl}(l_r,l_p) = f_{bl}(l,l) = \lim\limits_{r\to\infty} f_{bl}(l_r,l)$.

Proposition 1.12: [7] Let $(E, f_{bl}, K \ge 1)$ be a bMLS and $\{l_r\} \in E$ such that for $l \in E$, $\lim_{r\to\infty} f_{bl}(l_r,l) = 0$. Then,

i) l is unique,

ii) $\frac{1}{K} f_{bl}(l,w) \le \lim\limits_{r\to\infty} f_{bl}(l_r,w) \le K f_{bl}(l,w)$ $\forall$ $w \in E$.

Lemma 1.13: *[8] Let $(E, f_{bl}, K \ge 1)$ be a bMLS and $\{l_r\} \in E$ with $f_{bl}(l_r,l_{r+1}) \le \lambda f_{bl}(l_{r-1},l_r)$ for $\lambda \in [0,1)$ and $r \in \mathbb{N}$. Then, $\{l_r\}$ is Cauchy with $\lim_{r,p\to\infty} f_{bl}(l_r,l_p) = 0$.*

2. Main Results

In this paper, we offer a type-A $(\dot{\zeta}, \dot{\alpha}, \dot{\beta})$ -contractive map in b-metric-like spaces. These novel ideas are used to establish certain original fixed point theorems, and the results extend numerous findings in the literature.

Definition 2.1 For any $p \in \mathbb{N}$, define $\dot{\Upsilon}_p$ as the set of functions $\dot{\zeta} : [0,\infty)^p \to [0,\infty)$ such that

$(\dot{\zeta}_1)$ $\dot{\zeta}(s_1, s_2, ..., s_p) < \max\{s_1, s_2, ..., s_p\}$ if $(s_1, s_2, ..., s_p) \ne (0,0,...,0)$;

$(\dot{\zeta}_2)$ if $\{s_i^{(r)}\}_{r\in\mathbb{N}}$, $1 \le i \le p$, are p sequences in $[0,\infty)$ such that

$$\lim_{r\to\infty} \sup s_i^{(r)} = s_i < \infty \text{ for all } i = 1 \text{ to } p, \text{ then,}$$

$$\liminf_{r\to\infty} \dot{\zeta}(s_1^{(r)}, s_2^{(r)}, \ldots, s_p^{(r)}) \le \dot{\zeta}(s_1, s_2, \ldots, s_p).$$

Definition 2.2: Let $(E, f_{bl}, K \ge 1)$ be a bMLS. Then map $\dot{G}: E \to E$ is called type-A $(\dot{\zeta}, \dot{\alpha}, \dot{\beta})$-contraction, if there are $\dot{\alpha}, \dot{\beta}$: $E \times E \to [0,\infty)$, $\dot{\psi} \in \dot{\Psi}$, $\dot{\zeta} \in \dot{\Upsilon}_4$, $\varepsilon > 1$ such that

$$\dot{\alpha}(l,\dot{G}l)\dot{\beta}(w,\dot{G}w)\dot{\psi}(K^{\varepsilon} f_{bl}(\dot{G}l,\dot{G}w)) \le \dot{\psi}\left(K^{\varepsilon-1}\dot{\zeta}\left(f_{bl}(l,w), f_{bl}(l,\dot{G}l), f_{bl}(w,\dot{G}w), \frac{f_{bl}(l,\dot{G}w)+f_{bl}(\dot{G}l,w)-f_{bl}(w,w)}{2K}\right)\right), \tag{2.1}$$

for $l, w \in E$.

Theorem 2.3: Let $(E, f_{bl}, K \ge 1)$ be a bMLS and $\dot{G}: E \to E$ be a map such that the following conditions hold:

1. $\dot{G}$ is type-A $(\dot{\zeta}, \dot{\alpha}, \dot{\beta})$-contraction;
2. $\dot{G}$ is $(\dot{\alpha}, \dot{\beta})$-admissible;
3. there is $l_0 \in E$ with $\dot{\alpha}(l_0, \dot{G}l_0) \ge 1$ and $\dot{\beta}(l_0, \dot{G}l_0) \ge 1$;
4. E is $(\dot{\alpha}, \dot{\beta})$-regular w.r.t. $\dot{G}$ or $\dot{G}$ is continuous;
5. $f_{bl}(l,\dot{G}w) + f_{bl}(\dot{G}l,w) \ge f_{bl}(w,w)$.

Then $\dot{G}$ admits a fixed point. Also, if $\dot{G}l = l$ and $\dot{\beta}(l,l) \ge 1$, then $\dot{G}$ admits a single fixed point.

Proof: Define the sequence $\{l_r\} \in E$ as $l_r = \dot{G}l_{r-1}$ for $r \ge 1$. Assume that elements of $\dot{G}$ are consecutively distinct else $\dot{G}$ admits a fixed point. Using induction and the conditions (ii) and (iii), we now arrive to:

$$\dot{\alpha}(l_r, l_{r+1}) \ge 1 \text{ and } \dot{\alpha}(l_r, l_{r+1}) \ge 1 \ \forall\, r \ge 1.$$

Now we claim that $\{l_r\}$ is Cauchy for $r \in \mathbb{N}$,

$$\dot{\psi}(K^{\varepsilon} f_{bl}(l_r, l_{r+1}))$$

$$\le \dot{\alpha}(l_{r-1}, l_r)\dot{\beta}(l_r, l_{r+1})\dot{\psi}(K^{\varepsilon} f_{bl}(l_r, l_{r+1}))$$

$$= \dot{\alpha}(l_{r-1}, \dot{G}l_{r-1})\dot{\beta}(l_r, \dot{G}l_r)\dot{\psi}(K^{\varepsilon} f_{bl}(\dot{G}l_{r-1}, \dot{G}l_r))$$

$$\le \dot{\psi}\left(K^{\varepsilon-1}\dot{\zeta}\left(f_{bl}(l_{r-1}, l_r), f_{bl}(l_{r-1}, l_r), f_{bl}(l_r, l_{r+1}), \frac{f_{bl}(l_{r-1},l_{r+1})+f_{bl}(l_r,l_r)-f_{bl}(l_r,l_r)}{2K}\right)\right)$$

which implies that

$$K^{\varepsilon} f_{bl}(l_r, l_{r+1}) \leq K^{\varepsilon-1}\dot{\zeta}\left(f_{bl}(l_{r-1}, l_r), f_{bl}(l_{r-1}, l_r), f_{bl}(l_r, l_{r+1}), \frac{f_{bl}(l_{r-1}, l_{r+1})}{2K}\right)$$

$$\Rightarrow f_{bl}(l_r, l_{r+1}) \leq \frac{1}{K}\dot{\zeta}\left(f_{bl}(l_{r-1}, l_r), f_{bl}(l_{r-1}, l_r), f_{bl}(l_r, l_{r+1}), \frac{f_{bl}(l_{r-1}, l_{r+1})}{2K}\right) \tag{2.2}$$

$$< \frac{1}{K}\max\left\{f_{bl}(l_{r-1}, l_r), f_{bl}(l_{r-1}, l_r), f_{bl}(l_r, l_{r+1}), \frac{f_{bl}(l_{r-1}, l_{r+1})}{2K}\right\}$$

$$= \frac{1}{K}\max\left\{f_{bl}(l_{r-1}, l_r), \frac{f_{bl}(l_{r-1}, l_{r+1})}{2K}\right\}$$

$$\leq \frac{1}{K}\max\left\{f_{bl}(l_{r-1}, l_r), \frac{f_{bl}(l_{r-1}, l_{r+1})}{2K}\right\}, \tag{2.3}$$

Now, $f_{bl}(l_{r-1}, \dot{G}l_r) + f_{bl}(\dot{G}l_r, l_r) = f_{bl}(l_{r-1}, l_{r+1}) + f_{bl}(l_r, l_r) \geq f_{bl}(l_r, l_r)$

Therefore, we obtain

$$f_{bl}(l_r, l_{r+1}) \leq \frac{1}{K}\max\left\{f_{bl}(l_{r-1}, l_r), \frac{f_{bl}(l_{r-1}, l_r)+f_{bl}(l_r, l_{r+1})}{2K}\right\}, \tag{2.4}$$

which implies for $r \geq 1$, $f_{bl}(l_r, l_{r+1}) < \frac{1}{K} f_{bl}(l_{r-1}, l_r)$.

Case 1: If $K > 1$, then l_r is Cauchy according to Lemma 1.13 in light of (2.4).

Case 2: If $K = 1$, the sequence $\{f_{bl}(l_r, l_{r+1})\}$ is monotonically decreasing and is bounded below (by (2.4)), where

$$l' = \limsup_{r\to\infty} \frac{f_{bl}(l_{r-1}, l_{r+1})}{2} \leq \limsup_{r\to\infty} \frac{f_{bl}(l_{r-1}, l_r)+f_{bl}(l_r, l_{r+1})}{2} = l.$$

Now, $l \leq \dot{\zeta}(l, l, l, l') < \max\{l, l, l, l'\} = l$, a contradiction, thus,

$$\lim_{r\to\infty} f_{bl}(l_r, l_{r+1}) = 0. \tag{2.5}$$

Also, $f_{bl}(l_r, l_r) \leq f_{bl}(l_r, l_{r+1}) + f_{bl}(l_{r+1}, l_r)$,
By (2.5) and taking $\limsup r \to \infty$, we get

$$\lim_{r\to\infty} f_{bl}(l_r, l_r) = 0. \tag{2.6}$$

If $\{l_r\}$ is not Cauchy, then for $\varepsilon > 0$ and for each $r \in \mathbb{N}$, there exists $p_r > r_r \geq r$ such that

$$f_{bl}(l_{p_r}, l_{r_r}) \geq \varepsilon. \tag{2.7}$$

Assume as well that $p_r > r_r$ is the lowest natural that allows (2.7) to hold. Now,

$$\begin{aligned} \varepsilon &\leq f_{bl}(l_{p_r}, l_{r_r}) \\ &\leq f_{bl}(l_{p_r}, l_{p_r-1}) + f_{bl}(l_{p_r-1}, l_{r_r}) \\ &< f_{bl}(l_{p_r}, l_{p_r-1}) + \varepsilon \\ &< f_{bl}(l_r, l_r - 1) + \varepsilon, \end{aligned}$$

so, by taking $\lim r \to \infty$ and using (2.6),

$$f_{bl}(l_{p_r}, l_{r_r}) = \varepsilon. \tag{2.8}$$

Let's say there are infinitely many r such that $f_{bl}(l_{p_r}, \dot{G}l_{r_r}) + f_{bl}(\dot{G}l_{p_r}, l_{r_r}) < f_{bl}(l_{r_r}, l_{r_r})$.

Using (2.5) and taking $\limsup r \to \infty$, we get

$$\lim_{r\to\infty}(f_{bl}(l_{p_r}, l_{r_r}) + f_{bl}(\dot{G}l_{p_r}, l_{r_r})) = 0.$$

which implies

$$\lim_{r\to\infty} f_{bl}(l_{p_r}, l_{r_r+1}) = \lim_{r\to\infty} f_{bl}(\dot{G}l_{p_r+1}, l_{r_r}) = 0.$$

Now, $\varepsilon = \lim_{r\to\infty} f_{bl}(l_{p_r}, l_{r_r}) \leq \lim_{r\to\infty}\sup(f_{bl}(l_{p_r}, l_{r_r+1}) + f_{bl}(\dot{G}l_{r_r+1}, l_{r_r})) = 0,$

This seems incongruous. As a result, there is $r_0 \in \mathbb{N}$ such that for $r \geq r_0$,

$$f_{bl}(l_{p_r}, \dot{G}l_{r_r}) + f_{bl}(\dot{G}l_{p_r}, l_{r_r}) \geq f_{bl}(l_{r_r}, l_{r_r}).$$

For all $r \geq r_0$, consider

$$\begin{aligned} \dot{\psi}(f_{bl}(l_{p_{r+1}}, l_{r_r+1})) &\leq \dot{\alpha}(l_{p_r}, l_{p_r+1})\dot{\beta}(l_{r_r}, l_{r_r+1})\dot{\psi}(f_{bl}(l_{p_r+1}, l_{r_r+1})) \\ &= \dot{\alpha}(l_{p_r}, \dot{G}l_{p_r})\dot{\beta}(l_{r_r}, \dot{G}l_{r_r})\dot{\psi}(f_{bl}(\dot{G}l_{p_r}, \dot{G}l_{r_r})) \\ &\leq \dot{\psi}(\dot{\zeta}(f_{bl}(l_{p_r}, l_{r_r}), f_{bl}(l_{p_r}, l_{p_r+1}), f_{bl}(l_{r_r}, l_{r_r+1}), \\ &\quad \tfrac{f_{bl}(l_{p_r}, l_{r_r+1}) + f_{bl}(l_{p_r+1}, l_{r_r}) - f_{bl}(l_{r_r}, l_{r_r})}{2})), \end{aligned}$$

that implies $f_{bl}(l_{p_r+1}, l_{r_r+1}) \leq \dot{\zeta}(f_{bl}(l_{p_r}, l_{r_r}), f_{bl}(l_{p_r}, l_{p_r+1}), f_{bl}(l_{r_r}, l_{r_r+1}),$

$$\tfrac{f_{bl}(l_{p_r}, l_{r_r+1}) + f_{bl}(l_{p_r+1}, l_{r_r}) - f_{bl}(l_{r_r}, l_{r_r})}{2})$$

Now,

$$f_{bl}(l_{p_r}, l_{r_r}) \le f_{bl}(l_{p_r}, l_{p_r+1}) + f_{bl}(l_{p_r+1}, l_{r_r+1}) + f_{bl}(l_{r_r+1}, l_{r_r})$$
$$\le f_{bl}(l_{p_r}, l_{p_r+1}) + f_{bl}(l_{r_r+1}, l_{r_r}) + \dot{\zeta}(f_{bl}(l_{p_r}, l_{r_r}), f_{bl}(l_{p_r}, l_{p_r+1}),$$
$$f_{bl}(l_{r_r}, l_{r_r+1}), \tfrac{f_{bl}(l_{p_r}, l_{r_r+1}) + f_{bl}(l_{p_r+1}, l_{r_r})}{2}),$$

Using (2.5) and (2.8) as well as imposing $\liminf r \to \infty$, we obtain $\varepsilon \le 0+0+\dot{\zeta}(\varepsilon, 0, 0, \varepsilon')$, where

$$\varepsilon' = \limsup_{r\to\infty} \tfrac{f_{bl}(l_{p_r}, l_{r_r+1}) + f_{bl}(l_{p_r+1}, l_{r_r}) - f_{bl}(l_{r_r}, l_{r_r})}{2}$$
$$\le \limsup_{r\to\infty} \tfrac{f_{bl}(l_{p_r}, l_{r_r}) + f_{bl}(l_{r_r}, l_{r_r+1}) + f_{bl}(l_{p_r+1}, l_{p_r}) + f_{bl}(l_{p_r}, l_{r_r}) - 0}{2}$$
$$= \tfrac{\varepsilon+0+0+\varepsilon}{2}$$
$$= \varepsilon.$$

Hence, $\varepsilon \le \dot{\zeta}(\varepsilon, 0, 0, \varepsilon') < \max\{\varepsilon, 0, 0, \varepsilon'\} = \varepsilon$; a contradictory situation.

So, $\{l_r\}$ is Cauchy in $(E, f_{bl}, K \ge 1)$. But as $(E,\ f_{bl},\ K \ge 1)$ is complete, therefore, there is $l \in E$ such that $l_r \to l$ and

$$f_{bl}(l,l) = \lim_{r\to\infty} f_{bl}(l_r, l) = \lim_{r,p\to\infty} f_{bl}(l_r, l_p) = 0.$$

If we assume $\dot{G}$ to be continuous, then $l_r \to l \Rightarrow l_{r+1} = \dot{G}l_r \to \dot{G}l$.

But, unicity of limit implies $\dot{G}l = l$ or if E is $(\dot{\alpha}, \dot{\beta})$-regular w.r.t. $\dot{G}$, then $\dot{\alpha}(l, \dot{G}l) \ge 1$ and $\dot{\beta}(l, \dot{G}l) \ge 1$.

Let's take $\dot{G}l \ne l$. Now

$$\dot{\psi}(K^{\varepsilon} f_{bl}(\dot{G}l_r, \dot{G}l)) \le \dot{\alpha}(l_r, \dot{G}l_r)\dot{\beta}(l_r, \dot{G}l_r)\dot{\psi}(K^{\varepsilon} f_{bl}(\dot{G}l_r, \dot{G}l))$$
$$\le \dot{\psi}(K^{\varepsilon-1}\dot{\zeta}(f_{bl}(l_r, l), f_{bl}(l_r, l_{r+1}), f_{bl}(l, \dot{G}l),$$
$$\tfrac{f_{bl}(l_r, \dot{G}l) + f_{bl}(l, \dot{G}l_r) - f_{bl}(l,l)}{2K})),$$

that implies

$$K^{\varepsilon} f_{bl}(\dot{G}l_r, \dot{G}l) \le K^{\varepsilon-1}\dot{\zeta}\left(f_{bl}(l_r, l), f_{bl}(l_r, l_{r+1}), f_{bl}(l, \dot{G}l), \tfrac{f_{bl}(l_r, \dot{G}l) + f_{bl}(l, \dot{G}l_r) - f_{bl}(l,l)}{2K}\right)$$

i.e., $f_{bl}(l_{r+1}, \dot{G}l) \le \frac{1}{K}\dot{\zeta}(f_{bl}(l_r, l), f_{bl}(l_r, l_{r+1}), f_{bl}(l, \dot{G}l), \tfrac{f_{bl}(l_r, \dot{G}l) + f_{bl}(l, \dot{G}l_r) - f_{bl}(l,l)}{2K}),$

Applying $\liminf r \to \infty$ and by Proposition 1.12, we get

$$\tfrac{1}{K} f_{bl}(l, \dot{G}l) \le \tfrac{1}{K}\dot{\zeta}(0, 0, f_{bl}(l, \dot{G}l), l),$$

$$\Rightarrow f_{bl}(l,\dot{G}l) \leq \dot{\zeta}(0,0,f_{bl}(l,\dot{G}l),l),$$

where

$$l = \limsup_{r\to\infty} \frac{f_{bl}(l_r,\dot{G}l)+f_{bl}(l,l_{r+1})}{2K} \leq \limsup_{r\to\infty} \frac{Kf_{bl}(l,\dot{G}l)+0}{2K} = \frac{f_{bl}(l,\dot{G}l)}{2}.$$

Thus, $f_{bl}(l,\dot{G}l) \leq \dot{\zeta}(0,0,f_{bl}(l,\dot{G}l),l) < \max\{0,0,f_{bl}(l,\dot{G}l),l\} = f_{bl}(l,\dot{G}l)$,

This is contradictory; as a result, $\dot{G}l = l$.

If $\dot{G}w = w$ for $w \in E$, therefore, given the circumstance, $\dot{\beta}(w,w) \geq 1$. Assume that $l \neq w$ and consider

$$\begin{aligned}\dot{\psi}(K^{\varepsilon} f_{bl}(l,w)) &= \dot{\psi}(K^{\varepsilon} f_{bl}(\dot{G}l,\dot{G}w)) \\ &\leq \dot{\alpha}(l,\dot{G}l)\dot{\beta}(w,\dot{G}w)\dot{\psi}(K^{\varepsilon} f_{bl}(\dot{G}l,\dot{G}w)) \\ &\leq \dot{\psi}(K^{\varepsilon-1}\dot{\zeta}(f_{bl}(l,w), f_{bl}(l,\dot{G}l), f_{bl}(w,\dot{G}w), \\ &\quad \frac{f_{bl}(l,\dot{G}w)+f_{bl}(w,\dot{G}l)-f_{bl}(l,l)}{2K})),\end{aligned}$$

that implies

$$\begin{aligned}K^{\varepsilon} f_{bl}(l,w) &\leq K^{\varepsilon-1}\dot{\zeta}\left(f_{bl}(l,w), f_{bl}(l,\dot{G}l), f_{bl}(w,\dot{G}w), \frac{f_{bl}(l,\dot{G}w)+f_{bl}(w,\dot{G}l)-f_{bl}(l,l)}{2K}\right) \\ \Rightarrow f_{bl}(l,w) &\leq \frac{1}{K}\dot{\zeta}\left(f_{bl}(l,w),0,0,\frac{f_{bl}(l,w)}{K}\right) \\ &< \frac{1}{K}\max\left\{f_{bl}(l,w),0,0,\frac{f_{bl}(l,w)}{K}\right\} \\ &= \frac{f_{bl}(l,w)}{K},\end{aligned}$$

which is contradictory. Therefore, $l = w$.

Example 2.4: Let $E = [0,+\infty)$ and define $f_{bl} : E\times E \to [0,+\infty)$ by $f_{bl}(l,w) = (l+w)^2$ $\forall$ $l,$ $w \in E.$ Then, f_{bl} is bML, but not a b-metric on E.

Define $\dot{G}: E \to E$ as $\dot{G}(l) = \frac{l}{2}$ and $\dot{\alpha},\dot{\beta} : E\times E \to [0,+\infty)$, $\dot{\psi} \in \dot{\Psi}$, $\dot{\zeta} \in \dot{\Upsilon}_4$ as:

$$\dot{\alpha}(l,w) = \dot{\beta}(l,w) = \begin{cases} l & \text{if } l,w \in E \\ 0 & \text{otherwise,}\end{cases} \tag{2.9}$$

$\dot{\psi}(l) = l$ and $\dot{\zeta}(s_1,s_2,s_3,s_4) = \frac{1}{2}\max\{s_1,s_2,s_3,s_4\}$.

For $l,w \in E$; (2.1) is satisfied. Also, $\dot{G}$ is $(\dot{\alpha},\dot{\beta})$-admissible; E is $(\dot{\alpha},\dot{\beta})$-regular w.r.t. $\dot{G}$; $\dot{\alpha}(l_0,\dot{G}l_0) \geq 1$ and $\dot{\beta}(l_0,\dot{G}l_0) \geq 1$ for $l_0 = 1 \in E$.

Thus, all claims of Theorem 2.3 are met and 0 is sole fixed point of $\dot{G}$.

References

[1] A. Meir and E. Keeler, "A thorem on contractive mappings", *Journal of Mathematical Analysis and applications,* vol. 281, pp. 326-329 (1969).

[2] B. Samet, C. Vetro and P. Vetro, "Fixed point theorems for -contractive type mappings", *Nonlinear Analysis: Theory, Methods and Applications,* vol. 75, no. 4, pp. 2154-2165 (2012).

[3] B. E. Rhoades, "Some theorems on weakly contractive maps", *Nonlinear Analysis: Theory, Methods & Applications,* vol. 47, pp. 2683-2694 (2001).

[4] C. B. Nadler, "Multivalued contraction mappings", *Pacific Journal of Mathematics,* vol. 30, pp. 475-488 (1969).

[5] D. Wardowski, "Fixed points of a new type of contractive mappings in complete metric spaces," *Fixed Point Theory and Applications,* vol. 2012, no. 94 (2012).

[6] L. B. Ciric, "A generalization of Banachs contraction principle", *Proceeding of the American Mathematical Society,* vol. 45, pp. 267-273 (1974).

[7] M. A. Alghamdi, N. Hussain, and P. Salimi, "Fixed point and coupled fixed point theorems on b-metric like spaces," *Journal of Inequalities and Applications*, vol. 2013, no. 402 (2013).

[8] M. D. Sen, N. Nicolic, T. Dosenovic, M. Pavlovic and S. Radenovic, "Some Results on (s-q)-Graphic Contraction Mapping in b-Metric-like Spaces", *Mathematics,* vol. 7, no. 12. pp. 1190 (2019).

[9] N. Mlaiki, N. Dedovic, H. Aydi, M. G. Filipovic, B. B. Mohsin and S. Radenovic, "Some new observations on Geraghty and Ciric type results in b-metric spaces", *Mathematics*, vol. 7, no, 7, pp. 643 (2019).

[10] R. Kannan, "Some results on fixed points-II", *The American Mathematical Monthly,* vol. 76, no. 4 pp. 405-408 (1969).

[11] R. Pant and R. Panicker, "Geraghty and Ciric type fixed point theorems in b-metric spaces", *Journal of Nonlinear Science and Applications,* vol. 9, pp. 5741-5755 (2016).

[12] S. Banach, "Sur les operations dans les ensembles abstraits et leur application aux equations integrals". *Fundamenta Mathematicae,* vol. 3, pp. 133-181 (1922).

[13] S. Chandok, "Some fixed point theorems for (alpha,beta)-admissible Geraghty type contractive mappings and related results", *Mathematical Sciences,* vol. 9, pp. 127-135 (2015).

[14] S. Czerwik, "Nonlinear set-valued contraction mapping in b-metric spaces". Atti del Seminaro Matematico e Fisico dell'Universita di Modena e Reeggio Emilia, vol. 46, pp. 263-276 (1998).

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