

Existence and uniqueness of linear and non-linear fractional differential equations

Anil Kumar Menaria
Department of Mathematics
Bhupal Nobles' University
Udaipur 313002
Rajasthan
India

Abstract

The existence and uniqueness of solutions for linear and nonlinear fractional differential equations (FDE) involving Miller-Ross sequential fractional derivatives (SFD) can be established through mathematical analysis and specific conditions, which allows direct applications with Riemann-Liouville fractional derivatives, the Caputo fractional derivatives and the Grunwald Letnikov derivatives, which is a particular case of the Miller-Ross SFD.

Subject Classification (2020): 34A08, 26A33, 34G10, 34G20.

Keywords: Fractional differential equations, Miller-ross sequential fractional derivatives, Linear and non-linear differential equations, Riemann-liouville differential equations.

1. Introduction

The existence and uniqueness of solutions for linear and nonlinear FDE depend on various factors, such as the specific form of the equation and the conditions imposed. For linear fractional differential equations, under certain conditions on the coefficients and initial values, solutions can exist and be unique. Nonlinear FDE, on the other hand, may be more complicated.

2. Miller-Ross SFD and its Laplace Transform:

The Miller-Ross SFD is a type of fractional derivative that extends the classical Caputo fractional derivative. Let us introduce the following notation for the Miller-Ross SFD[5]:

$$\left. \begin{aligned} {}_{\eta}\mathcal{D}_{\xi}^{\zeta_m} &= {}_{\eta}\mathcal{D}_{\xi}^{\varphi_m} {}_{\eta}\mathcal{D}_{\xi}^{\varphi_{m-1}} \dots {}_{\eta}\mathcal{D}_{\xi}^{\varphi_1} \\ {}_{\eta}\mathcal{D}_{\xi}^{\zeta_{m-1}} &= {}_{\eta}\mathcal{D}_{\xi}^{\varphi_{m-1}} {}_{\eta}\mathcal{D}_{\xi}^{\varphi_{m-2}} \dots {}_{\eta}\mathcal{D}_{\xi}^{\varphi_1} \end{aligned} \right\} \dots (2.1)$$

Where $\zeta_m = \sum_{j=1}^m \varphi_j$; $0 < \varphi_j \leq 1, j = 1, 2, \dots, m$.

Now by Laplace transform of (2.1), the sequential derivative [8], we obtained

$$\mathcal{L}\left\{{}_0\mathcal{D}_\xi^{\zeta_m} f(\xi); s\right\} = s^{\zeta_m} F(s) - \sum_{k=0}^{m-1} s^{\zeta_m - \zeta_{m-k}} \left[{}_0\mathcal{D}_\xi^{\zeta_{m-k-1}} f(\xi) \right]_{\xi=0} \quad \dots (2.2)$$

$$\text{Where } {}_\eta\mathcal{D}_\xi^{\zeta_{m-k-1}} = {}_\eta\mathcal{D}_\xi^{\varphi_{m-k-1}} {}_\eta\mathcal{D}_\xi^{\varphi_{m-k-2}} \dots {}_\eta\mathcal{D}_\xi^{\varphi_1}$$

3. Fractional Derivative of Power Function $(t - a)^\omega$:

To obtain the R-L fractional derivative ${}_\eta\mathcal{D}_\xi^p f(\xi)$ of the power function [4] $f(\xi) = (\xi - \eta)^\omega$, where ω is a real number, first we take R-L derivative [6]

$${}_\eta\mathcal{D}_\xi^p f(\xi) = \frac{d^n}{d\xi^n} \left\{ {}_\eta\mathcal{D}_\xi^{-(n-p)} f(\xi) \right\}; \quad n-1 \leq p < n$$

Substituting $\varphi = n - p$ and $f(\xi) = (\xi - \eta)^\omega$ in the above equating and simplifying, we get

$${}_\eta\mathcal{D}_\xi^p \{(\xi - \eta)^\omega\} = {}_\eta\mathcal{D}_\xi^{-\varphi} \{(\xi - \eta)^\omega\} = \frac{\Gamma(1+\omega)}{\Gamma(1+\omega-p)} (\xi - \eta)^{\omega-p} \quad \dots (3.1)$$

And the only restriction for $f(\xi) = (\xi - \eta)^\omega$ is its integrability, namely $\omega > -1$.

4. Linear Fractional Differential Equations:

To find the existence and uniqueness solutions of linear FDE, first we consider the following problem [1]:

$${}_0\mathcal{D}_\xi^{\zeta_n} y(\xi) + \sum_{j=0}^{n-1} p_j(\xi) {}_0\mathcal{D}_\xi^{\zeta_{n-3}} y(\xi) + p_n(\xi) y(\xi) = f(\xi); \quad \dots (4.1)$$

where, $(0 < \xi < T < \infty)$ and

$$\left. \begin{aligned} & \left[{}_0\mathcal{D}_\xi^{\zeta_{k-1}} y(\xi) \right]_{\xi=0} = \beta_k; \quad k = 1, 2, \dots, n \\ & {}_\eta\mathcal{D}_\xi^{\zeta_k} = {}_\eta\mathcal{D}_\xi^{\varphi_k} {}_\eta\mathcal{D}_\xi^{\varphi_{k-1}} \dots {}_\eta\mathcal{D}_\xi^{\varphi_1} \\ & {}_\eta\mathcal{D}_\xi^{\zeta_{k-1}} = {}_\eta\mathcal{D}_\xi^{\varphi_{k-1}} {}_\eta\mathcal{D}_\xi^{\varphi_{k-2}} \dots {}_\eta\mathcal{D}_\xi^{\varphi_1} \\ & \zeta_k = \sum_{j=1}^k \varphi_j; \quad 0 < \varphi_j \leq 1; \quad k, j \in \mathcal{N} \\ & \text{and } f(\xi) \in \mathcal{L}_1(0, T), \text{ i.e. } \int_0^T |f(\xi)| d\xi < \infty \end{aligned} \right\} \quad \dots (4.2)$$

We assume $f(\xi) = 0$ for $\xi > T$, then $p_k(\xi) = 0, k \in \mathcal{N}$.

Theorem 1: If $f(\xi) \in \mathcal{L}_1(0, T)$, then the linear fractional differential equation ${}_0\mathcal{D}_\xi^{\zeta_n} y(\xi) = f(\xi)$ has unique solution $y(\xi) \in \mathcal{L}_1(0, T)$, which satisfies the initial conditions given in equation (4.2).

Proof: Given linear fractional differential equation is,

$${}_0\mathcal{D}_\xi^{\zeta_n} y(\xi) = f(\xi)$$

using Laplace transform, $\mathcal{L} \left\{ {}_0\mathcal{D}_\xi^{\zeta_n} y(\xi) \right\} = \mathcal{L} \{ f(\xi) \}$

Now by Laplace transform of the Miller-Ross SFD from equation (2.2), we obtained

$$s^{\zeta_n} Y(s) - \sum_{k=0}^{n-1} s^{\zeta_n - \zeta_{n-k}} \left\{ {}_0\mathcal{D}_{\xi}^{\zeta_{n-k}-1} y(\xi) \right\}_{\xi=0} = F(s), \quad \dots (4.3)$$

Where $Y(s)$ and $F(s)$ are Laplace transforms of $y(\xi)$ and $f(\xi)$. Now from (4.2), we find

$$Y(s) = s^{-\zeta_n} F(s) + \sum_{k=0}^{n-1} \beta_{n-k} s^{-\zeta_{n-k}} \quad \dots (4.4)$$

Now taking inverse Laplace transforms gives,

$$y(\xi) = \frac{1}{\Gamma(\zeta_n)} \int_0^{\xi} (\xi - \tau)^{\zeta_n-1} f(\tau) d\tau + \sum_{k=0}^{n-1} \frac{\beta_{n-k}}{\Gamma(\zeta_{n-k})} \xi^{\zeta_{n-k}-1} \quad \dots (4.5)$$

and putting $i = n - k$,

$$y(\xi) = \frac{1}{\Gamma(\zeta_n)} \int_0^{\xi} (\xi - \tau)^{\zeta_n-1} f(\tau) d\tau + \sum_{k=0}^n \frac{\beta_i}{\Gamma(\zeta_i)} \xi^{\zeta_i-1} \quad \dots (4.6)$$

taking R-L fractional derivative (3.1), and using $\frac{1}{\Gamma(-m)} = 0, m = 0, 1, 2, \dots$

We get,

$${}_0\mathcal{D}_{\xi}^{\zeta_k} \left(\frac{\xi^{\zeta_i-1}}{\Gamma(\zeta_i)} \right) = \begin{cases} \frac{\xi^{\zeta_i-\zeta_k-1}}{\Gamma(\zeta_i-\zeta_k)}, & (k < i) \\ 0, & k \geq i \end{cases} \quad \dots (4.7)$$

and

$${}_0\mathcal{D}_{\xi}^{\zeta_{k-1}} \left(\frac{\xi^{\zeta_i-1}}{\Gamma(\zeta_i)} \right) = \begin{cases} \frac{\xi^{\zeta_i-\zeta_k}}{\Gamma(1+\zeta_i-\zeta_k)}, & (k < i) \\ 1, & (k = i) \\ 0, & (k > i) \end{cases} \quad \dots (4.8)$$

Where $k, i \in \mathcal{N}$. It is clear from (4.6) that $y(\xi) \in \mathcal{L}_1(0, T)$. From (4.7) and (4.8), we find direct substitution of the function $y(\xi)$ which is define in (4.6) with initial condition (4.2) shows that $y(\xi)$ is one of the solutions of them, and therefore, the existence of the solution is proved.

Remark: Let us suppose that given problem has two solutions such that $y_1(\xi)$ and $y_2(\xi)$, then we define a new function $z(\xi) = y_1(\xi) - y_2(\xi)$, which will satisfy the equation ${}_0\mathcal{D}_{\xi}^{\zeta_n} z(\xi) = 0$, with no initial conditions such that the Laplace transform of $z(\xi)$ is $Z(s) = 0$, hence $z(\xi) = 0$ belongs to domain $(0, T)$, which gives unique solution.

5. Non-Linear Fractional Differential Equations:

Now for nonlinear FDE in terms of the Miller-Ross SFD [2], we consider the initial value problem,

$${}_0\mathcal{D}_{\xi}^{\zeta_n} y(\xi) = f(\xi, y), \quad \dots (5.1)$$

$$\left[{}_0\mathcal{D}_{\xi}^{\zeta_{k-1}} y(\xi) \right]_{\xi=0} = \beta_k; \quad k \in \mathcal{N} \quad \dots (5.2)$$

where,

$$\left. \begin{aligned} \eta \mathcal{D}_\xi^{\zeta_k} &= \eta \mathcal{D}_\xi^{\varphi_k} \eta \mathcal{D}_\xi^{\varphi_{k-1}} \dots \eta \mathcal{D}_\xi^{\varphi_1} \\ \eta \mathcal{D}_\xi^{\zeta_{k-1}} &= \eta \mathcal{D}_\xi^{\varphi_{k-1}} \eta \mathcal{D}_\xi^{\varphi_{k-2}} \dots \eta \mathcal{D}_\xi^{\varphi_1} \\ \zeta_k &= \sum_{j=1}^k \varphi_j ; 0 < \varphi_j \leq 1 ; k, j \in \mathcal{N} \end{aligned} \right\} \quad \dots (5.3)$$

Let the function, $f(\xi, y) \in G$ of (ξ, y) , and $R(h, K) \subset G$, then it will satisfy the following inequalities [7]:

$$\left| \xi^{1-\zeta_1} y(\xi) - \sum_{i=1}^n \beta_i \frac{\xi^{\zeta_i - \zeta_1}}{\Gamma(\zeta_i)} \right| \leq K ; 0 < \xi < h.$$

Theorem 2: If $f(\xi, y) \in G$ be a real valued continuous function, satisfying the Lipschitz condition with respect to y , i. e.,

$$|f(\xi, y_1) - f(\xi, y_2)| \leq A|y_1 - y_2|,$$

Such that, $|f(\xi, y)| \leq \mathcal{M} < \infty ; \forall (\xi, y) \in G$. Let also, $K \geq \frac{\mathcal{M} h^{\zeta_n - \zeta_1 + 1}}{\Gamma(1 + \zeta_n)}$

Then show that there exists a continuous and unique solution $y(\xi)$ of the problem given in the equation (5.1) and (5.2) in the domain $R(h, K)$.

Proof: To prove this theorem, firstly we reduce the problem (5.1) and (5.2) to an equivalent fractional integral equation. Using (4.6) and the composition rule, we obtain,

$$y(\xi) = \sum_{i=1}^n \frac{\beta_i}{\Gamma(\zeta_i)} \xi^{\zeta_i - 1} + \frac{1}{\Gamma(\zeta_n)} \int_0^\xi (\xi - \tau)^{\zeta_n - 1} f(\tau, y(\tau)) d\tau \quad \dots (5.4)$$

Here, if $y(t)$ satisfies the problem (5.1) and (5.2), then it must satisfy the equation (5.4). Now we define the sequence of functions $y_0(\xi), y_1(\xi), y_3(\xi), \dots$, by the following relationship,

$$y_0(\xi) = \sum_{i=1}^n \frac{\beta_i}{\Gamma(\zeta_i)} \xi^{\zeta_i - 1} \quad \dots (5.5)$$

and for $m \in \mathcal{N}$, we have

$$y_m(\xi) = \sum_{i=1}^n \frac{\beta_i}{\Gamma(\zeta_i)} \xi^{\zeta_i - 1} + \frac{1}{\Gamma(\zeta_n)} \int_0^\xi (\xi - \tau)^{\zeta_n - 1} f(\tau, y_{m-1}(\tau)) d\tau ; \quad \dots (5.6)$$

Now we show that $\lim_{m \rightarrow \infty} y_m(\xi)$ exists and $y(\xi)$ is the required solution of the equation (5.4).

Firstly, by the principle of mathematical induction that for $0 < \xi \leq h$ we have $y_m(\xi) \in R(h, K)$ for all m .

$$\begin{aligned} \left| \xi^{1-\zeta_1} y_m(\xi) - \sum_{i=1}^n \beta_i \frac{\xi^{\zeta_i - \zeta_1}}{\Gamma(\zeta_i)} \right| &= \left| \frac{\xi^{1-\zeta_1}}{\Gamma(\zeta_n)} \int_0^\xi (\xi - \tau)^{\zeta_n - 1} f(\tau, y_{m-1}(\tau)) d\tau \right| \\ &\leq \frac{\mathcal{M} \xi^{\zeta_n - \zeta_1 + 1}}{\Gamma(1 + \zeta_n)} \leq \frac{\mathcal{M} h^{\zeta_n - \zeta_1 + 1}}{\Gamma(1 + \zeta_n)} \leq K \end{aligned} \quad \dots (5.7)$$

And for the same reasons we have the same inequality for $y_1(\xi)$:

$$\left| \xi^{1-\zeta_1} y_1(\xi) - \sum_{i=1}^n \beta_i \frac{\xi^{\zeta_i - \zeta_1}}{\Gamma(\zeta_i)} \right| \leq \frac{\mathcal{M} h^{\zeta_n - \zeta_1 + 1}}{\Gamma(1 + \zeta_n)} \leq K$$

Continuously, it can be obtained by induction that for all m ,

$$|y_m(\xi) - y_{m-1}(\xi)| \leq \frac{\mathcal{M}A^{m-1}\xi^m\zeta_n}{\Gamma(1+m\sigma_n)} \quad \dots (5.8)$$

For $m = 1$, we have from equation (5.8):

$$|y_1(\xi) - y_0(\xi)| \leq \frac{\mathcal{M}\xi\zeta_n}{\Gamma(1+\zeta_n)} ; 0 < \xi \leq h \quad \dots (5.9)$$

Let us suppose that

$$|y_{m-1}(\xi) - y_{m-2}(\xi)| \leq \frac{\mathcal{M}A^{m-2}\xi^{(m-1)}\zeta_n}{\Gamma(1+(m-1)\zeta_n)} ; 0 < \xi \leq h \quad \dots (5.10)$$

Then, using (5.6), (5.10) and R-L fractional derivative (3.1), we get

$$\begin{aligned} |y_m(\xi) - y_{m-1}(\xi)| &\leq \frac{A}{\Gamma(\zeta_n)} \int_0^\xi (\xi - \tau)^{\zeta_n} |y_{m-1}(\tau) - y_{m-2}(\tau)| d\tau \\ &\leq \frac{\mathcal{M}A^{m-1}}{\Gamma(1+(m-1)\zeta_n)} \frac{1}{\Gamma(\zeta_n)} \int_0^\xi (\xi - \tau)^{\zeta_n-1} \tau^{(m-1)\zeta_n} d\tau \\ &= \frac{\mathcal{M}A^{m-1}}{\Gamma(1+(m-1)\zeta_n)} {}_0\mathcal{D}_\xi^{-\zeta_n} \xi^{(m-1)\zeta_n} \\ &= \frac{\mathcal{M}A^{m-1}}{\Gamma(1+(m-1)\zeta_n)} \frac{\Gamma(1+(m-1)\zeta_n) \xi^{(m-1)\zeta_n+\zeta_n}}{\Gamma(1+(m-1)\zeta_n+\zeta_n)} \\ &= \frac{\mathcal{M}A^{m-1} \xi^m\zeta_n}{\Gamma(1+m\zeta_n)} \quad \dots (5.11) \end{aligned}$$

Which shows that the equation (5.8) true for all m .

Consider the following series:

$$y^*(\xi) = \lim_{m \rightarrow \infty} (y_m(\xi) - y_0(\xi)) = \sum_{j=1}^{\infty} (y_j(\xi) - y_{j-1}(\xi)) \quad \dots (5.12)$$

From (5.8), for $0 < \xi \leq h$ the absolute value of its terms is less than the corresponding terms of the convergent series,

$$\mathcal{M} \sum_{j=1}^{\infty} \frac{\mathcal{M}A^{j-1}h^j\sigma_n}{\Gamma(1+j\sigma_n)} = \frac{\mathcal{M}}{A} \{E_{\zeta_n,1}(Ah_n^\zeta) - 1\} \quad \dots (5.13)$$

Where $E_{\lambda,\mu}(z)$ is Mittag-Leffler function [3], and the series (5.12) converges uniformly. Obviously, each term $(y_j(\xi) - y_{j-1}(\xi))$ of (5.12) is a continuous in the interval $0 < \xi \leq h$. Hence, the summation of series (5.12), $y^*(\xi)$ is a continuous for $0 < \xi \leq h$, and $y(\xi) = \lim_{m \rightarrow \infty} y_m(\xi) = y_0(\xi) + y^*(\xi)$ is a continuous for $0 < \xi \leq h$.

Now, for uniqueness of the solution, let the equation (5.4) has one more solution *i.e.* $\tilde{y}(\xi)$, continuous in the interval $0 < \xi \leq h$. Then from the equation (5.4), we define a new function $z(\xi) = y(\xi) - \tilde{y}(\xi)$, which satisfies the equation

$$z(\xi) = \frac{1}{\Gamma(\zeta_n)} \int_0^\xi (\xi - \tau)^{\zeta_n-1} f(\tau, z(\tau)) d\tau, \quad \dots (5.14)$$

When $\xi \rightarrow 0 \Rightarrow z(t) = 0$. Therefore, $z(\xi)$ is continuous in the interval $0 \leq \xi \leq h$. Then we can write $|z(\xi)| < B$, for $0 \leq \xi \leq h$, here B is constant, and from equation (5.14), we find

$$|z(\xi)| \leq \frac{AB\xi\zeta_n}{\Gamma(1+\zeta_n)} ; 0 \leq \xi \leq h \quad \dots (5.15)$$

By repeating this process j times, gives

$$|z(\xi)| \leq \frac{A^j_B \xi^{j\zeta_n}}{\Gamma(\zeta_n)} ; j \in \mathcal{N} \quad \dots (5.16)$$

The general term of the Mittag-Leffler function $E_{\zeta_n,1}(A\xi^{\zeta_n})$, for all ξ , $\lim_{j \rightarrow \infty} \frac{A^j_B \xi^{j\zeta_n}}{\Gamma(\zeta_n)} = 0$.

Taking the limit of (5.16) as $j \rightarrow \infty$, we summarize that $z(\xi) = 0$, and $\tilde{y}(\xi) = y(\xi)$ for $0 < \xi \leq h$. Which shows the uniqueness of the solution and ends the proof of theorem 2.

6. Conclusion

Existence and uniqueness theorems play a crucial role in the study of linear and nonlinear FDE. In the field of linear FDE, these theorems ensure the existence and uniqueness of solutions, providing a solid mathematical foundation in the field of viscoelastic materials where we understand the behavior of materials with both elastic and viscous properties involve linear fractional differential equations.

In the context of nonlinear FDE, existence and uniqueness theorems become powerful tools for analyzing complex dynamical systems. They establish the conditions under which solutions to nonlinear fractional equations exist and are unique within a certain domain. This is particularly relevant in fields like biology, where nonlinear fractional models describe phenomena like population dynamics or the spread of diseases. The theorems guide researchers in formulating and solving these equations, ensuring the reliability and uniqueness of the obtained solutions. Overall, the application of existence and uniqueness theorems in both linear and nonlinear fractional differential equations significantly advance our understanding and ability to model diverse phenomena across various scientific disciplines.

References

- [1] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, p. 261 (1999).
- [2] R. Hilfer, *Applications of Fraction Calculus in Physics*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, p. 87 (2000).
- [3] H. M. Kilbas, J. J. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, Amsterdam (2006).
- [4] A. A. Chikrii and I. I. Matychyn, "On an analogue of the Cauchy formula for linear systems of arbitrary fractional order," *Dopovidi Natsional'noi Akademii Nauk Ukrainy*, vol. 1, pp. 50–55 (2007).

- [5] A. A. Chikrii and I. I. Matichin, "Presentation of solutions of linear systems with fractional derivatives in the sense of Riemann–Liouville, Caputo, and Miller–Ross," *J. Autom. Inform. Sci.*, vol. 40, no. 6, pp. 1–11 (2008).
- [6] C. Bai, "Impulsive periodic boundary value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative," *J. Math. Anal. Appl.*, vol. 384, pp. 211–231 (2011).
- [7] H. Fazli, J. J. Nieto, and F. Bahrami, "On the existence and uniqueness results for nonlinear sequential fractional differential equations," *Appl. Comput. Math.*, vol. 17, pp. 36–47 (2018).
- [8] A. Anwar, S. Muhammad, B. Z. Mian, and S. Kamal, "Existence of solution to fractional differential equations with fractional integral type boundary conditions," *Math. Methods Appl. Sci.*, vol. 44, pp. 1615–1627 (2020).

Received July, 2024