Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 8, pp. 1869–1874
DOI : 10.47974/JIM-2052

Application of Marichev-Saigo-Maeda fractional differential and integral operators on composition of two special functions

Harish Nagar *
Department of Mathematics
Chandigarh University
Gharuan 140301
Mohali, Punjab
India

Shristi Mishra †
Department of Mathematics
University Institute of Sciences
Chandigarh University
Gharuan 140301
Mohali, Punjab
India

Abstract

This manuscript utilizes the Marichev-Saigo-Maeda (MSM) fractional differential (F.D) and fractional integral (F.I) operators to analyze the composition of two functions i.e., Struve function (S.F) and generalized function (G.F). The outcome is expressed in terms of the 'generalized Wright function' and the hypergeometric function (H.F).

Subject Classification: *44C20, 44A20.*

Keywords: *Marichev-Saigo-Maeda fractional operator, Generalized Wright function, Struve function, Gauss hypergeometric function (G.H.F).*

1. Introduction

In this manuscript, let $\mathbb{R}$ and $\mathbb{C}$ stands for the groups of real and complex numbers, respectively, and also let $\mathbb{R}^{+} \coloneqq (0.\infty)$.

1.1 *M.S.M. Operator*

"Saigo" introduced F.I and F.D operators [1-3] involving the G.H.F in the kernel. For $\mu, \nu, \xi \in \mathbb{C}$ and $x \in \mathbb{R}^{+}$, with $Re(\mu) > 0$ [4] 'generalized functional integral operators' associated with G.H.F are defined by,

* *E-mail:* **drharishngr@gmail.com (Corresponding Author)**
† *E-mail:* **mshristi69@gmail.com**

$$(I_{0+}^{\mu,\nu,\xi} g)(x) = \frac{x^{-\mu-\nu}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} F_1\left(\mu+\nu, -\xi; \mu; 1-\frac{t}{x}\right) g(t)dt \qquad (1)$$

And,

$$\left(I_{-}^{\mu,\nu,\xi} g\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty \frac{(t-x)^{\mu-1}}{t^{\mu+\nu}} F_1\left(\mu+\nu, -\xi;\ \mu; 1-\frac{x}{t}\right) g(t)dt \qquad (2)$$

Where $2F_1\ (\mu, \nu; \xi; z)$ is a G.H.F defined [5] for $z \in \mathbb{C}, |z| < 1\ \mu, \nu \in \mathbb{C}, \xi \in \mathbb{C}/Z_0^-$ by

$$\Rightarrow {}_2F_1(\mu, \nu; \xi; z) = \sum_{t=0}^{\infty} \frac{(\mu)_t (\nu)_t}{(\xi)_t} \frac{z^t}{t!}$$

Fractional differential operator is

$$\left(D_{0+}^{\mu,\nu,\xi} g\right)(x) = \left(\frac{d}{dx}\right)^q \left(I_{0+}^{-\mu+q,-\nu-q,\mu+\xi-q}\ g\right)(x) \qquad (3)$$

And

$$\left(D_{-}^{\mu,\nu,\xi}\ g\right)(x) = \left(\frac{d}{dx}\right)^q (I_{0-}^{-\mu+q,-\nu-q,\mu+\xi} g)(x) \qquad (4)$$

Where $m = [Re(\mu)] + 1$ and $[Re(\mu)]$ represents the integral value of $Re(\mu)$.

1.2 *Struve Function*

Struve function is denoted as [6, 7] $H_\varsigma(z)$, and is defined as,

$$H_\varsigma(z) = \sum_{t=0}^{\infty} \frac{(-1)^t \left(\frac{z}{2}\right)^{\varsigma+2t+1}}{\Gamma\left(t+\frac{3}{2}\right)\Gamma\left(\varsigma+t+\frac{3}{2}\right)}\ , z \in \mathbb{C}\ 5$$

Where, $\varsigma \in \mathbb{R}$, and this equation is known as Struve equation of order ς

2. Required Result

Here, we need these particular MSM integral operators [8] to achieve the MSM fractional integration of a composite of two functions, i.e., Struve function $H_\varsigma(z)$ and generalized special function $G_{\alpha,\beta,s}[a, z]$.

Let $\zeta, \mu'', \delta, \nu'', \xi, \alpha \in \mathbb{C}$ such that $Re(\zeta) > 0$

$$\left(I_{0+}^{\zeta,\mu'',\delta,\nu'',\xi} d^{\alpha-1}\right)(d) = \frac{\Gamma(\alpha)\Gamma(-\mu''+\nu''+\alpha)\Gamma(-\zeta-\mu''-\delta+\xi+\alpha)}{\Gamma(\nu''+\alpha)\Gamma(-\zeta-\mu''+\xi+\alpha)\Gamma(-\mu''-\delta+\xi+\alpha)} \times d^{-\zeta-\mu''+\xi+\alpha-1} \qquad (6)$$

$$\left(I_{-}^{\zeta,\mu'',\delta,\nu'',\xi}\ d^{-\alpha}\right)(d) = \frac{\Gamma(-\nu+\alpha)\Gamma(\zeta+\mu''-\xi+\alpha)\ \Gamma(\zeta+\nu''-\xi+\alpha)}{\Gamma(\alpha)\Gamma(\zeta-\delta+\alpha)\ \Gamma(\zeta+\mu''+\nu''-\xi+\alpha)} \times d^{-\zeta-\mu''+\xi-\alpha} \qquad (7)$$

To obtain the MSM fractional differentiation of composition of two functions [9] i.e., Struve function $H_\varsigma(z)$ and generalized special function $G_{\alpha,\beta,s}[a, z]$, [10] the following result will be used.

Let $\zeta, \mu'', \delta, \nu'', \xi, \alpha \in \mathbb{C}$.

$$\left(D_{0+}^{\zeta,\mu'',\delta,\nu'',\xi} d^{\alpha-1}\right)(d) = \frac{\Gamma(\alpha)\Gamma(-\delta+\mu+\alpha)\Gamma(\zeta+\mu''+\nu''-\xi+\alpha)}{\Gamma(-\delta+\alpha)\Gamma(\zeta+\mu''-\xi+\alpha)\Gamma(\zeta+\nu''-\xi+\alpha)} \times d^{\zeta+\mu''-\xi+\alpha-1} \qquad (8)$$

$$(D_{-}^{\zeta,\mu'',\delta,\nu'',\xi}\ d^{-\alpha})(d) = \frac{\Gamma(\nu''+\alpha)\Gamma(-\zeta-\mu''+\xi+\alpha)\ \Gamma(-\mu''-\delta+\xi+\alpha)}{\Gamma(\alpha)\Gamma(-\mu''+\nu''+\alpha)\ \Gamma(-\zeta-\mu''-\delta+\xi+\alpha)} \times d^{\zeta+\mu''-\xi-\alpha} \qquad (9)$$

3. Main Results

Now we will present a formula for the MSM fractional integration and differentiation operators associated with the composition of two functions i.e., Struve function $H_\varsigma(z)$ and generalized special function $G_{\alpha,\beta,s}[a,z]$.

Theorem 3.1: *Let* $\zeta,\mu'',\delta,\nu'',\xi\in\mathbb{C}$ *and* $Re(\alpha s-\beta)>0$ *be such that* $Re(\xi)>0$. *Also let* $a\in\mathbb{C}$ *then for* $z>0$.

$$\left(I_{0+}^{\zeta,\mu'',\delta,\nu'',\xi}G_{\alpha,\beta,s}[a,z]\,H_\varsigma\right)(z)=\frac{z^{-\zeta-\mu''+\xi+s\alpha-\beta+\varsigma}}{2^{\varsigma+1}\Gamma(s)}$$

$$\times\,{}_4\psi_6\left[\begin{matrix}(s,1), & \begin{pmatrix}s\alpha-\beta\\+\varsigma\\+1,\alpha+2\end{pmatrix}, & \begin{pmatrix}-\mu''+\nu''\\+s\alpha\\-\beta+\\\varsigma+1,\\\alpha+2\end{pmatrix}, & \begin{pmatrix}-\mu-\mu''-\delta+\\\xi+s\alpha\\-\beta+\\\varsigma+1,\\\alpha+2\end{pmatrix}\\ \begin{pmatrix}s\alpha-\\\beta,\alpha\end{pmatrix},\left(\frac{3}{2},1\right), & \left(\varsigma+\frac{3}{2},1\right),\begin{pmatrix}\nu''+s\alpha\\-\beta\\+\varsigma+1,\\\alpha+2\end{pmatrix}, & \begin{pmatrix}-\mu-\mu''+\xi\\+s\alpha\\-\beta+\\\varsigma-1,\\\alpha+2\end{pmatrix}, & \begin{pmatrix}-\mu''-\delta+\\\xi\\+s\alpha-\beta+\\\varsigma+1,\\\alpha+2\end{pmatrix}\end{matrix};\left(\frac{-az^{\alpha+2}}{4}\right)\right] \quad 3.1$$

Proof: On using Eq.6 we get

$$\left(I_{0+}^{\zeta,\mu'',\delta,\nu'',\xi}G_{\alpha,\beta,s}[a,z]\,H_\varsigma\right)(z)=\sum_{t=0}^{\infty}\frac{(s)_t(a)^t}{\Gamma(t\alpha+s\alpha-\beta)t!}\times\frac{(-1)^t}{\Gamma\left(t+\frac{3}{2}\right)\Gamma\left(\varsigma+t+\frac{3}{2}\right)}\times\frac{1}{2^{\varsigma+2t+1}}\times\left(I_{0+}^{(\zeta,\mu'',\delta,\nu'',\xi)}\left(z^{t\alpha+s\alpha-\beta-1+\varsigma+2t+1}\right)\right)$$

$$=\frac{z^{-\zeta-\mu''+\xi+s\alpha-\beta+\varsigma}}{2^{\varsigma+1}\Gamma(s)}\sum_{t=0}^{\infty}\frac{\Gamma(t+s)\Gamma(t\alpha+s\alpha-\beta+\varsigma+2t+1)\,\Gamma(-\mu''+\nu''+t\alpha+s\alpha-\beta+\varsigma+2t+1)\,\Gamma(-\zeta-\mu''-\delta+\xi+t\alpha+s\alpha-\beta+\varsigma+2t+1)}{\Gamma(t\alpha+s\alpha-\beta)\Gamma\left(t+\frac{3}{2}\right)\Gamma\left(\varsigma+t+\frac{3}{2}\right)\Gamma(\nu''+t\alpha+s\alpha-\beta+\varsigma+2t+1)\,\Gamma(-\zeta-\mu''+\xi+t\alpha+s\alpha-\beta+\varsigma+2t+1)\,\Gamma(-\mu''-\delta+\xi+t\alpha+s\alpha-\beta+\varsigma+2t+1)};\left(\frac{-az^{\alpha+2}}{4}\right)^t\times\frac{1}{t!}$$

Theorem 3.2: *Let* $\zeta,\mu'',\delta,\nu'',\xi\in\mathbb{C}$ *and* $Re(\alpha s-\beta)>0$ *be such that* $Re(\xi)>0$. *Also let* $a\in\mathbb{C}$ *then for* $z>0$.

$$\left(I_{0-}^{\zeta,\mu'',\delta,\nu'',\xi}G_{\alpha,\beta,s}[a,z]\,H_\varsigma\right)(z)=\frac{z^{-\zeta-\mu''+\xi+s\alpha-\beta+\varsigma}}{2^{\varsigma+1}\Gamma(s)}$$

$${}_4\psi_6\left[\begin{matrix}(s,1), & \begin{pmatrix}-\delta-s\alpha\\+\beta\\-\varsigma,-(\alpha+2)\end{pmatrix}, & \begin{pmatrix}\zeta+\mu''\\-\xi\\-s\alpha+\beta-\varsigma,\\-(\alpha+2)\end{pmatrix}, & \begin{pmatrix}\zeta+\nu''-\\\xi-s\alpha\\+\beta-\varsigma,\\-(\alpha+2)\end{pmatrix}\\ \begin{pmatrix}s\alpha-\\\beta,\alpha\end{pmatrix},\left(\frac{3}{2},1\right), & \left(\varsigma+\frac{3}{2},1\right),\begin{pmatrix}-s\alpha+\\\beta-\varsigma\\,-(\alpha+2)\end{pmatrix}, & \begin{pmatrix}\zeta-\delta-\\s\alpha+\beta-\\\varsigma,\\-(\alpha+2)\end{pmatrix}, & \begin{pmatrix}\zeta+\mu''+\\\delta-\xi\\-s\alpha+\beta\\-\varsigma,\\-(\alpha+2)\end{pmatrix}\end{matrix};\left(\frac{-az^{\alpha+2}}{4}\right)\right]$$

Proof: On using Eq.7, we get the required result and the proof is similar to proof of theorem 1.

Theorem 3.3: *Let* $\zeta, \mu'', \delta, \nu'', \xi \in \mathbb{C}$ *and* $Re(\alpha s - \beta) > 0$ *be such that* $Re(\xi) > 0$. *Also let* $a \in \mathbb{C}$ *then for* $z > 0$

$$(D_{0+}^{\zeta,\mu'',\delta,\nu'',\xi} G_{\alpha,\beta,s}[a,z]\, H_{\varsigma})(z) = \frac{z^{\zeta+\mu''-\xi+s\alpha-\beta+\varsigma}}{2^{\varsigma+1}\Gamma(s)}$$

$$\times {}_4\psi_6\left[\begin{matrix} (s,1), & \begin{pmatrix} s\alpha - \\ \beta+\varsigma \\ +1, \\ \alpha+2 \end{pmatrix}, \begin{pmatrix} -\delta+\mu \\ +s\alpha \\ -\beta+\varsigma+1, \\ \alpha+2 \end{pmatrix}, & \begin{pmatrix} \zeta+\mu''+\nu'' \\ -\xi+s\alpha \\ -\beta+ \\ \varsigma+1, \\ \alpha+2 \end{pmatrix} \\ \begin{pmatrix} s\alpha - \\ \beta, \alpha \end{pmatrix}, \left(\frac{3}{2},1\right), & \left(\varsigma+\frac{3}{2},1\right), \begin{pmatrix} -\delta+ \\ s\alpha-\beta \\ +\varsigma+1, \\ \alpha+2 \end{pmatrix}, \begin{pmatrix} \zeta+\mu'' \\ -\xi+s\alpha \\ -\beta+ \\ \varsigma-1, \\ \alpha+2 \end{pmatrix}, & \begin{pmatrix} \zeta+\nu'' \\ -\xi \\ +s\alpha-\beta+ \\ \varsigma+1, \\ \alpha+2 \end{pmatrix} \end{matrix} ;\ \left(\frac{-az^{\alpha+2}}{4}\right)\right]$$

Proof: On using Eq.5 and Eq.7 we get

$$(D_{0+}^{\zeta,\mu'',\delta,\nu'',\xi} G_{\alpha,\beta,s}[a,z]\, H_{\varsigma})(z) = \sum_{t=0}^{\infty} \frac{(s)_t (a)^t}{\Gamma(t\alpha+s\alpha-\beta)t!} \times \frac{(-1)^t}{\Gamma\left(t+\frac{3}{2}\right)\Gamma\left(\varsigma+t+\frac{3}{2}\right)} \times \frac{1}{2^{\varsigma+2t+1}} \times \left(D_{0+}^{(\zeta,\mu'',\delta,\nu'',\xi)}\left(z^{t\alpha+s\alpha-\beta-1+\varsigma+2t+1}\right)\right)$$

$$= \frac{z^{\zeta+\mu''-\xi+s\alpha-\beta+\varsigma}}{2^{\varsigma+1}\Gamma(s)} \sum_{t=0}^{\infty} \frac{\begin{matrix}\Gamma(t+s)\Gamma(t\alpha+s\alpha-\beta+\varsigma+2t+1) \\ \Gamma(-\delta+\zeta+t\alpha+s\alpha-\beta+\varsigma+2t+1) \\ \Gamma(\zeta+\mu''+\nu''-\xi+t\alpha+s\alpha-\beta+\varsigma+2t+1)\end{matrix}}{\begin{matrix}\Gamma(t\alpha+s\alpha-\beta)\Gamma\left(t+\frac{3}{2}\right)\Gamma\left(\varsigma+t+\frac{3}{2}\right) \\ \Gamma(-\delta+t\alpha+s\alpha-\beta+\varsigma+2t+1) \\ \Gamma(\zeta+\mu''-\xi+t\alpha+s\alpha-\beta-\varsigma+2t+1) \\ \Gamma(\zeta+\nu''-\xi+t\alpha+s\alpha-\beta+\varsigma+2t+1)\end{matrix}} ; \left(\frac{-az^{\alpha+2}}{4}\right)^t \times \frac{1}{t!}$$

Theorem 3.4: *Let* $\zeta, \mu'', \delta, \nu'', \xi \in \mathbb{C}$ *and* $Re(\alpha s - \beta) > 0$ *be such that* $Re(\xi) > 0$. *Also let* $a \in \mathbb{C}$ *then for* $z > 0$

$$(D_{0-}^{\zeta,\mu'',\delta,\nu'',\xi} G_{\alpha,\beta,s}[a,z]\, H_{\varsigma})(z) = \frac{z^{\zeta+\mu''-\xi+s\alpha-\beta+\varsigma}}{2^{\varsigma+1}\Gamma(s)}$$

$$\times {}_4\psi_6\left[\begin{matrix} (s,1), & \begin{pmatrix} \nu'' - \\ s\alpha+\beta \\ -\varsigma, \\ -(\alpha+2) \end{pmatrix}, \begin{pmatrix} -\zeta-\mu'' \\ +\xi \\ -s\alpha+ \\ \beta-\varsigma, \\ -(\alpha+2) \end{pmatrix}, & \begin{pmatrix} -\mu''+\delta \\ +\xi-s\alpha \\ +\beta-\varsigma, \\ -(\alpha+2) \end{pmatrix} \\ \begin{pmatrix} s\alpha - \\ \beta, \alpha \end{pmatrix}, \left(\frac{3}{2},1\right), & \left(\varsigma+\frac{3}{2},1\right), \begin{pmatrix} -s\alpha+ \\ \beta-\varsigma \\ ,-(\alpha+2) \end{pmatrix}, \begin{pmatrix} -\mu''+\nu'' \\ -s\alpha+\beta- \\ \varsigma, \\ -(\alpha+2) \end{pmatrix}, & \begin{pmatrix} -\zeta-\mu'' \\ -\delta+\xi \\ -s\alpha+\beta \\ -\varsigma, \\ -(\alpha+2) \end{pmatrix} \end{matrix} ;\ \left(\frac{-az^{\alpha+2}}{4}\right)\right]$$

Proof: By applying Eq.5, we obtain the desired outcome, which resembles the proof of theorem 3.

3.1. *Special Case:*

If we take $\varsigma = -1, \alpha + 2 = \alpha, -a = 4a$ and $t = 0$ in equation 3.1. we get the required result reduces to the known result of [11] theorem 3.1.

$$= \frac{z^{-\zeta-\mu''+\xi+s\alpha-\beta-1}}{\Gamma(s)} \times {}_3\psi_3 \left[\begin{matrix} (s,1), & ,\begin{pmatrix} -\mu''+\nu'' \\ +s\alpha \\ -\beta+ \\ \varsigma+1, \\ \alpha+2 \end{pmatrix}, & \begin{pmatrix} -\zeta-\mu''-\delta+ \\ \xi+s\alpha \\ -\beta+ \\ \varsigma+1, \\ \alpha+2 \end{pmatrix} \\ \begin{pmatrix} \nu''+s\alpha \\ -\beta \\ +\varsigma+1, \\ \alpha+2 \end{pmatrix} & \begin{pmatrix} -\zeta-\mu''+\xi \\ +s\alpha \\ -\beta+ \\ \varsigma-1, \\ \alpha+2 \end{pmatrix}, & \begin{pmatrix} -\mu''-\delta+ \\ \xi \\ +s\alpha-\beta+ \\ \varsigma+1, \\ \alpha+2 \end{pmatrix} \end{matrix} ; (az^{\alpha}) \right]`$$

And rest all the theorems of this paper can be reduced to the known result of [35] using the above same conditions.

4. Conclusion

The results provided here are generic, and they can be used to generate a variety of integral formulas involving the MSM fractional integral (F.I) and differential operators. Fractional operators' nonlocal feature and memory effect are spawning a slew of new and crucial applications in a variety of sectors of science and engineering. The fractional operator's theory has long been acknowledged as a useful tool for modelling complex issues.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations. Elsevier, North Holland (2006).

[2] E. A. Mansour, E. A. Kuffi, and S. A. Mehdi, "Complex SEE integral transform in solving Abel's integral equation," *Journal of Interdisciplinary Mathematics,* vol. 25, no. 5, pp. 1307–1314 (2022).

[3] S. A. Mehdi, E. A. Kuffi, and J. A. Jasim, "Solving ordinary differential equations using a new general complex integral transform," *Journal of Interdisciplinary Mathematics,* vol. 25, no. 6, pp. 1919–1932 (2022).

[4] M. Saigo and N. Maeda, "More generalization of fractional calculus," in Transform Methods and Special Functions, Varna, Bulgaria, pp. 386–400 (1996).

[5] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, Integrals and Series: More Special Functions, vol. 3, Gordon and Breach, New York (1990).

[6] R. M. Aarts and A. J. Janssen, "Approximation of the Struve function 1H 1 occurring in impedance calculations," *The Journal of the Acoustical Society of America*, vol. 113, no. 5, pp. 2635–2637 (2003).

[7] R. M. Ali, S. R. Mondal, and K. S. Nisar, "Monotonicity properties of the generalized Struve functions," *Journal of the Korean Mathematical Society*, vol. 54, no. 2, pp. 575–598 (2017).

[8] M. Saigo, "A remark on integral operators involving the Gauss hypergeometric functions," *Math. Rep. Kyushu Univ.,* vol. 11, no. 2, pp. 135–143 (1978).

[9] K. K. Kataria and P. Vellaisamy, "Some fractional calculus results associated with the I-function," *Matematicae* (Catania), vol. 70, no. 2, pp. 173–190 (2015).

[10] R. M. Aarts and A. J. Janssen, "Approximation of the Struve function H1 occurring in impedance calculations," *The Journal of the Acoustical Society of America*, vol. 113, no. 5, pp. 2635–2637 (2003).

[11] S. Kabra, H. Nagar, K. S. Nisar, and V. K. Vyas, "Marichev-Saigo-Maeda fractional operators on generalized function [G]_ρ,η,γ [a,z]," *Journal of Mathematics in Engineering, Science and Aerospace,* vol. 11, no. 2, pp. 371–379 (2020).

Received July, 2024