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B-multiplicative metric spaces : Coupled coincidence points with application

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Abstract

In this manuscript, we use g-couplings for b-multiplicative metric spaces for demonstrating the presence of the coupled coincidence point. Additionally, we demonstrate the presence of a fixed point with t-property in an ordered b-MMS. A few examples and an application are also provided to support the veracity of our findings.

Subject Classification: *54H25, 47H10.*

Keywords: *g-coupling, b-multiplicative metric spaces, Coupled coincidence points.*

1. Introduction

In 1922, Banach [11] constituted the seminal principles of fixed-point theory within realm of metric spaces. This pivotal contribution laid the groundwork for a fundamental concept in mathematics. Subsequent writers expanded upon the Banach contraction principle. The first person to study fixed points in ordered sets was Turinici [6]. A large body of research on fixed points in ordered metric spaces can be found here ([7]-[10]).

Guo *et al.* [13] derived coupling fixed points after which a number of writers subsequently reported in this direction. Multiplicative triangle inequality was introduced as a main notion to replace the standard triangular inequality.

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A broader interpretation of a metric space known as a b-metric space, was first proposed as well as modified respectively by Bakhtin [4] and Czerwik [12]. Numerous related extensive findings can be found in literature. Then the idea behind b-metric space and multiplicative metric has been combined and a new metric space has been introduced by Usman Ali *et al.*[5] named as b-multiplicative metric space. Later, Rashid *et al.* [15] expanded on the concept of coupling and proposed the notion of g-couplings.

In this study, we have now shown that coupled coincidence points exist for g-couplings in b-multiplicative metric space and that strong coupled coincidence points are unique. We present examples to demonstrate our findings, along with an application for solving integral equations.

2. Preliminaries

Some terminology and findings are recounted for the reader's ease.

Definition 2.1L [12] Assume a real integer $K > 1$and a nonempty set W. When $i, u, f \in W$ satisfy the following axioms, the function $B_f : W \times W \to [0, \infty]$ is referred to as a b-metric space.

(1) $B_f(i,u) = 0 \Leftrightarrow i = u$;

(2) $B_f(i,u) = B_f(u,i)$;

(3) $B_f(i,f) \leq K[B_f(i,u) + B_f(u,f)]$.

The pair (W, B_f) is referred to as b-metric space.

Definition of b-metric space makes it abundantly evident that it is a standard metric space generalization.

Definition 2.2: [1] Assume that W is a nonempty set and that M_f: W × W → $\mathbb{R}$ meets these requirements; it is referred to as a multiplicative metric:

(1) $M_f(i,u) \geq 1$ for all $i, u \in W$ with i, u and $M_f(i,u) = 1 \; iff \; i = u$;

(2) $M_f(i,u) = M_f(u,i)$ for all $c, u \in W$;

(3) $M_f(i,f) \leq [s(i,u).s(u,f)]$ for all $i, f, u \in W$.

Then $\left(W, M_f\right)$ is referred as multiplicative metric space.

Definition 2.3: [5] Assume that W is a nonempty set and that $r \geq 1$ a real number. A mapping $M_b : W \times W \to \mathbb{R}^+$ if the following criteria met, is referred to as b-multiplicative metric:

(1) $M_b(i,u) > 1$ for all i, u with j , u and $M_b(i,u) = 1 \; iff \; i = u$;

(2) $M_b(i,y) = M_b(u,i)$ for all $i, u \in W$;

(3) $M_b(i,f) \leq [m_b(i,u).M_b(u,f)]r$ for all $i, u, f \in W$.

If $r > 1, s \in W$, then $BM_b(s,r) = \{w \in W : M_b(w,s) \leq r\}$ is referred to as closed ballin (W, M_b).

Definition 2.4: [14] Let D_C: W × W → W be a mapping for any nonempty set W. If $D_C(j,u) = c$ and $D_C(j,u) = u$, then an element $(j,u) \in W \times W$ is considered as coupled fixed point of D_C.

Definition 2.5: [2] For any nonempty set W, let $D_C : W \times W \to W$ be a mapping. Then an element $(j,u) \in W \times W$ is strong coupled fixed point of D_C, if , $j = u; D_C(j,j) = j$.

Definition 2.6: [3] For any nonempty set W, let $D_C : W \times W \to W$and $g : W \to W$ be two mappings. Then an element $(j,u) \in W \times W$ is described as coupled coincident point of D_C & g, if $D_C(j,u) = gj$ and $D_C(u,j) = gu$.

Definition 2.7: [15]For any nonempty set W, let $D_C : W \times W \to W$ and $g : W \to W$ be two mappings. Then, for subsets L, T of W, H is g-coupling if $D_C(j,u) \in g(L) \cap T$ and $D_C(u,j) \in g(T) \cap L$ for $j \in L$ and $u \in T$.

3. Results

Theorem 3.1: *Given b-multiplicative metric* (G,s)*, assume thatDhas two non-empty subsets* L *and* T*and let* D *be a g-coupling concerning a pair of subsets within* G*. Assume also that* q *is invariantto* L *and* T*, and that* $g(L), g(T)$ *are multiplicative closed subsets of* (G,s)*. Also,*

1) $L \cap T \neq \phi$;
2) The coupled coincidence point of D and q will be located in $L \times T$.

Additionally, if$q$ is one-to-one map on $L \cup T$, there exists distinct strong coupled coincidence point in $L \times T$ between D and q.

Proof: Existence of g-coupling in D is given, we aim to show the presence of a coupled coincidence point in $L \times T$ between D and q. Let d be the b-multiplicative metric on the space. Consider the set $L \times T$ consisting of all ordered pairs (j,u), where $j \in L$ and $u \in T$. Define the function $F : L \times T \to \Re^+$ as $F(j,u) = D(j,u) + q(j).q(u)$

We aim to show the presence of a point (j_0, u_0) in $L \times T, F(j_0, u_0) = 0$. As D has g-coupling, for each $(j,u) \in L \times T$, we have:

$$0 \leq D(j,u) \leq q(j).q(u) \leq d(j,u).$$

Thus, $0 \leq F(j,u) \leq 2q(j).q(u) \leq 2d(j,u)$.Now, if q is an injective map on $L \cup T$, it implies that for any distinct points $e, v \in L \cup T, q(e) \neq q(v)$, where $q(e)$ denotes the image of e in q. Let

$$d_{max} = max\, d(j,u) : j \in L, u \in T$$

d_{max} must be finite as L and T are nonempty subsets of the metric space. Now, consider any $(j_0, u_0) \in L \times T$. We have:

$$0 \leq F(j_0, u_0) \leq 2q(j_0).q(u_0) \leq 2d(j_0, u_0) \leq 2d_{max}.$$

By the Intermediate Value Theorem (IVT) for real functions, as F is continuous function from $L \times T$ to $\Re^+$and $0 \leq F(j_0, u_0) \leq 2d_{max}$, there must exist a point (j_0, u_0) in $L \times T$ such that $F(j_1, u_1) = 0$. Thus, $(j_1, u_1) \in L \times T$ and hence, $D(j_1, u_1) + q(j_1).q(u_1) = 0$, $D(j_1, u_1) = -q(j_1).q(u_1)$

Since $0 \leq D(j_1, u_1) \leq q(j_1).q(u_1) \leq d(j_1, u_1)$, it follows that

$$D(j_1, u_1) = -q(j_1).q(u_1) \leq 2d(j_1, u_1)$$

This indicates that a point $(j_1, u_1)$is present in $L \times T$ such that $D(j_1, u_1) \leq 2d(j_1, u_1)$, meeting the conditions needed for D and q to have paired coincidence points in $L \times T$. As a result, $L \times T$ contains a coincidence point of D and q under the specified circumstances.Furthermore, if q is injective on $L \cup T$, then there present a single strong coupled coincidence point between D as well as q in $L \times T$ at (j_1, u_1) fulfilling $D(j_1, u_1) \leq 2d(j_1, u_1)$.

Example 3.2: Let $G = \mathbb{R}$, metric $s(j, u) = e^{|j-u|}$. Take $L = [0, 2]$ and $T = [0, 3]$.D be defined as $D(j, u) = \frac{j+u}{10}$, where $j, u \in G$. We define $q : G \to G$ by $q(j) = \frac{j}{2}$. Obviously, $q(L) = [0, 1]$ and $q(T) = \left[0, \frac{3}{2}\right]$, $q(L)$ and $q(T)$ are closed multiplicative subsets of G. Also, L and T are invariant by q as $q(L) \subseteq L$ and $q(T) \subseteq T$.

Now, $q(L) \cap T = [0, 1], q(T) \cap L = \left[0, \frac{3}{2}\right]$, hence for every $j \in L, u \in T$. Also, $0 \leq D(j, u) \leq 1/2$, $0 \leq D(u, j) \leq 1/2, D(j, u) \in q(L) \cap T$and $D(u, j) \in q(T) \cap L$. It shows the existence of g-coupling in D with respect to L and T. Similarly, if $j, o \in L$ and $u, a \in T$,

$$S\big(D(j, o), D(o, a)\big) = e^{\left|\frac{j+u}{10}\right| - \left|\frac{j+u}{10}\right|} s\big(D(j, u), D(u, a)\big) \leq \max e^{\left|\frac{j+u}{10}\right| - \left|\frac{j+u}{10}\right|}$$

Hence, all the conditions of the theorem hold. Therefore, $(j, u) \in L \times T$ such that $D(j, u) = q(j)$ and $D(u, j) = q(u)$, that is $\frac{j+u}{10} = \frac{j}{2} \frac{u+j}{10} = \frac{u}{2}$. Then $j = u = 0$. Therefore, $(0, 0)$ acts as individual g-coupled coincidence point of D as well as q, because of the existence of injective map in g.

4. Application

Assume $K = C([0, 1], \mathbb{R})$ as a compilation of every continuous function on $[0, 1]$ and furnished with metric $j(o, p) = e^{sup|\varphi(t) - \zeta(t)|}$ for all $n \in [0, 1]$. (K, j) is unquestionably a b-multiplicative metric space. Let us now examine the integral equation that follows:

$$\varphi(\mu) = h(\mu) + \int_0^1 k(\mu, \varsigma)[f_1(\varsigma, \varphi(\varsigma)) + f_2(\varsigma, \varphi(\varsigma))]d\varsigma$$

where $h \in C([0, 1]$. For $0 < \alpha, \beta \leq 1/4$, we have,

$$\big|f_1\big(\mu, \varphi(\xi)\big) - f_1\big(\mu, \zeta(\xi)\big)\big| \leq \alpha\, |\varphi(\xi) - \zeta(\xi)|, \big|f_2\big(\mu, \varphi(\xi)\big) - f_2\big(\mu, \zeta(\xi)\big)\big|$$
$$\leq \beta\, |\varphi(\xi) - \zeta(\xi)|, k(\mu, \chi) \leq 1/2.$$

Theorem 4.1: *By using above conditions, the integral equation individual solution is discovered in* $C([0,1],\mathbb{R})$.

Proof: Let $D : K \times K \to K$ and $q : K \to K$ be defined as

$$D(\varphi,\zeta)(\mu) = h(\mu) + \int_0^1 \frac{k(\mu,\varsigma)}{2}\left[f_1\left(\varsigma,\varphi(\varsigma)\right) + f_2\left(\varsigma,\varphi(\varsigma)\right)\right]d\varsigma + \int_0^1 \frac{k(\mu,\varsigma)}{2}\left[f_1(\varsigma,\zeta(\varsigma)) + f_2(\varsigma,\zeta(\varsigma))\right]d\varpi$$

Taking $T = L$, we have $q(L) \subset L$ and $q(T) \subseteq T$. Also, $q(L)$ and $q(T)$ are b-multiplicative K's closed subsets. Therefore, g-coupling exists in K. Hence, $q(L) \cap T = L$ as well as $q(T) \cap L = T$. Let $\varphi,\tau \in L$ and $\zeta,\vartheta \in T$, we have

$$|D(\varphi,\zeta)(\mu) - D(\vartheta,\tau)(\mu)|$$

$$\leq 1/4 \left| \int_0^1 [f_1(\varsigma,\varphi(\varsigma)) - f_1(\varsigma,\vartheta(\varsigma)) + f_2(\varsigma,\varphi(\varsigma)) - f_2(\varsigma,\vartheta(\varsigma)) - f_1(\varsigma,\varphi(\varsigma)) - f_1(\varsigma,\vartheta(\varsigma)) + f_1(\varsigma,\varphi(\varsigma)) - f_1(\varsigma,\vartheta(\varsigma))]\, d\varrho \right|$$

Now, using $e^{|r-j|} \leq max\, e^{|r|^2}, e^{|j|^2}$, we have

$$e\,|D(\varphi,\zeta)(\mu) - D(\vartheta,\tau)(\mu)| \leq max\, e^{\left[\frac{1}{4[\alpha|\varphi-\vartheta|+\beta|\varphi-\vartheta|]}\right]^2}, e^{\left[\frac{1}{4[\alpha|\zeta-\tau|+\beta|\zeta-\tau|]}\right]^2}$$

$$= [max\, e^{|\varphi-\vartheta|}, e^{|\varphi-\vartheta|}]^2$$

We can now declare that the theorem's requirements are all met. Also, a unique $j(\tau) \in L$ exists, thus $F\left(j(\tau),j(\tau)\right) = q\left(j(\tau)\right) = j(\tau)$, because g is one-one.Thus, L contains the integral equation's individual solution.

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