Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 8, pp. 1817–1826
DOI : 10.47974/JIM-2046

Fibonacci statistical convergence of double sequences of order $\bar{\alpha}$ in intuitionistic fuzzy normed spaces (IFNS)

Sonal Sharma *
Meenakshi Chawla †
Department of Mathematics
Chandigarh University
Gharuan 140301
Mohali, Punjab
India

Abstract

The main objective of this work is to study and develop the theory of Fibonacci statistical convergence of double sequences of order $\bar{\alpha}$ in IFNS. We examine the Fibonacci statistically Cauchy double sequences of order $\bar{\alpha}$ and Fibonacci statistically completeness of double sequences of order $\bar{\alpha}$ in IFNS.

Subject Classification: *40A35, 40A05.*

Keywords: *Statistical convergence, Fibonacci statistical convergence, Double sequences, Intuitionistic fuzzy normed spaces.*

1. Introduction and Preliminaries

Fast [1] and Schoenberg [2] separately introduced the idea of statistical convergence. Statistical convergence has been studied throughout history and under various names in the fields of number theory, ergodic theory, and Fourier analysis theory, Banach space. Statistical convergence is suggested using the idea of the natural density of subsets of $\mathbb{N}$, the set of positive integers.

For every subset $\tau \subset \mathbb{N}$, the natural density Δ is represented by $\Delta(\tau)$ and is defined as

$$\Delta(\tau) = \lim_{p\to\infty} \frac{1}{p} |\{k < p : k \in \tau\}|$$

Mursaleen and Osama [3] developed the idea previously mentioned to encompass double sequences of scalars. Numerous fields, including

* *E-mail:* `26.sharmasonal@gmail.com (Corresponding Author)`
† *E-mail:* `chawlameenakshi7@gmail.com`

mathematics, science, engineering, and medicine, have been transformed by fuzzy logic. This idea was presented by Zadeh [4]. Fuzzy logic emerged as a significant field of study in many mathematical disciplines, like the theory of functions, metric and topological spaces, etc

Atanassov [5] studied intuitionistic fuzzy sets, which are suitable in this kind of circumstance. The concept of an IFNS was first presented by Park [6]. Moreover, the concept of IFNS is provided by Saadati and Park [7]. The field of IFNS has advanced significantly since the studies of [8], [9], [10], [11], [12], and [13].

Kara and Başarir [14] used the Fibonacci sequence for the first time in their theory of sequence spaces. Then, using the Fibonacci sequence f_n for n in the range of 1, 2, 3, ..., Kara [15] specify the Fibonacci difference matrix $\hat{F}$ and provided the updated sequence spaces connected to matrix domain $\hat{F}$.

Many mathematicians [16], [17], [18], [19], [20], [21], [22], [23], [27], [29] have produced excellent papers on the Fibonacci matrix after [14] and [15]. Fibonacci type statistical convergence was defined by Kirişçi and Karaisa [24], who also looked into some fundamental characteristics. Kirişçi [25] then investigated the statistical convergence of Fibonacci on IFNS. The lacunary statistical convergence of the Fibonacci sequence on IFNS was studied by Kiši [26], [28].

Definition 1.1: Let $\tau \subset \mathbb{N} \times \mathbb{N}$ and let $\tau\ (p,q)$ be the number of (a,b) in τ s.t. $a \leq p$ and $b \leq q$. Then, the lower asymptotic density of a set $\tau \subseteq \mathbb{N} \times \mathbb{N}$ is identifies with

$$\Delta_2(\tau) = \lim_{p,q} inf\ \frac{\tau(p,q)}{pq}$$

We say that τ has a double natural density and is defined by

$$\Delta_2(\tau) = \lim_{p,q} \frac{\tau(p,q)}{pq}$$

if the sequence $\frac{\tau(p,q)}{pq}$ has a limit in Pringsheim's sense.

For the sake of brevity, we substitute $\bar{\alpha}$ for (i,j) and $\bar{\vartheta}$ for (k,l) throughout the paper, taking i, j, k, and l ∈ (0, 1], unless otherwise noted. Moreover, we define

$$\bar{\vartheta} > \bar{\alpha} \Leftrightarrow i \leq k\ and\ j \leq l$$
$$\bar{\vartheta} \geq \bar{\alpha} \Leftrightarrow i < k\ and\ j < l$$
$$\bar{\alpha} \cong \bar{\vartheta} \Leftrightarrow i = k\ and\ j = l$$
$$\bar{\vartheta} \in (0,1] \Leftrightarrow k,l \in (0,1]$$
$$\bar{\alpha} \in (0,1] \Leftrightarrow i,j \in (0,1]$$
$$\bar{\alpha} \cong 1\ if\ \ i = j = 1$$
$$\bar{\vartheta} \cong 1\ if\ \ k = l = 1$$
$$\bar{\alpha} > 1\ if\ \ j > 1 \text{ and } i > 1$$

Additionally in the next session, we write $S_2^{\overline{\alpha}}$ to represent $S_2^{i,j}$ and $S_2^{\overline{\vartheta}}$ to represent $S_2^{k,l}$. The $\overline{\alpha}$ – double density of the set $\tau \subseteq \mathbb{N} \times \mathbb{N}$ has now defined as

$$\Delta_2^{\overline{\alpha}}(\tau) = \lim_{p,q} \frac{\tau(p,q)}{p^i q^j}$$

Definition 1.2: Let be given $y = (y_{ab})$ is statistical convergent of double sequences to the number y_0 if for each $\epsilon > 0 \ \exists \ \eta > 0$.

$$\lim_{p,q\to\infty} \frac{1}{pq} |\{(a,b): a \leq p, b \leq q: |y_{ab} - y_0| \geq \epsilon\}| = 0$$

Where the amount of elements in the enclosed set is indicated by the vertical bars.

Definition 1.3: Let be given $y = y_{ab} \in \omega^2$ and $\overline{\alpha} \in (0, 1]$. For every $\epsilon > 0$, the sequence (y_{ab}) is considered statistically convergent of order $\overline{\alpha}$ if there exists a complex number y_0.

$$\lim_{p,q\to\infty} \frac{1}{p^i q^j} |(a,b): a \leq p, b \leq q: |y_{ab} - y_0| \geq \epsilon| = 0$$

Definition 1.4: The term Fibonacci numbers refers to the series of numbers (f_y) for y = 1, 2, ... that are found by

$$f_y = f_{y-1} + f_{y-2} \quad y \geq 2$$

The characteristics of Fibonacci number are given by

$$\lim_{y\to\infty} \frac{F_{y+1}}{f_y} = \frac{1+\sqrt{5}}{2} = \alpha, \ (Golden \quad ratio)$$

$$\sum_{n=0}^{y} f_n = f_{y+2} - 1 \ (y \in N),$$

$$\sum_n \frac{1}{f_n} \quad converges,$$

$$f_{y-1} f_{y+1} - f_y^2, y \geq 1. \quad Cassini\, formula$$

For each $y \in \mathbb{N}$, we define $\hat{F}$=$(\hat{F}_{yn})$ as an infinte matrix by

$$\hat{f}_{yn} = \begin{cases} -\frac{f_y+1}{f_y}, & (n = y-1) \\ \frac{f_y}{f_y+1}, & (n = y) \\ 0, (0 \leq n < y-1 \ or \ n > y) \end{cases}$$

Definition 1.5: A binary operation $\circ$:[0,1]×[0,1]→ [0,1]. When all of the following criteria are satisfied, the operation $\circ$ is referred as a continuous t-norm.

- When $\zeta \leq \xi$ and $\rho \leq \gamma \ \forall \ \zeta, \xi, \rho, \gamma \in [0,1]$ then $\zeta \circ \xi \leq \rho \circ \gamma$,
- $\circ$ is continuous,
- $\zeta \circ 1 = \forall \ \zeta \in [0,1]$,
- $\circ$ is commutative and associative.

Definition 1.6: A binary operation $\diamond:[0,1]\times[0,1]\to[0,1]$. When all of the following criteria are satisfied, the operation $\diamond$ is referred as a continuous s-norm.

- When $\zeta \leq \xi$ and $\rho \leq \gamma$ $\forall\, \zeta,\xi,\rho,\gamma \in [0,1]$ then $\zeta \diamond \xi \leq \rho \diamond \gamma$,
- $\diamond$ is continuous,
- $\zeta \diamond 0 = \forall\ \zeta \in [0,1]$,
- $\diamond$ is commutative and associative.

Definition 1.7: [7]If Y is a vector space, (φ, ϖ) are fuzzy sets on Y $(0, \infty)$, $\circ$ is a continuous t-norm, and $\diamond$ is a continuous s-norm, that satisfy the following criteria, then the five tuples $(Y,\varphi,\varpi,\circ,\diamond)$ is known as IFNS (i.e intuitionistic fuzzy normed space) whenever ξ, $\varrho > 0$ and a, b $\in$ Y.

- $\varphi(a,\xi) + \varpi(a,\xi) \leq 1$
- $\varphi(a,\xi) > 0$
- $\varpi(a,\xi) < 1$
- $\varphi(a,\xi) = 1\ if\ a = 0$
- $\varpi(a,\xi) = 0\ if\ a = 0$
- $\varphi(\chi a,\xi) = \varphi\left(a,\frac{\xi}{|\alpha|}\right) foreach\ \chi \neq 0$
- $\varpi(\chi a,\xi) = \varpi\left(y,\frac{\xi}{|\alpha|}\right) foreach\ \chi \neq 0$
- $\varphi(a,\xi) \circ \varphi(b,\varrho) \leq \varphi(a+b,\xi+\varrho)$
- $\varpi(a,\xi) \diamond \varpi(b,\varrho) \geq \varpi(a+b,\xi+\varrho)$
- $\varphi(a,.): (0,\infty) \to [0,1]$ is continuous
- $\varpi(a,.): (0,\infty) \to [0,1]$ is continuous
- $\lim_{\xi\to\infty}\ \varphi(a,\xi) = 1\ and \lim_{\xi\to\infty}\ \varphi(a,\xi) = 0$
- $\lim_{\xi\to\infty}\ \varpi(a,\xi) = 0\ and \lim_{\xi\to\infty}\ \varpi(a,\xi) = 1$

Definition 1.8: Let $(Y,\varphi,\varpi,\circ,\diamond)$ be an IFNS. If for every $\epsilon > 0$ $\exists$ $\varrho > 0$ then the double Fibonacci sequence y = $(\hat{F}_{y_{ab}})$ is statistically convergent to y_0 in Y, then

$$\lim_{p,q\to\infty}\frac{1}{p^i q^j}\left|\left\{p \geq a, q \geq b: \varphi\left(\hat{F}_{y_{ab}} - y_0, \varrho\right) \leq 1-\epsilon \text{ or } \varpi\left(\hat{F}_{y_{ab}} - y_0, \varrho\right) \geq \epsilon\right\}\right| = 0$$

i.e. $S_2^{\overline{\alpha}}(\tau_\epsilon) = 0$ that is

$$\tau_\epsilon = \left\{p \geq a, q \geq b: \varphi\left(\hat{F}_{y_{ab}} - y_0, \varrho\right) \leq 1-\epsilon \text{ or } \varpi\left(\hat{F}_{y_{ab}} - y_0, \varrho\right) \geq \epsilon\right\}$$

2. Main Result

The Fibonacci statistical convergence of double sequences of order $\bar{\alpha}$ in IFNS is defined in this section. Initially, the following definition of Fibonacci statistical convergence of double sequences of order $\bar{\alpha}$ is given:

Definition 2.1: Let $\bar{\alpha} \in (0,1]$ and $y = (\hat{F}_{y_{ab}}) \in \omega^2$ be given. For every $\epsilon > 0$, the sequence $\hat{F}_{y_{ab}}$ is considered Fibonacci statistical convergence of order $\bar{\alpha}$ if $\exists$ a complex number y_0

$$\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|(a,b): a \le p, b \le q: \left|\hat{F}_{y_{ab}} - y_0\right| \ge \epsilon\right| = 0$$

Example 2.2: Let $(Y, \|\,.\,\|)$ be a normed space and for all $\xi, \zeta \in [0,1]$, $\xi \circ \zeta = \xi\zeta$ and $\xi \diamond \zeta = \min\{\xi + \zeta, 1\}$. For all $y \in Y$ and every $\varrho > 0$. Consider $\varphi(y, \varrho) = \frac{\varrho}{\varrho + \|y\|}$, $\varpi(y, \varrho) = \frac{\|y\|}{\varrho + \|y\|}$. Then (Y, $\varphi, \varpi, \circ, \diamond$) is an IFNS. Define $\hat{F}_{y_{ab}} = ({f^2}_{ab+1}) = (1, 2^2, 3^2, 5^2 \ldots.)$

Since ${f^2}_{ab+1} \to \infty$ as $\kappa \to \infty$ and $\hat{F}_{y_{ab}} = (1,0,0 \ldots.)$ then $\hat{F}_{y_{ab}} = S_2^{\alpha}$. Consider $\kappa_{ab}(\varepsilon, \varrho) = \{\kappa \le ab: \varphi_F(\hat{F}_{y_{ab}}, \varrho) \le 1 - \varepsilon \text{ } or \text{ } \varpi_F(\hat{F}_{y_{ab}}, \varrho) \ge \varepsilon\}$ for $\varepsilon \in (0,1)$ and for any $f > 0$.

When ab become sufficiently large, the quantity $\varphi_F(\hat{F}_{y_{ab}}, \varrho)$ become less then $1 - \varepsilon$ and similarly $\varpi_F(\hat{F}_{y_{ab}}, \varrho)$ become greater then ε. So for $\varepsilon > 0$ and $\varrho > 0$. $\kappa_\varepsilon(\hat{F}_{y_{ab}}) = 0$

Lemma 2.3: *Take (Y,φ, ϖ,∘,⋄) be an IFNS. The subsequent requirements are equivalent for each* $\epsilon > 0$ *and* $\varrho > 0$

- $S_2^{\bar{\alpha}} - lim\hat{F}_{y_{ab}} = y_0$
- $\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|\{a \le p, b \le q: \varphi(\hat{F}_{y_{ab}} - y_0, \varrho) \le 1 - \epsilon\}\right| =$
  $\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|\{a \le p, b \le q: \varphi(\hat{F}_{y_{ab}} - y_0, \varrho) \ge \epsilon\}\right| = 0$
- $\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|\{a \le p, b \le q: \varphi(\hat{F}_{y_{ab}} - y_0, \varrho) > 1 - \epsilon\}\right|$ and $\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|\{a \le p, b \le q: \varphi(\hat{F}_{y_{ab}} - y_0, \varrho) < \epsilon\}\right| = 1$
- $\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|\{a \le p, b \le q: \varphi(\hat{F}_{y_{ab}} - y_0, \varrho) > 1 - \epsilon\}\right| =$
  $\lim_{p,q\to\infty} \frac{1}{p^i q^j}\left|\{a \le p, b \le q: \varphi(\hat{F}_{y_{ab}} - y_0, \varrho) < \epsilon\}\right| = 1$
- $S_2^{\bar{\alpha}} - \lim \varphi\left(\hat{F}_{y_{ab}} - y_0, \varrho\right) = 1$ and $S_2^{\bar{\alpha}} - \lim \varpi\left(\hat{F}_{y_{ab}} - y_0, \varrho\right) = 0$

Theorem 2.4: *Take (Y,φ, ϖ,∘,⋄) be an IFNS. Then* $S_2^{\bar{\alpha}} - lim\hat{F}_{y_{ab}} = y_0$ *is unique, where the double Fibonacci sequence y =* $(\hat{F}_{y_{ab}})$ *is statistically convergent in IFNS.*

Proof: For $\mathcal{L}_{11} \ne \mathcal{L}_{22}$ where $\mathcal{L}_{11}, \mathcal{L}_{22} \in Y$.

Suppose that $S_2^{\bar{\alpha}} - \lim \hat{F}_{y_{ab}} = \mathcal{L}_{11}$ and $S_2^{\bar{\alpha}} - \lim \hat{F}_{y_{ab}} = \mathcal{L}_{22}$. Select $\zeta > 0$ so that, for a given $\epsilon > 0$, (1-ζ)∘ $(1 - \zeta) > 1 - \epsilon$ and $\zeta \circ \zeta < \epsilon$. Thus, describe the following set for any $\varrho > 0$ as follows1:

$$\tau_{\varphi,1}(\zeta, \varrho) = \left\{(a,b) \in \mathbb{N} \times \mathbb{N}: \varphi\left(\hat{F}_{y_{ab}} - \mathcal{L}_{11}, \frac{\varrho}{2}\right) \le 1 - \zeta\right\}$$

$$\tau_{\varphi,2}(\zeta,\varrho)=\left\{(a,b)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}_{y_{ab}}-\mathcal{L}_{22},\frac{\varrho}{2}\right)\le 1-\zeta\right\}$$
$$\tau_{\varpi,1}(\zeta,\varrho)=\left\{(a,b)\in\mathbb{N}\times\mathbb{N}:\varpi\left(\hat{F}_{y_{ab}}-\mathcal{L}_{11},\frac{\varrho}{2}\right)\ge \zeta\right\}$$
$$\tau_{\varpi,2}(\zeta,\varrho)=\left\{(a,b)\in\mathbb{N}\times\mathbb{N}:\varpi\left(\hat{F}_{y_{ab}}-\mathcal{L}_{22},\frac{\varrho}{2}\right)\ge \zeta\right\}$$

Since $S_2^{\overline{\alpha}}-\lim\hat{F}_{y_{ab}}=\mathcal{L}_{11}$, by lemma (i) we have

$$S_2^{\overline{\alpha}}\left(\tau_{\varphi,1}(\zeta,\varrho)\right)=S_2^{\overline{\alpha}}\left(\tau_{\varpi,1}(\zeta,\varrho)\right)=0\ \forall\ \varrho>0$$

Let$\tau_{\varphi,\varpi}(\zeta,\varrho)=\left[(\tau_{\varphi,2}(\zeta,\varrho))\cup(\tau_{\varphi,1}(\zeta,\varrho))\right]\cap\left[(\tau_{\varpi,2}(\zeta,\varrho))\cup(\tau_{\varpi,1}(\zeta,\varrho))\right]$.
Hence observed that

$$S_2^{\overline{\alpha}}(\tau_{\varphi,\varpi}(\zeta,\varrho))=0$$
$$S_2^{\overline{\alpha}}(\mathbb{N}\times\mathbb{N}/(\tau_{\varphi,\varpi}(\zeta,\varrho))=1$$

If (a, b)$\in\mathbb{N}\times\mathbb{N}/(\tau_{\varphi,\varpi}(\zeta,\varrho))$, then there are two scenarios that could occur:

Case 1: (a, b)$\in\mathbb{N}\times\mathbb{N}/\ \tau_{\varphi,1}(\zeta,\varrho)\cup\ \tau_{\varphi,2}(\zeta,\varrho)$ and

Case 2: (a, b)$\in\mathbb{N}\times\mathbb{N}/\ \tau_{\varpi,1}(\zeta,\varrho)\cup\ \tau_{\varpi,2}(\zeta,\varrho)$

Firstly consider case (1), we have

$$\varphi(\mathcal{L}_{11}-\mathcal{L}_{22})\ge\varphi(\hat{F}_{y_{ab}}-\mathcal{L}_{11},\tfrac{\varrho}{2})\circ(\hat{F}_{y_{ab}}-\mathcal{L}_{22},\tfrac{\varrho}{2})>(1-\zeta)\circ(1-\zeta)$$

As $(1-\zeta)\circ(1-\zeta)>1-\epsilon$, it follows that $\varphi(\mathcal{L}_{11}-\mathcal{L}_{22},\varrho)>1-\epsilon$, where $\epsilon>0\ \forall\ \varrho>0$, we get $\varphi(\mathcal{L}_{11}-\mathcal{L}_{22},\varrho)=1$. From that $\mathcal{L}_{11}=\mathcal{L}_{22}$ is observed.

In other case, if (a, b)$\in\mathbb{N}\times\mathbb{N}/\ \tau_{\varpi,1}(\zeta,\varrho)\cup\ \tau_{\varpi,2}(\zeta,\varrho)$, then we write

$$\varpi(\mathcal{L}_{11}-\mathcal{L}_{22},\varrho)<\varpi\left(\hat{F}_{y_{ab}}-\mathcal{L}_{11},\frac{\varrho}{2}\right)\diamond\varpi(\hat{F}_{y_{ab}}-\mathcal{L}_{22},\tfrac{\varrho}{2})<\zeta\diamond\zeta$$

We see that $\varpi(\mathcal{L}_{11}-\mathcal{L}_{22},\zeta)<\epsilon$. By using fact $\zeta\diamond\zeta$, $\varphi(\mathcal{L}_{11}-\mathcal{L}_{22},\zeta)=0$

Thus $\mathcal{L}_{11}=\mathcal{L}_{22}$.Hence we conclude that $S_2^{\overline{\alpha}}-lim\hat{F}_{y_{ab}}=y_0$ is unique.

Theorem 2.5: *Take an IFNS* $(Y,\varphi,\varpi,\circ,\diamond)$ *and y =* $(\hat{F}_{y_{ab}})$ *be an increasing sequence. If* $(\varphi,\varpi)-lim\hat{F}_{y_{ab}}=y_0$ *then* $S_2^{\overline{\alpha}}-lim\hat{F}_{y_{ab}}=y_0$

Proof: Suppose $(\varphi,\varpi)-lim\hat{F}_{y_{ab}}=y_0$ then $\exists$ a number $a_0,b_0\in\mathbb{N}\times\mathbb{N}$ $\forall\ \epsilon>0\ and\ \xi>0\ s.t.$

$$\varphi\left(\hat{F}_{y_{ab}}-y_0,\xi\right)>1-\epsilon$$
$$\varpi\left(\hat{F}_{y_{ab}}-y_0,\xi\right)<\epsilon\ \forall\ a\ge a_0,\ b\ge b_0$$

Hence, there are only a limited number of terms in the set n

$$\{(a,b)\in N\times N:\varphi\left(\hat{F}_{y_{ab}}-y_0,\xi\right)\le 1-\epsilon\ or\ \varpi(\hat{F}_{y_{ab}}-y\_\{0\},\xi)\ge\epsilon\ \}$$

Since the density of every finite $\subset \mathbb{N} \times \mathbb{N}$ is zero and hence

$$S_2^{\overline{\alpha}}\{(a,b) \in N \times N: \varphi(\hat{F}_{y_{ab}} - y_0, \xi) \leq 1 - \epsilon \; or \; \varpi(\hat{F}_{y_{ab}} - y\_\{0\}, \xi) \geq \epsilon\} = 0$$

$$S_2^{\overline{\alpha}} - lim\hat{F}_{y_{ab}} = y_0$$

Theorem 2.6: *Take an IFNS* $(Y, \varphi, \varpi, \circ, \diamond)$. *A sequence* $y = (\hat{F}_{y_{ab}})$ *is known as Fibonacci statistical Cauchy of order* $\overline{\alpha}$ *of double sequences in IFN*(φ, ϖ) *iff sequence is Fibonacci statistical convergent of order* $\overline{\alpha}$ *of the double sequences in IFNS.*

Proof: $S_2^{\overline{\alpha}}(\varphi, \varpi) - lim\hat{F}_{y_{ab}} = y_0$ if $\epsilon > 0$ select $\omega > 0 \; s.t.$

$$1 - \boldsymbol{\omega} < (1 - \boldsymbol{\epsilon}) \circ (1 - \boldsymbol{\epsilon}) \text{ and } \boldsymbol{\omega} > \boldsymbol{\epsilon} \diamond \boldsymbol{\epsilon}$$

For $\varrho > 0$, we have

$$S_2^{\overline{\alpha}}\big(A(\varphi, \varpi)\big) = S_2^{\overline{\alpha}}\{(a,b) \in I_n: \varphi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) \leq 1 - \epsilon \; and \; \varpi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) \geq \epsilon\} = 0$$

Which implies that

$$S_2^{\overline{\alpha}}\big(A^c(\varphi, \varpi)\big) = S_2^{\overline{\alpha}}\{(a,b) \in I_n: \varphi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) > 1 - \epsilon \; and \; \varpi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) < \epsilon\} = 1$$

Let $n, m \in (\, A^c(\varphi, \varpi)\,)$. Then $\varphi(\hat{F}_{y_{nm}} - y_0, \varrho) > 1 - \epsilon \; and \; \varpi(\hat{F}_{y_{nm}} - y_0, \varrho) > \epsilon$

Now let $\big(D(\varphi, \varpi)\big) = \{(a,b) \in I_n: \varphi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}}, \varrho) \leq 1 - \omega \; or \; \varpi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}}, \varrho) \geq \omega\}$

Required to prove $D(\varphi, \varpi) \subset A(\varphi, \varpi)$

Let $k \in D(\varphi, \varpi) - A(\varphi, \varpi)$, then

$\varphi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}}, \varrho) \leq 1 - \omega \; and \; \varphi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) > 1 - \epsilon$, in particular

$$\varphi\left(\hat{F}_{y_{nm}} - y_0, \frac{\varrho}{2}\right) > 1 - \epsilon, \text{then}$$

$1 - \omega \geq \varphi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}}, \varrho) \geq \varphi(\widehat{F_{y_{nm}}} - y_0, \frac{\varrho}{2}) \circ \varphi(\hat{F}_{y_{nm}} - y_0, \frac{\varrho}{2}) > (1 - \epsilon) \circ (1 - \epsilon) > 1 - \omega$

Which is not possible

$$D(\varphi, \varpi) \subset A(\varphi, \varpi)$$

By $S_2^{\overline{\alpha}}\big(A(\varphi, \varpi)\big)$=0

In light of this, y is statistically twice as Cauchy as Fibonacci in relation to the IFNS (φ, ϖ).

Definition 2.7: An IFNS $(Y, \varphi, \varpi, \circ, \diamond)$ is said to be $\hat{F}S_2^{\overline{\alpha}} - complete$ if every $\hat{F}S_2^{\overline{\alpha}}$ − Cauchy sequence with respect to the IFN(φ, ϖ) is $\hat{F}S_2^{\overline{\alpha}}$ − convergent with respect to the IFN(φ, ϖ)

Theorem 2.8: *Let y be a sequence defined by* $\hat{F}_{y_{ab}}$. *Then* $S_2^{\overline{\alpha}}((\varphi,\varpi))$ *-completeness holds for all IFNS* $(Y,\varphi,\varpi,\circ,\diamond)$.

Proof: Let a sequence y $=\hat{F}_{y_{ab}}$. For each $\epsilon > 0$, choose $\zeta > 0$, $S_2^{\overline{\alpha}}$ is not convergent but Cauchy in IFN(φ,ϖ) s.t.

$$1-\epsilon < (1-\zeta)\circ(1-\zeta) \quad \text{and} \quad \epsilon > \zeta \diamond \zeta.$$

Now

$$\varphi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}},\varrho) \geq \varphi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) \circ \varphi\left(\hat{F}_{y_{nm}} - y_0, \frac{\varrho}{2}\right) > (1-\epsilon)\circ (1-\epsilon) > 1-\zeta$$

$$\varpi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}},\varrho) \leq \varpi\left(\hat{F}_{y_{ab}} - y_0, \frac{\varrho}{2}\right) \diamond \varpi\left(\hat{F}_{y_{nm}} - y_0, \frac{\varrho}{2}\right) < \epsilon \diamond \epsilon < \zeta$$

Considering y is not $S_2^{\overline{\alpha}}(\varphi,\varpi)$-convergent. Thus, $S_2^{\overline{\alpha}}(B(\varphi,\varpi))$=0, where

$$B(\epsilon,\varrho) = \{(a,b) \in I_n : \varpi(\hat{F}_{y_{ab}} - \hat{F}_{y_{nm}}) \leq 1-\omega\}$$

$S_2^{\overline{\alpha}}B(\epsilon,\varrho)$ =1

Which is inconsistency, given that y was Cauchy in IFNS by $S_2^{\overline{\alpha}}$. Thus in terms of the IFNS, y must be $S_2^{\overline{\alpha}}$ −convergent. Therefore, $S_2^{\overline{\alpha}}(\varphi,\varpi)$-completeness holds ∀ IFNS $(Y,\varphi,\varpi,\circ,\diamond)$.

Theorem 2.9: Let $\overline{\vartheta},\overline{\alpha} \in (0,1]$ such that $\overline{\vartheta} \geq \overline{\alpha}$. Then $S_2^{\overline{\alpha}}\, S_2$, and for some $\overline{\vartheta}$ and $\overline{\alpha}$, s.t. $\overline{\vartheta} > \overline{\alpha}$, the inclusion is strict.

Proof: Assume that $\bar{\vartheta},\bar{\alpha} \in (0,1]$; if $\overline{\vartheta} > \overline{\alpha}$ and k ≥ i and l ≥ j, then

$$\frac{1}{p^k q^l}\left|\{(a,b); a \leq p, b \leq q, |\hat{F}_{y_{ab}} - y_0| \geq \epsilon\}\right| \geq \frac{1}{p^i q^j} \left|\{(a,b); a \leq p, b \leq q, |\hat{F}_{y_{ab}} - y_0| \geq \epsilon\}\right|$$

Consequently, $S_2^{\overline{\alpha}} \subseteq S_2$ for each $\epsilon > 0$. To demonstrate the inclusion's strictness, let's look at the sequence y= $\hat{F}_{y_{ab}}$ given by

$$\text{y} = \hat{F}_{y_{ab}} = \begin{cases} 1, a = p^2, b = q^2 \\ 0, \text{in any case} \end{cases} \quad p,q = 1,2,3\ldots..$$

Thus, $S_2^{\overline{\alpha}} - lim\hat{F}_{y_{ab}} = i.e\ y \in S_2^{\overline{\alpha}}\ for\ \overline{\vartheta} \in (1,2,1]\ (i.e\ for\ \frac{1}{2} < l \leq 1\ and\ \frac{1}{2} < k \leq 1)$ but $y \notin S_2^{\overline{\alpha}}\hat{F}\ for\ \overline{\alpha}\left(0,\frac{1}{2}\right]\ i.e\ for\ 0 < j < \frac{1}{2}\ and\ 0 < i < \frac{1}{2}$. By utilising $\overline{\vartheta} \cong 1$ in this theorem, we can derive the subsequent outcome.

Corollary 2.10: Let $\bar{\alpha},\bar{\vartheta} \in (0,1]$ *be given then*

$$S_2^{\overline{\alpha}} = S_2^{\overline{\vartheta}}\ \ iff\ \bar{\alpha}\ \cong \bar{\vartheta}$$
$$S_2^{\overline{\alpha}} = S_2\ \ iff\ \bar{\alpha} \cong 1$$

Conclusion

This paper provides a more comprehensive definition of the Fibonacci statistical convergence of double sequences of order $\bar{\alpha}$ in comparison to the concept of Fibonacci statisical convergence of double sequences, defined by KIŞI [30]. In terms of completeness and Cauchy convergence criteria, the idea is more broadly expressed. The classical findings regarding this kind of convergence also apply to double sequences.

References

[1] H. Fast, "Sur la convergence statistique," *Colloq. Math.*, vol. 2, pp. 241–244 (1951).

[2] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *Amer. Math. Monthly,* vol. 66, pp. 361–375 (1959).

[3] M. Mursaleen and O. H. H. Edely, "Statistical convergence of double sequences," *J. Math. Anal. Appl.*, vol. 288, pp. 223–231 (2003).

[4] L. A. Zadeh, "Fuzzy sets," Inform. Control, vol. 8, pp. 338–353 (1965).

[5] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets and Systems*, vol. 20, no. 1, pp. 87–96 (1986).

[6] J. H. Park, "Intuitionistic fuzzy metric spaces," *Chaos Solitons Fractals,* vol. 22, pp. 1039–1046 (2004).

[7] R. Saadati, "On the intuitionistic fuzzy topological spaces," *Chaos Solitons Fractals,* vol. 27, pp. 331–344 (2006).

[8] S. Karakuş, "Statistical convergence on intuitionistic fuzzy normed spaces," *Chaos Solitons Fractals*, vol. 35, pp. 763–769 (2008).

[9] M. Mursaleen, "Statistical convergence of double sequences in intuitionistic fuzzy normed space," *Chaos Solitons Fractals*, vol. 41, no. 5, pp. 2414–2421 (2009).

[10] M. Mursaleen, "On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space," *J. Comp. Appl. Math.*, vol. 233, no. 2, pp. 142–149 (2009).

[11] E. Savaş, "Certain summability methods in intuitionistic fuzzy normed spaces," *J. Intell. Fuzzy Syst.,* vol. 27, no. 4, pp. 1621–1629 (2014).

[12] E. Savaş, "A generalized statistical convergence in intuitionistic fuzzy normed spaces," *Science Asia*, vol. 41, pp. 289–294 (2015).

[13] P. Debnath, "Lacunary ideal convergence in intuitionistic fuzzy normed linear space," *Comput. Math. Appl.,* vol. 63, no. 3, pp. 708–715 (2012).

[15] E. E. Kara, "Some topological and geometrical properties of new Banach sequence space," *J. Inequal. Appl.*, vol. 38 (2013).

[16] V. A. Khan, "Intuitionistic fuzzy I-convergent Fibonacci difference sequence spaces," *J. Inequal. Appl.*, vol. 202, p. 7 (2019).

[17] M. İlkan, "A new Banach space defined by Euler totient matrix operator," *Operators and Matrices*, vol. 13, no. 2, pp. 527–544 (2019).

[18] E. E. Kara, "On some Banach sequence spaces derived by a new band matrix," *British J. Math. Comput. Sci.*, vol. 9, no. 2, pp. 141–159 (2015).

[19] E. E. Kara, "Compactness of matrix operators on some sequence spaces derived by Fibonacci numbers," *Kragujev. J. Math.*, vol. 39, no. 2, pp. 217–230 (2015).

[20] E. E. Kara, "Some new paranormed difference sequence spaces derived by Fibonacci numbers," *Miskolc Math. Notes,* vol. 16, no. 2, pp. 907–923 (2015).

[21] E. E. Kara, "Some properties of generalized Fibonacci sequence space," *Linear Multilinear Algebra,* vol. 64, no. 11, pp. 2208–2223 (2016).

[22] A. Alotaibi, "Compact operators on some Fibonacci difference sequence space," *J. Inequal. Appl.,* vol. 203 (2015).

[23] S. Demiriz, "On the Fibonacci almost convergent sequence space and Fibonacci core," *Kyungpook Math. J.,* vol. 55, pp. 355–372 (2015).

[24] M. Kirişci, "Fibonacci statistical convergence and Korovkin type approximation theorems," *J. Inequal. Appl.,* vol. 229 (2017).

[25] M. Kirişci, "Fibonacci statistical convergence on intuitionistic fuzzy normed spaces," *J. Intell. Fuzzy Syst.*, vol. 36, pp. 5597–5604 (2019).

[26] O. Kisi, "Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces," unpublished (2019).

[27] R. Çolak, "Statistical convergence of order α," in Modern Methods in Analysis and Its Applications, Anamaya Publishers, New Delhi, India, pp. 121–129 (2010).

[28] O. Kisi, "On Fibonacci lacunary statistical convergence of double sequences in intuitionistic fuzzy normed linear spaces," *Int. J. Adv. Appl. Math. Mech.,* vol. 7, no. 4, pp. 64-71 (2020).

[29] G. N.Bilgin, "Fibonacci lacunary statistical convergence of order gamma in IFNLS," *Int. J. Adv. Appl. Math. and Mech.,* vol. 8(4), pp. 28-36 (2021).

[30] A. Alotaibi, " On lacunary statistical convergence of double sequences with respect to the intuitionistic fuzzy normed spaces," *Int. J. Contemp. Math. Sciences,* vol. 5(42), pp. 2069-2078 (2010).

Received July, 2024