

Common fixed point theorems satisfying rational contraction in partially ordered metric space

Isha *

Naveen Mani [†]

Anjana [§]

Department of Mathematics

Chandigarh University

Gharuan 140301

Mohali, Punjab

India

Reeta Bhardwaj [‡]

Department of Mathematics

Amity University

Gurugram 122001

Haryana

India

Abstract

Our main purpose in this article to established some results on fixed point for two maps which are non-continuous, non-compatible and non- commutative. We obtain our result without using completeness or Cauchy's criteria, and by utilizing a contraction mapping where maps are weakly increasing. Lastly, we provide a two examples with graphical illustrations that supports our primary findings.

Subject Classification: 46TXX, 47H10, 54H25.

Keywords: Fixed point, Partially ordered metric spaces, Weakly increasing-mapping, Contraction.

* E-mail: isha.dimple696@gmail.com (Corresponding Author)

[†] E-mail: naveenmani81@gmail.com

[§] E-mail: anjanabhardwaj1999@gmail.com

[‡] E-mail: bhardwajreeta84@gmail.com

1. Historical Background

The fixed point theorem has applications in several spaces, notably “partially ordered metric spaces”, which are significant in numerous fields of Mathematics. Studying mappings that preserve a certain order structure on the underlying space is essential for introducing fixed point theorems in “partially ordered metric spaces”. The present growth of fixed point theory is driven by its capacity to establish the existence of solutions to integral as well as differential equations.

The Banach fixed point theorem [1] is the most renowned area in nonlinear analysis. It has introduced some fascinating ideas for exploring contractions, which have been further investigated by other scholars. Dass and Gupta [2] provided one of the well-known extensions of the Banach's result, generalizing the Banach contraction as a rational expression. In literature, several authors have extended and generalizes these results for various contractions in metric spaces, some of them are [3-12].

2. Preliminaries

Definition 2.1: [13] A mapping $S, T : H \rightarrow H$, where $H \neq \emptyset$ and partially ordered relation $\leq$ on H , is said to be non-decreasing if for each $\eta, \zeta \in H$, $\eta \leq \zeta$ implies $S(\eta) \leq S(\zeta)$.

Definition 2.2: [13] A mapping $S, T : H \rightarrow H$, where $H \neq \emptyset$ and partially ordered relation $\leq$ on H , if for each $\eta \in E$, $S\eta \leq TS\eta$ and $T\eta \leq ST\eta$, then the map S and T are weakly increasing.

Remark 2.3: It is obvious that weakly increasing mappings doesn't implies that it is non-decreasing.

Definition 2.4: [14] An control function (commonly known as altering distance function), denoted as $\xi : [0, \infty) \rightarrow [0, \infty)$, that satisfies the properties are as follows:

- (i) ξ is nondecreasing, continuous and,
- (ii) $\xi(\tau) = 0$ iff $\tau = 0$.

3. Main Results

To simplify, we will use the abbreviation "PoMs" for “partially ordered metric space”.

Theorem 3.1: Consider a triplet $(H, d, \leq)$ as a PoMs and mappings $S, T : H \rightarrow H$ are Weakly-Increasing-mappings (WIm), and for every comparable pair of elements $(\eta, \zeta) \in H$, it satisfies

$$\xi(d(S\eta, T\zeta)) \leq \lambda(M(\eta, \zeta)) \quad (1)$$

where,

$$M(\eta, \zeta) = \max \left\{ \frac{d(\zeta, T\zeta)d(\eta, S\eta)}{1+d(\eta, \zeta)}, \frac{d(\zeta, T\zeta)[1+d(\eta, S\eta)]}{1+d(\eta, \zeta)} \right\}, \quad (2)$$

ξ and $\lambda: [0, \infty) \rightarrow [0, \infty)$ be an altering distance and continuous function respectively, such that

$$0 < \lambda(\varrho) < \xi(\varrho), \forall \varrho > 0. \quad (3)$$

Therefore, both S and T possess one common fixed point.

Proof: Set $\eta_0 \in H$ be any arbitrary point with initial approximation. Define $S\eta_0 = \eta_1$ and $T\eta_1 = \eta_2$. In general, construct sequences $\{\eta_i\}, \{\zeta_i\} \in H$ such that

$$\eta_{2i+1} = S\eta_{2i} = \zeta_{2i}, \eta_{2i+2} = T\eta_{2i+1} = \zeta_{2i+1}, \forall i \in N. \quad (4)$$

Since S and T are weakly increasing function

$$\zeta_0 = \eta_1 = S\eta_0 \leq T\eta_1 = \zeta_1 \cdots$$

Continuing in this process, we obtain

$$\zeta_0 \leq \zeta_1 \leq \zeta_2 \cdots \leq \zeta_{2i} \leq \zeta_{2i+1} \leq \cdots$$

Assume that

$$\zeta_{2i} = \zeta_{2i+1}, \text{ for no } i \in N. \quad (5)$$

Since η_{2i} and η_{2i+1} are comparable, using eq. (1), (2) and (3),

$$\xi(d(\zeta_{2i}, \zeta_{2i+1})) = \xi(d(S\eta_{2i}, T\eta_{2i+1})) \leq \lambda(M(\eta_{2i}, \eta_{2i+1})) \quad (6)$$

where

$$\begin{aligned} M(\eta_{2i}, \eta_{2i+1}) &= \max \left\{ \frac{\frac{d(\eta_{2i+1}, T\eta_{2i+1})d(\eta_{2i}, S\eta_{2i})}{1+d(\eta_{2i}, \eta_{2i+1})}}{\frac{d(\eta_{2i+1}, T\eta_{2i+1})[1+d(\eta_{2i}, S\eta_{2i})]}{1+d(\eta_{2i}, \eta_{2i+1})}} \right\} \\ &\leq \max \left\{ \frac{\frac{d(\eta_{2i+1}, \eta_{2i+2})d(\eta_{2i}, \eta_{2i+1})}{1+d(\eta_{2i}, \eta_{2i+1})}}{\frac{d(\eta_{2i+1}, \eta_{2i+2})[1+d(\eta_{2i}, \eta_{2i+1})]}{1+d(\eta_{2i}, \eta_{2i+1})}} \right\} \\ &\leq \max \left\{ \frac{\frac{d(\eta_{2i+1}, \eta_{2i+2})d(\eta_{2i}, \eta_{2i+1})}{1+d(\eta_{2i}, \eta_{2i+1})}}{d(\eta_{2i+1}, \eta_{2i+2})} \right\} \\ &\leq d(\zeta_{2i}, \zeta_{2i+1}) \end{aligned}$$

From (6), we obtain

$$\xi(d(\zeta_{2i}, \zeta_{2i+1})) = \xi(d(S\eta_{2i}, T\eta_{2i+1})) \leq \lambda(d(\eta_{2i}, \eta_{2i+1})) < \xi(d(\zeta_{2i}, \zeta_{2i+1}))$$

This is a contradiction. Thus our assumption in (5) is false. Therefore, $\zeta_{2i} = \zeta_{2i+1}$ for some $i \in N$, let's say $i = k$, i.e., for some $i = k$, we've $\zeta_{2k} = \zeta_{2k+1}$ and $\rho = \eta_{2k+1}$, this implies by referring (4) that $T\rho = \rho$. Further, let us set $\eta = \zeta = \rho$ in (1), we obtain

$$\xi(d(S\rho, \rho)) = \xi(d(S\rho, T\rho)) \leq \lambda(M(\rho, \rho)) \quad (7)$$

where

$$M(\rho, \rho) = \max \left\{ \frac{\frac{d(\rho, T\rho)d(\rho, S\rho)}{1+d(\rho, \rho)}}{\frac{d(\rho, T\rho)[1+d(\rho, S\rho)]}{1+d(\rho, \rho)}} \right\}$$

$$\leq \max \left\{ \frac{\frac{d(\rho, \rho)d(\rho, S\rho)}{1+d(\rho, \rho)}, \frac{d(\rho, \rho)[1+d(\rho, S\rho)]}{1+d(\rho, \rho)} \right\} = 0$$

Then, from equation (7), we have $\xi(d(S\rho, \rho)) \leq \lambda(0)$. But from equation (3) and the continuity of ξ and λ implies that $\lambda(0) = 0$. Since $\xi(0) = 0$, thus, we obtain $S\rho = \rho$. Consequently, we get $S\rho = T\rho = \rho$.

Next, to demonstrate the uniqueness of fixed points, we have a sufficient condition to prove.

Theorem 3.2: *In addition, Theorem 3.1, let if for any pair of elements ζ and η in H , there exists an element z in H that can be compared to both η and ζ . A single fixed point is invariant under both the mappings S and T .*

Proof: The existence of a fixed point may be deduced from Theorem 3.1. Let's assume the contrary for the proof of uniqueness. Take π and σ be two common fixed points of S and T respectively, such that $S\pi = T\pi = \pi$ and $S\sigma = T\sigma = \sigma$. Now, we will discuss two separate scenarios.

Case 1: Firstly, if π is comparable to σ , then $S\pi = \pi$ is comparable to $\sigma = T\sigma$. Then, $S\pi = \sigma$.

Hence, from (1)

$$\xi(d(\pi, \sigma)) = \xi(d(S\pi, T\sigma)) \leq \lambda(M(\pi, \sigma)) \quad (8)$$

where

$$\begin{aligned} M(\pi, \sigma) &= \max \left\{ \frac{\frac{d(\sigma, T\sigma)d(\pi, S\pi)}{1+d(\sigma, \pi)}, \frac{d(\sigma, T\sigma)[1+d(\pi, S\pi)]}{1+d(\sigma, \pi)} \right\} \\ &\leq \max \left\{ \frac{\frac{d(\sigma, \sigma)d(\pi, \sigma)}{1+d(\sigma, \pi)}, \frac{d(\sigma, \sigma)[1+d(\pi, \sigma)]}{1+d(\sigma, \pi)} \right\} = 0. \end{aligned}$$

Thus, from (8) $\xi(d(\pi, \sigma)) \leq 0$. Hence, our assumption contradicts, thus $d(\pi, \sigma) = 0$, i.e., $\pi = \sigma$.

Case 2: On the other hand, if π is not comparable to σ . Then, $\exists c \in H$ such that c is comparable to π and also comparable to σ . This implies that $Tc = c$ is comparable to $\pi = T\pi$ and also comparable to $S\sigma = \sigma$. By (1),

$$\xi(d(\sigma, Tc)) = \xi(d(S\sigma, Tc)) \leq \lambda(M(\sigma, c)). \quad (9)$$

Where

$$M(\sigma, c) = \max \left\{ \frac{\frac{d(c, Tc)d(\sigma, S\sigma)}{1+d(\sigma, c)}, \frac{d(c, Tc)[1+d(\sigma, S\sigma)]}{1+d(\sigma, c)} \right\}$$

$$\leq \max \left\{ \frac{d(c,c)d(\sigma,\sigma)}{1+d(\sigma,c)}, \frac{d(c,c)[1+d(\sigma,\sigma)]}{1+d(\sigma,c)} \right\} = 0$$

Thus, from (9), $\xi(d(\sigma, Tc)) \leq 0$, so $d(\sigma, Tc) = 0$ i.e. $Tc = \sigma$. Like this, we may demonstrate that $d(\pi, Tc) = 0$ i.e., $Tc = \pi$. Thus, $\pi = \sigma$, hence proved.

Example 1: Let $H = [1, \infty) \cup \{0\}$ with usual metric d . Define self-maps $S, T : H \rightarrow H$ are weakly increasing and partial order $\leq$ in H as $\eta \leq \zeta$ for all $\eta, \zeta \in H$ such that $S\eta = \frac{1}{2}\eta$ and $T\eta = \sqrt{\eta}$, $\forall \eta \in H$.

Define a function ξ (altering distance) and a function $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$\xi(\varrho) = \varrho; \lambda(\varrho) = \frac{9\varrho}{10}, \text{ for all } \varrho > 0.$$

Consequently, the requirements of Theorem 3.1 are verified. Also S and T admits unique common fixed point in H (at $\eta = 0$).

Example 2: Let $H = [1, \infty) \cup \{0\}$ with usual metric d . Define self-maps $S, T : H \rightarrow H$ are weakly increasing and partial order $\leq$ in H as $\eta \leq \zeta$ for all $\eta, \zeta \in H$ such that $S\eta = \frac{\eta}{2}$ and $T\eta = \frac{\eta}{\eta+1}$, $\forall \eta \in H$.

Define a function ξ (altering distance) and a function $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$\xi(\varrho) = \varrho; \lambda(\varrho) = \frac{4\varrho}{5}, \text{ for all } \varrho > 0.$$

Evidently, λ is a continuous function and also verify with the condition $0 < \lambda(\varrho) < \xi(\varrho)$. Thus, all the assumptions made in Theorem 3.1 are verified. Also S and T admits unique common fixed point (at $\eta = 0 \in H$), see Figure 1.

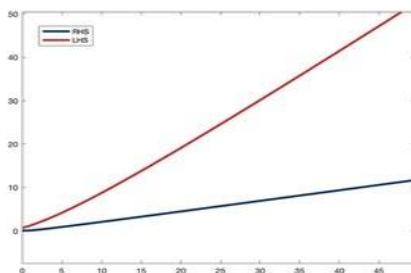


Figure 1
Graph of Example 2

Conclusion

In this paper, we concluded that without the help of compatibility and commutative property fixed point theorems for noncontinuous selfmapping were established. We used contraction mapping with weakly increasing parameters to find the common fixed point in “partially ordered metric spaces” without using Cauchy’s criterion.

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Received July, 2024