

A new generalized common fixed point theorem

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Abstract

In this article, we extend a recent result of Chandra et al. [1] for two maps having a novel generalized contractive condition of the Suzuki type on a complete metric space. We render an example to demonstrate that the results presented are generalizations of similar conclusions in the available literature.

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1. Introduction

The use of fixed point theory in a wide range of fields, including physics, electronics, computer science, social sciences, topology, economic equilibrium, biological populations, traffic flow, control systems, fractal geometry, social network dynamics, population genetics, and electric circuits, has brought it extremely prevalent in recent years. The fundamental Banach contraction principle (BCP) gives the guarantees of the existence as well as the uniqueness of the solution to the problem. Various authors proved fixed point theorems and developed iterations to find the solution to the problem [2, 8] and provided various applications.

2. Preliminaries and Definitions

To keep things brief, we adhere to the following function throughout the paper.

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$$\text{Define } \mu: [0, 1) \rightarrow \left(\frac{1}{2}, 1\right] \text{ as } \mu(a) = \begin{cases} 1 & \text{if } 0 \leq a \leq \frac{1}{2}(\sqrt{5} - 1), \\ \frac{1-a}{a^2} & \text{if } \frac{1}{2}(\sqrt{5} - 1) \leq a \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+a} & \text{if } \frac{1}{\sqrt{2}} \leq a < 1. \end{cases}$$

2.1 Symbols utilized

$\mathcal{L}$ - Subset of $\mathcal{R}$ which is non-empty; $(\mathcal{L}, \omega)$ - metric space; $\exists$ - there exists; $\in$ - belongs to.

2.2 Abbreviations and Acronyms

FP - Fixed Point; CFP - Common Fixed Point; CMS - Complete Metric Space.

The innovative and powerful extension of the classical BCP is the Suzuki contraction theorem [11]. This theorem's inherent simplicity and wide applicability have made it an invaluable tool in nonlinear analysis.

Theorem 2.1: [11] *Let $\mathcal{O}: \mathcal{L} \rightarrow \mathcal{L}$ with $\mathcal{L}$ a CMS. Assume $\exists a \in [0, 1)$ in such a way that for all $c, e \in \mathcal{L}$, $\mu(a) \omega(c, \mathcal{O}c) \leq \omega(c, e)$ implies $\omega(\mathcal{O}c, \mathcal{O}e) \leq a \omega(c, e)$. Then $\mathcal{O}$ has a unique FP "d". Moreover, $\lim_{n \rightarrow \infty} \mathcal{O}^n r = d$ for any r in $\mathcal{L}$.*

Mot and Petrusel [9] created a new contraction type called (τ_1, τ_2, τ_3) Mot-Petrusel-Suzuki contraction, which broadly extended the work of Suzuki [11].

Theorem 2.2: *Let $\mathcal{O}: \mathcal{L} \rightarrow \mathcal{L}$ with $\mathcal{L}$ a CMS. Suppose that $\exists \tau_1, \tau_2, \tau_3 \in [0, 1)$ with $\tau_1 + \tau_2 + \tau_3 < 1$ in such a way that for every $c, e \in \mathcal{L}$,*

$$\frac{1-\tau_2-\tau_3}{1+\tau_1} \omega(c, \mathcal{O}c) \leq \omega(c, e) \text{ implies}$$

$$\omega(\mathcal{O}c, \mathcal{O}e) \leq \tau_1 \omega(c, e) + \tau_2 \omega(c, \mathcal{O}c) + \tau_3 \omega(e, \mathcal{O}e).$$

Then $\mathcal{O}$ admit a unique FP.

For a novel type of contraction known as generalized Kikkawa-Suzuki-Kannan-type contraction, Kikkawa and Suzuki [6] gave the subsequent result.

Theorem 2.3: *Let $\mathcal{O}: \mathcal{L} \rightarrow \mathcal{L}$ with $\mathcal{L}$ a CMS. Assume $\exists a \in [0, 1)$ in such a way that for all $c, e \in \mathcal{L}$,*

$$\mu(a) \omega(c, \mathcal{O}c) \leq \omega(c, e) \text{ implies } \omega(\mathcal{O}c, \mathcal{O}e) \leq a \max \{\omega(c, \mathcal{O}c), \omega(e, \mathcal{O}e)\}.$$

Then $\mathcal{O}$ has a unique FP.

Chandra et al. [1] established a new FP result by using a novel type of mapping condition, as shown in the following result.

Theorem 2.4: *Let $\mathcal{O}: \mathcal{L} \rightarrow \mathcal{L}$ with $\mathcal{L}$ a CMS. Suppose that $\exists \tau_1, \tau_2, \tau_3 \in [0, 1)$ with $\tau_1 + \tau_2 + \tau_3 = a < 1$ in such a way that for every $c, e \in \mathcal{L}$,*

$$\mu(a) \min \{\omega(c, \mathcal{O}c), \omega(c, \mathcal{O}e)\} \leq \omega(c, e) \text{ implies}$$

$\omega(\mathcal{O}c, \mathcal{O}e) \leq \tau_1 \omega(c, e) + \tau_2 \max \{ \omega(c, \mathcal{O}c), \omega(e, \mathcal{O}e) \} + \tau_3 [\omega(c, \mathcal{O}e) + \omega(e, \mathcal{O}c)]$. Then $\mathcal{O}$ has a unique FP.

Many researchers worked along these lines (see, for instance, [1], [3-7], [9], [11]). For a complete comparison of various contractive maps, one may refer to [10]. Consider a new contractive condition for a pair of maps which we use in our main result

$$\omega(\mathcal{J}c, \mathcal{O}e) \leq \tau_1 \omega(c, e) + \tau_2 \max \{ \omega(c, \mathcal{J}c), \omega(e, \mathcal{O}e) \} + \tau_3 [\omega(c, \mathcal{O}e) + \omega(e, \mathcal{J}c)]. \quad (2.1)$$

In this work, we derive a CFP theorem (cf. Theorem 3.1) generalizing Theorem 2.3 and Theorem 2.4. As a consequence, we obtain a generalization of Theorems 2.1 and Theorem 2.2.

3. Main Results

The primary CFP theorem in this part is as follows.

Theorem 3.1: Let $\mathcal{L}$ be a CMS and $\mathcal{J}, \mathcal{O}: \mathcal{L} \rightarrow \mathcal{L}$. Suppose that $\exists \tau_1, \tau_2, \tau_3 \in [0, 1)$ with $\tau_1 + \tau_2 + 2\tau_3 < 1$ in such a way that for all $c, e \in \mathcal{L}$,

$$\left(\frac{1-\tau_2-\tau_3}{1+\tau_1} \right) \min \{ \omega(c, \mathcal{J}c), \omega(e, \mathcal{O}e), \omega(e, \mathcal{J}c), \omega(c, \mathcal{O}e) \} \leq \omega(c, e) \text{ implies}$$

$$\omega(\mathcal{J}c, \mathcal{O}e) \leq \tau_1 \omega(c, e) + \tau_2 \max \{ \omega(c, \mathcal{J}c), \omega(e, \mathcal{O}e) \} + \tau_3 [\omega(c, \mathcal{O}e) + \omega(e, \mathcal{J}c)].$$

Then $\mathcal{J}$ and $\mathcal{O}$ have unique CFP.

Proof: Let $b_0 \in \mathcal{L}$. Define $b_{2n+1} = \mathcal{J}(b_{2n})$, $b_{2n+2} = \mathcal{O}(b_{2n+1})$, $n = 0, 1, 2, \dots$

Now if $\omega(b_{2n}, b_{2n+1}) \leq \omega(b_{2n+1}, b_{2n+2})$, for each $n = 0, 1, 2, \dots$

$$\begin{aligned} \text{Then } & \left(\frac{1-\tau_2-\tau_3}{1+\tau_1} \right) \min \left\{ \omega(b_{2n}, \mathcal{J}b_{2n}), \omega(b_{2n+1}, \mathcal{O}b_{2n+1}), \right. \\ & \left. \omega(b_{2n+1}, \mathcal{J}b_{2n}), \omega(b_{2n}, \mathcal{O}b_{2n+1}) \right\} \\ & = \left(\frac{1-\tau_2-\tau_3}{1+\tau_1} \right) \min \left\{ \omega(b_{2n}, b_{2n+1}), \omega(b_{2n+1}, b_{2n+2}), \right. \\ & \quad \left. \omega(b_{2n}, b_{2n+2}) \right\} \leq \omega(b_{2n}, b_{2n+1}). \end{aligned}$$

So by given assumption,

$$\begin{aligned} \omega(\mathcal{J}b_{2n}, \mathcal{O}b_{2n+1}) & \leq \tau_1 \omega(b_{2n}, b_{2n+1}) + \tau_2 \max \left\{ \omega(b_{2n}, \mathcal{J}b_{2n}), \right. \\ & \quad \left. \omega(b_{2n+1}, \mathcal{O}b_{2n+1}) \right\} \\ & \quad + \tau_3 [\omega(b_{2n}, \mathcal{O}b_{2n+1}) + \omega(b_{2n+1}, \mathcal{J}b_{2n})]. \end{aligned}$$

$$\begin{aligned} \text{Thus, } \omega(b_{2n+1}, b_{2n+2}) & \leq \tau_1 \omega(b_{2n}, b_{2n+1}) + \tau_2 \omega(b_{2n+1}, b_{2n+2}) \\ & \quad + \tau_3 [\omega(b_{2n}, b_{2n+1}) + \omega(b_{2n+1}, b_{2n+2})]. \end{aligned}$$

$$\text{Therefore, } \omega(b_{2n+1}, b_{2n+2}) \leq \left(\frac{\tau_1 + \tau_3}{(1-\tau_2-\tau_3)} \right) \omega(b_{2n}, b_{2n+1}).$$

$$\text{Analogously, } \omega(b_{2n+2}, b_{2n+3}) \leq \left(\frac{\tau_1 + \tau_2 + \tau_3}{(1-\tau_3)} \right) \omega(b_{2n+1}, b_{2n+2}).$$

$$\text{Let } r = \left(\frac{\tau_1 + \tau_3}{(1-\tau_2-\tau_3)} \right) \text{ and } t = \left(\frac{\tau_1 + \tau_2 + \tau_3}{(1-\tau_3)} \right).$$

$$\text{So } \omega(b_{2n+1}, b_{2n+2}) \leq r \omega(b_{2n}, b_{2n+1}), \quad (3.1)$$

$$\text{And } \omega(b_{2n+2}, b_{2n+3}) \leq t \omega(b_{2n+1}, b_{2n+2}). \quad (3.2)$$

By (3.1), (3.2) and induction, we have for each $n = 0, 1, 2, \dots$

$$\begin{aligned}\omega(b_{2n+1}, b_{2n+2}) &\leq r(rt)^n \omega(b_0, b_1) \text{ and} \\ \omega(b_{2n+2}, b_{2n+3}) &\leq (rt)^{n+1} \omega(b_0, b_1).\end{aligned}$$

Since $r, t < 1$ and

$$\sum_{n=0}^{\infty} \omega(b_n, b_{n+1}) \leq (1+r) \sum_{n=0}^{\infty} (rt)^n \omega(b_0, b_1). \quad (3.3)$$

If $\omega(b_{2n+1}, b_{2n+2}) \leq \omega(b_{2n}, b_{2n+1})$,

Then

$$\begin{aligned}\left(\frac{1-\tau_2-\tau_3}{1+\tau_1}\right) \min\left\{\omega(b_{2n}, \mathcal{J}b_{2n}), \omega(b_{2n+1}, \mathcal{O}b_{2n+1}), \right. \\ \left. \omega(b_{2n+1}, \mathcal{J}b_{2n}), \omega(b_{2n}, \mathcal{O}b_{2n+1})\right\} \leq \left(\frac{1-\tau_2-\tau_3}{1+\tau_1}\right) \omega(b_{2n}, b_{2n+1}) \\ \leq \omega(b_{2n}, b_{2n+1}).\end{aligned}$$

Then by the same procedure as above, we get (3.3). This implies $\{b_n\}$ is Cauchy. By completeness of $(\mathcal{L}, \omega)$, $\{b_n\}$ converges to some point c in $\mathcal{L}$.

We shall now prove c is FP of $\mathcal{J}$ and $\mathcal{O}$. As $b_n \rightarrow c$, so $\exists n_0 \in N$ such that

$$\omega(c, b_n) \leq \frac{1}{3} \omega(c, e) \text{ for } e \neq c \text{ and all } n \geq n_0. \text{ Then we have}$$

$$\begin{aligned}\left(\frac{1-\tau_2-\tau_3}{1+\tau_1}\right) \omega(b_{2n}, \mathcal{J}b_{2n}) \leq \omega(b_{2n}, \mathcal{J}b_{2n}) \leq \omega(c, b_{2n}) + \omega(b_{2n+1}, c) \\ \leq \omega(c, e) - \omega(b_{2n}, c) \leq \omega(b_{2n}, e).\end{aligned} \quad (3.4)$$

$$\begin{aligned}\text{Since } \left(\frac{1-\tau_2-\tau_3}{1+\tau_1}\right) \min\{\omega(b_{2n}, \mathcal{J}b_{2n}), \omega(e, \mathcal{O}e), \omega(b_{2n}, \mathcal{O}e), \omega(e, \mathcal{J}b_{2n})\} \\ \leq \omega(b_{2n}, \mathcal{J}b_{2n}) \leq \omega(b_{2n}, e).\end{aligned} \quad (3.5)$$

So by (3.5) and the given assumption,

$$\begin{aligned}\omega(\mathcal{J}b_{2n}, \mathcal{O}e) &\leq \tau_1 \omega(b_{2n}, e) + \tau_2 \max\{\omega(b_{2n}, \mathcal{J}b_{2n}), \omega(e, \mathcal{O}e)\} \\ &\quad + \tau_3 [\omega(b_{2n}, \mathcal{O}e) + \omega(e, \mathcal{J}b_{2n})] \\ &= \tau_1 \omega(b_{2n}, e) + \tau_2 \max\{\omega(b_{2n}, b_{2n+1}), \omega(e, \mathcal{O}e)\} + \tau_3 [\omega(b_{2n}, \mathcal{O}e) + \omega(e, \\ &\quad b_{2n+1})].\end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$(1 - \tau_3) \omega(c, \mathcal{O}e) \leq (\tau_1 + \tau_3) \omega(c, e) + \tau_2 \omega(e, \mathcal{O}e).$$

Now

$$\omega(e, \mathcal{O}e) \leq \omega(c, e) + \omega(c, \mathcal{O}e) \leq \omega(c, e) + \frac{(\tau_1 + \tau_3)}{(1 - \tau_3)} \omega(c, e) + \frac{\tau_2}{(1 - \tau_3)} \omega(e, \mathcal{O}e).$$

$$\text{Thus, } \frac{(1 - \tau_3 - \tau_2)}{(1 - \tau_3)} \omega(e, \mathcal{O}e) \leq \frac{(1 + \tau_1)}{(1 - \tau_3)} \omega(c, e),$$

$$\text{that is, } \frac{(1 - \tau_2 - \tau_3)}{(1 + \tau_1)} \omega(e, \mathcal{O}e) \leq \omega(c, e).$$

$$\text{Now } \frac{(1 - \tau_2 - \tau_3)}{(1 + \tau_1)} \min\{\omega(c, \mathcal{J}c), \omega(e, \mathcal{O}e), \omega(e, \mathcal{J}c), \omega(c, \mathcal{O}e)\}$$

$$\leq \frac{(1 - \alpha_2 - \alpha_3)}{(1 + \alpha_1)} \omega(e, \mathcal{O}e) \leq \omega(c, e)$$

implies

$$\omega(\mathcal{J}c, \mathcal{O}e) \leq \tau_1 \omega(c, e) + \tau_2 \max\{\omega(c, \mathcal{J}c), \omega(e, \mathcal{O}e)\} + \tau_3 [\omega(c, \mathcal{O}e) + \omega(e, \mathcal{J}c)].$$

Taking $e = b_{2n+1}$ and letting $n \rightarrow \infty$, we have

$$\omega(c, \mathcal{J}c) \leq (\tau_2 + \tau_3) \omega(c, \mathcal{J}c).$$

So $\omega(c, \mathcal{J}c) = 0$ implies $c = \mathcal{J}c$.

Analogously, we can show $c = \mathcal{O}c$.

To prove the uniqueness of FP, we suppose e be another FP of $\mathcal{O}$.

$$\text{As } \frac{(1-\tau_2-\tau_3)}{(1+\tau_1)} \min\{\omega(\mathcal{J}c, c), \omega(\mathcal{O}e, e), \omega(\mathcal{J}c, e), \omega(\mathcal{O}c, c)\} = 0 \leq \omega(c, e)$$

so by the given assumption,

$$\omega(c, e) \leq \tau_1 \omega(c, e) + \tau_2 \max\{\omega(e, \mathcal{O}e), \omega(c, \mathcal{J}c)\} + \tau_3 [\omega(e, \mathcal{J}c) + \omega(c, \mathcal{O}e)],$$

that is, $\omega(c, e) \leq (\tau_1 + 2\tau_3) \omega(c, e)$.

So $\omega(c, e) = 0$ implies, $c = e$.

Thus $\mathcal{J}$ and $\mathcal{O}$ have unique CFP.

Theorems 2.1 and 2.2 are generalized by the subsequent result.

Corollary 3.1: Let $\mathcal{O}: \mathcal{L} \rightarrow \mathcal{L}$ with $\mathcal{L}$ a CMS. Suppose that $\exists \tau_1, \tau_2, \tau_3 \in [0, 1)$ with $\tau_1 + \tau_2 + 2\tau_3 < 1$ such a way that for all $c, e \in \mathcal{L}$

$$\left(\frac{1-\alpha_2-\alpha_3}{1+\alpha_1} \right) \min\{\omega(c, \mathcal{O}c), \omega(e, \mathcal{O}c)\} \leq \omega(c, e)$$

implies

$$\omega(\mathcal{O}c, \mathcal{O}e) \leq \tau_1 \omega(c, e) + \tau_2 \max\{\omega(c, \mathcal{O}c), \omega(e, \mathcal{O}e)\} + \tau_3 [\omega(c, \mathcal{O}e) + \omega(e, \mathcal{O}c)].$$

Then $\mathcal{O}$ has a unique FP.

Proof: It take place from Theorem 3.1 when $\mathcal{O} = \mathcal{J}$.

Now we give an example which satisfied Theorem 3.1 but does not satisfied condition 2.1.

Example 3.1: Consider $\mathcal{L} = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\}$ with metric

$$\omega[(c_p, c_q), (e_p, e_q)] = |c_p - e_p| + |c_q - e_q|.$$

$$\mathcal{O}(c_p, c_q) = \begin{cases} (c_p, 0) & \text{if } c_q \geq c_p \\ (0, c_q) & \text{if } c_q < c_p \end{cases} \text{ and } \mathcal{J}(c_p, c_q) = \begin{cases} (0, c_p) & \text{if } c_q \geq c_p \\ (0, c_q) & \text{if } c_q < c_p \end{cases}.$$

Then $\mathcal{O}$ and $\mathcal{J}$ do not satisfy the condition (2.1) at $c = (4, 5)$, $e = (5, 4)$.

Nonetheless, it is simple to verify that the maps $\mathcal{O}$ and $\mathcal{J}$ fulfill all of Theorem 3.1's hypotheses.

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