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The Hamilton-Jacobi treatment of complex fields as constrained conformable fractional systems

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Abstract

We examine the fractional complex scalar field within a limited system and apply the Hamilton-Jacobi method to it. As a result, we obtain the simplified stage space Hamiltonian density excluding needing additional gauge fixing constraints or conformable fractional Lagrange multipliers. We also explore the quantization of this system. Alternatively, we can derive the path integral for these systems by integrating over the canonical phase space coordinates (φ,A), which eliminates the need for any gauge fixing constraints. On another note, our motivation to investigate the parameterization of conformable fractional time stems from the drawbacks of the conventional Hamiltonian approach in classical fields. Although

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the Lagrangian formulation maintains relativistic covariance, the unique role of time prevents the construction of a Hamiltonian formulation that treats x, y, z, and time symmetrically.

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1. Introduction

Fractional derivatives, which generalize classical calculus, have observed successful uses in various fields of science and engineering. Physicists have employed this tool to solve problems beyond classical means, establishing fractional calculus as a potent method for modeling intricate physical systems. However, traditional energy-based approaches fail to describe the behavior of non-conservative systems with internal damping. Throughout the past twenty years, fractional calculus has garnered significant focus, finding use in diverse fields such as biochemistry, physics, viscoelasticity, fluid mechanics, computer modeling, and simulating a fractional order mathematical model on COVID-19 [1–10]. Baleanu et al. [11] introduced fractional mechanics, applications in fractional calculus and mathematical analysis, and integral inequalities, presenting general Raina fractional integral inequalities and an integral identity for partially differentiable functions using an (l_1,l_1)-(l_2,l_2) convex function on coordinates [11]. In a recent research paper, Mohamed et al. [12] developed a novel family of integral operators for conformable fractional calculus, a refinement of classical calculus that allows for non-integer order differentiation and integration. The authors propose these new integral operators to circumvent some of the limitations of the existing operators in the field. The work offers a comprehensive mathematical analysis of the linearity, continuity, and boundedness of these unique operators. The authors also demonstrate how they can be applied to the solution of particular types of fractional differential equations, which are commonly employed to explain physical processes such as diffusion and wave propagation. In a similar vein, Khalil, Horani, Yousef, and Sababheh proposed the conformable fractional derivative and integral, which strictly adhere to the Leibniz rule and the chain rule, accommodating features lacking a first derivative and enabling conformable fractional derivatives of all orders below one [13]. This novel derivative presents a significant advancement, contrasting with current derivatives including the Riemann-Liouville and Caputo derivatives, which fail to adhere to the Leibniz and chain rules. This limitation restricts their effective application in physical systems with high-order derivatives. Notably, the results of fractional calculus reduce to those of traditional calculus when replacing the fractional order derivative with an integral order, an approach extended to classical fields with fractional derivatives [14]. Recent investigations into the Hamilton-Jacobi differential equation with partial derivatives and the WKB Model have utilized the canonical technique for systems with fractional derivatives [15]. This approach yields a generic framework for solving the HJPDE, resulting in Hamilton-Jacobi functions and equations akin to those found in classical field theory. The main features of this manuscript's contributions can be summarized as follows:

- The generalized Hamilton-Jacobi formulation can be obtained by rewriting the Hamilton-Jacobi Treatment of Complex Fields with Fractional Derivatives. For the first time, the complex fields and the Hamilton-Jacobs equation have been used to derive a fractional derivative Hamilton-Jacobs partial differential.
- The new equations have right- and left-hand derivatives, Developing them into more difficult for practical implementation. Consequently, we present a unique and highly effective method
- By extending our formulations, fractional derivatives can be used to create Hamilton-Jacobi formulations. A method for finding equations for generalized complex fields is proposed.

This study aims to analyze field systems as singular systems using the Hamilton-Jacobi approach with conformable fractions, considering that the reparametrization invariant theories contain vanishing Hamiltonians. We also explain using the canonical path of conformable fractional integral quantization to quantize this system.

This essay is Organized as Observes: A brief overview of the calculus of variations is presented in Section 2. Section 3 presents the fraction-conformable Hamilton-Jacobi approach. Section 4 presents a fractional approach to a complex scalar field as a limited system. Section 5 uses the fractional Hamilton-Jacobi formulation to obtain the evolution parameter "t." Section 6 presents the route integral quantization approach in its fractional form. Section 7 introduces the employment of Hamilton-Jacobi theory in constrained systems. The conclusions are presented in Section 8.

2. Calculus of fractions for variations

Now, let's look at the concept of conformable fractional calculus [16]. In the case where $0 < \alpha \leq 1$, the left conformable fractional derivative (CFD) has the following definition:

$$\mathrm{D}_{s|x}^{\alpha}\mathrm{f}(\mathrm{x}) = \lim_{\epsilon\to\infty} \frac{(x + \epsilon(x-s)^{1-\alpha}) - f(x)}{\epsilon} \tag{2.1}$$

and the right CFD is defined as

$$\mathrm{D}_{x|s'}^{\alpha}\mathrm{f}(\mathrm{x}) = -\lim_{\epsilon\to\infty} \frac{\left(\mathrm{x} + \epsilon\left(\mathrm{s}' - \mathrm{x}\right)^{1-\alpha}\right) - \mathrm{f}(\mathrm{x})}{\epsilon} \tag{2.2}$$

The Characterization of the left conformable fractional integral (CFI) in the scenario where $0 < \alpha \leq 1$ is:

$$\mathrm{I}_{s|x}^{\alpha}\mathrm{f}(\mathrm{x}) = \int_{s}^{x}(\xi - \mathrm{s})^{1-\alpha}\,\mathrm{f}(\xi)\,\mathrm{d}\xi \tag{2.3}$$

the appropriate CFI is described as

$$I^{\alpha}_{x|s'}f(\mathrm{x}) = \int_{\mathrm{x}}^{\mathrm{x}}(s' - \xi)^{1-\alpha}\, \mathrm{f}(\xi)\mathrm{d}\xi \tag{2.4}$$

The CFD and CFI adhere to the subsequent relationships:

$$\mathrm{D}^{\alpha}_{\mathrm{s|x}} I^{\alpha}_{s|x}\mathrm{f(x)} = \mathrm{f(x)} \tag{2.5}$$

$$I^{\alpha}_{x|s'}\, \mathrm{D}^{\alpha}_{\mathrm{x}|s'}\mathrm{f(x)} = \mathrm{f(x)} - \mathrm{f}(a) \tag{2.6}$$

Hereafter, we will focus on the case where s = 0. To express $\mathrm{D}^{\alpha}_{\mathrm{s|x}} = D^{\alpha}_{x}$, we will use the notation $\mathrm{D}^{\alpha}_{\mathrm{s|x}}$. Next, we shall define the conformable fractional derivative (CFD) in this context.

$$\mathrm{D}^{\alpha}_{\mathrm{x}}\mathrm{f(x)} = \lim_{\epsilon\to\infty} \frac{(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon} \tag{2.7}$$

Using Hôpital's rule, the concept of CFD can be restated as follows, given that $0 < \alpha \leq 1$ and D^{α}_{x} represents a right CFD:

$$\mathrm{D}^{\alpha}_{\mathrm{x}}\mathrm{f(x)} = \mathrm{f(x)}x^{1-x} \tag{2.8}$$

The following properties are satisfied by the CFD:

1. **Linearity**

$$\mathrm{D}^{\alpha}_{\mathrm{x}}\big(af(x) + bg(x)\big) = aD^{\alpha}_{x}f(x) + bg(x) \tag{2.9}$$

2. **Leibniz rule**

$$\mathrm{D}^{\alpha}_{\mathrm{x}}\big(f(x)g(x)\big) = [D^{\alpha}_{x}f(x)]g(x) + f(x)D^{\alpha}_{x}g(x) \tag{2.10}$$

3. **Chain rule**

$$\mathrm{D}^{\alpha}_{\mathrm{x}}f\big(g(x)\big) = \big[D^{\alpha}_{g(x)}f\big(g(x)\big)\big][D^{\alpha}_{x}g(x)]g(x)^{\alpha-1} \tag{2.11}$$

3. The conformable fractional Hamilton-Jacobi approach

In this section, we will go over the Hamilton-Jacobi technique for investigating constrained systems briefly. Consider an n-degree-of-freedom system. If it has r primary restrictions, the canonical formulation provides the Hamilton-Jacobi partial differential relations [17].

$$H'_{\alpha}\left(\chi_{\beta}, \phi_{\alpha}, \frac{\partial s}{\partial\phi_{\alpha}}, \frac{\partial s}{\partial\chi_{\beta}}\right) = 0 \;\alpha, \beta = 0, n - r + 1, n. \alpha = 1, \ldots n - r, \tag{3.1}$$

where

$$H'_{\alpha} = H_{\alpha}\big(\chi_{\beta}, \phi_{\alpha}, \pi_{\alpha}\big) + \pi_{\alpha}, \tag{3.2}$$

The canonical Hamiltonian is H_0. The motion equations are derived as a total differential equation in numerous parameters, as shown below:

$$\begin{cases} d\phi_{\alpha} = \frac{\delta H'_{\alpha}}{\delta\pi_{\alpha}} d\chi_{\alpha} = \begin{cases} d\pi_{\alpha} = -\frac{\delta H'_{\alpha}}{\delta\phi_{\alpha}} d\chi_{\alpha}\,. \\ d\pi_{\mu} = -\frac{\delta H'_{\alpha}}{\delta\chi_{\mu}} d\chi_{\mu,}\; \mu = 1, \ldots, r \end{cases} \\ \\ dz = \left(-H_{\alpha} + \pi_{\alpha} + \frac{\delta H'_{\alpha}}{\delta\pi_{\alpha}}\right) d\chi_{\alpha},. \end{cases} \tag{3.3}$$

where $z = S(\chi_\alpha, \phi_\alpha)$ and $\frac{\delta H}{\delta \chi}$ denote oscillations in H with respect to x. If and only if the pair of equations (2.3), and (2.4) can be integrated

$$dH_0' = 0, dH_\mu' = 0, \mu = 1, \dots, r. \tag{3.4}$$

If criteria (3.4) are not satisfied in the same way, they are viewed as additional limits, and the consistency conditions are re-examined. By repeating this process, a group of criteria can be produced. AS a result, the canonical formulation produces the set of canonical phase-space coordinates φ_a and π_a with respect to χ_α, as well as the canonical action integral defined in relation to of the canonical coordinates. The Hamiltonians H_a' may be regarded of as infinitesimal generators of the canonical transformation specified through the parameters χ_α. The path integral representation in this scenario is as follows[18-19]:

$$D(\phi_\alpha', \chi_\alpha', \phi_\alpha, \varphi_\alpha) = \int_{\phi_\alpha}^{\phi_\alpha'} (D\phi^\alpha)(D\pi^\alpha) exp\left[i\left\{\int_{\chi_\alpha}^{\chi_\alpha'}\left(-H_\alpha + \pi_\alpha \frac{\delta H_\alpha'}{\delta \pi_\alpha}\right) d^3x d\chi_\alpha\right\}\right]. \tag{3.5}$$

The integral (3.5) is an integration of the canonical phase space coordinates a W and Q.

4. A fractional approach of a complex scalar field as a limited system

Consider a complex scalar field with a Lagrangian density[20].

$$L = D_\mu^\alpha \phi^+ D_\mu^\alpha \phi - \mu_0^2 \phi^+ \phi - \lambda(\phi^+\phi)^2 \tag{4.1}$$

ϕ^+ and ϕ are functions of x_μ, with μ = 0,1,2,3 and t depending on the independent variable r. The action integral of this system is stated as

$$S[\phi, \phi^+] = \int \; [-D_i^\alpha \phi D_i^\alpha \phi^+ - \mu_0^2 \phi^+ \phi \lambda(\phi\phi^+)^2]\, dxdt \tag{4.2}$$

In this case, the subscript (i) denotes sum over (x,y,z). Assuming that $\mathcal{L}_\tau$ is Lagrangian density and that the time t--------τ is parameterized with $\frac{\partial \tau}{\partial t}$=0, the action integral (4.2) can be expressed as:

$$S[\phi, \phi^+] = \int \; L_\tau \; dxd\tau, \tag{4.3}$$

The singular Lagrangian density $\mathcal{L}_\tau$ can be presented as follows:

$$\mathcal{L}_\tau = \tau(D_\tau^\alpha \phi D_\tau^\alpha \phi^+) - \frac{D_i^\alpha \phi D_i^\alpha \phi^+}{D_t^\alpha \tau} - \frac{\mu_0^2 \phi\phi^+}{D_t^\alpha \tau} - \frac{\lambda(\phi\phi^+)^2}{D_t^\alpha \tau} \tag{4.4}$$

The generalized momenta conjugated to the generalized coordinate has the following definition:

$$\begin{cases} \pi = \frac{\partial L_\tau}{\partial\left(\frac{\partial \phi}{\partial \tau}\right)} = D_t^\alpha \tau \frac{\partial \phi^+}{\partial \tau}, .a \\ \pi^+ = \frac{\partial L_\tau}{\partial\left(\frac{\partial \phi^+}{\partial \tau}\right)} = D_t^\alpha \tau \frac{\partial \phi}{\partial \tau} \; .b \\ \pi_t = \frac{\partial L_\tau}{\partial D_t^\alpha \tau} = (D_t^\alpha \tau)^2 \; (D_\tau^\alpha \phi D_\tau^\alpha \phi^+) - D_i^\alpha \phi D_i^\alpha \phi^+ - \mu_0^2 \phi\phi^+ - \lambda(\phi\phi^+)^2 \; c \end{cases} \tag{4.5}$$

Equation (4.5) can be rewritten by using (4.3) and (4.4).

$$\pi_t = -[\pi\pi^+ - D_i^\alpha \phi D_i^\alpha \phi^+ - \mu_0^2 \phi\phi^+ - \lambda(\phi\phi^+)^2] = -H_t) \tag{4.6}$$

Therefore, the main restriction is

$$H_t' = \pi_t + H_t = 0, \tag{4.7}$$

where

$$H_t = \pi\pi^+ + D_i^\alpha \phi D_i^\alpha \phi^+ + \mu_0^2 \phi\phi^+ + \lambda(\phi\phi^+)^2 \tag{4.8}$$

In addition, the canonical Hamiltonian H has the following definition:

$$H_\tau = \int \left[-L_\tau + \pi \frac{\partial \phi}{\partial \tau} + \pi^+ \frac{\partial \phi^+}{\partial \tau} + \pi_t \frac{\partial t}{\partial \tau}\right] d^3x = 0 \tag{4.9}$$

The computations' outcomes show that H is zero. Consequently, the Hamilton-Jacobi collection of partial differential equations (HJPDE) is now expressed as follows:

$$\begin{cases} H_\tau' = \pi^0 + H_\tau = 0, \pi^0 = \frac{\partial S}{\partial \tau} = 0 \ a \\ H_t' = \pi_t + H_t = 0, \pi_t = \frac{\partial S}{\partial t} = 0 \ b \end{cases} \tag{4.10}$$

The expressions of motion are generated like a complete differential equation in multiple symbols in the following manner:

$$\begin{cases} d\phi = \frac{\delta H_\tau'}{\delta \pi} d\tau + \frac{\delta H_t'}{\delta \pi} dt = \pi^+ dt\,. \\ d\pi = -\frac{\delta H_\tau'}{\delta \phi} d\tau - \frac{\delta H_t'}{\delta \phi^+} dt = -[\mu_0^2 \phi^+ + 2\lambda\phi\phi^+ + D_i^\alpha \phi^+ (D_i^\alpha)^2 \phi] - \phi^+ dt, \\ d\phi^+ = \frac{\delta H_\tau'}{\delta \pi^+} d\tau + \frac{\delta H_t'}{\delta \pi^+} dt = \pi dt,. \\ d\pi^+ = -\frac{\delta H_\tau'}{\delta \phi^+} d\tau - \frac{\delta H_t'}{\delta \phi^+} dt = -\left[\mu_0^2 \phi + 2\lambda\phi\phi^+ + D_i^\alpha \phi D_i^{\alpha\,2} \phi^+\right] dt,. \\ d\pi_t = \frac{\delta H_\tau'}{\delta t} d\tau - \frac{\delta H_t'}{\delta t} dt = 0\,. \end{cases} \tag{4.11}$$

To ascertain whether or not the set of mathematical expressions (4.11) is integrable, single must consider the variance of the limitations. In light of this

$$dH_t' = d\pi_t + dH_t \tag{4.12}$$

Identical disappearance occurs, the equation set is integrable, and the canonical phase-space coordinates ϕ^+, ϕ, π^+, and π may be determined about (t).

5. The evolution parameter "t" is analyzed using fractional Hamilton-Jacobi.

We will examine the model (4.1) in this part by taking the time "t" as an evolution parameter. For this model, the Lagrangian is provided by:

$$L = (D_\tau^\alpha \phi D_\tau^\alpha \phi^+) - (D_i^\alpha \phi D_i^\alpha \phi^+) - \mu_0^2 \phi^+ \phi - \lambda(\phi\phi^+)^2 \tag{5.1}$$

The following formula yields the generalized canonical momenta:

$$\begin{cases} \pi = \frac{\partial L}{\partial (D_\tau^\alpha \phi)} = D_\tau^\alpha \phi^+ \\ \pi^+ = \frac{\partial L}{\partial (D_\tau^\alpha \phi^+)} = D_\tau^\alpha \phi \end{cases} \quad (5.2)$$

Calculating the canonical Hamiltonian is simply

$$H_t = \pi\pi^+ + D_i^\alpha \phi D_i^\alpha \phi^+ + \mu_0^2 \phi\phi^+ + \lambda(\phi\phi^+)^2 \quad (5.3)$$

Hamilton-Jacobi's partial differential equation is

$$\begin{cases} H_t' = \pi_t + H_t = 0 \; a \\ \pi_t = D_t^\alpha S = 0 \; b \end{cases} \quad (5.4)$$

Furthermore, the value derived from equations (5.4) agrees with the motion equation. We can find the canonical Hamiltonian in fractional form for the complex scalar field in fractional form H by using the Hamilton-Jacobi method, which does not require gauge knowledge. Equation(5.4) and equation (5.3) are equivalent, proving this.

6. Fractional form of path integral quantization method

We obtain the route integral quantization for this system employing the fractional path integral formulation. With equation (3.3), one can find the canonical action integral.

$$S[\phi, \phi^+] = \int \left[-H_t - \pi \frac{\delta H_t'}{\delta \pi} - \pi^+ \frac{\delta H_t'}{\delta \pi^+}\right] d^3x dt. \quad (6.1)$$

This yields the route integral of the complex scalar field system as

$$\left\langle \frac{I}{n} \right\rangle = \int \prod D\phi D\phi^+ D\pi D\pi^+ \, exp\left[i\left\{\int_{\chi_\alpha}^{\chi_\alpha'} \left(-H_t + \pi \frac{\delta H_t'}{\delta \pi} + \pi^+ \frac{\delta H_t'}{\delta \pi^+}\right) d^3x dt\right\}\right] \quad (6.2)$$

The integral (6.1) is an integration of the canonical phase space coordinates ϕ^+, ϕ, π^+, and π, as one may have noticed.

7. Lagrangane Form of Fractional Generalized Euler

We will introduce a fractional function to obtain the fractional form of the Euler-Lagrange relation.

$$J^\alpha[\emptyset] = I_{0|x_0}^\alpha f\big(y(x), D_x^\alpha y(x)\big) \quad (7.1)$$

To determine the condition for a local minimum $J^\alpha[\emptyset]$ if we examine a new fractional functional that depends on the parameter ϵ^α

$$J^\alpha[t] = I_{0|t_0}^\alpha f\big(y(t, \epsilon^\alpha), D_x^\alpha y(t, \epsilon^\alpha)\big) \quad (7.2)$$

Where

$$Y(t, \epsilon^\alpha) = y(t) + \epsilon^\alpha \eta(x) \quad (7.3)$$

$$D_t^\alpha y(t, \epsilon^\alpha) = D_t \; y(t) + \epsilon^\alpha D_t \; \eta(x) \quad (7.4)$$

And

$$\eta(0) = \eta(x0) = 0 \quad (7.5)$$

The condition for an extreme value is obtained as $D_t^\alpha J[\epsilon^\alpha] = 0$.Then, From the chain rule of the fractional derivative, we Obtain:

$$D_t^\alpha J[\epsilon^\alpha] = I_{0|t_0}^\alpha \left[D_Y^\alpha f D_\epsilon^\alpha Y Y^{\alpha-1} + D_{D_t^\alpha Y}^\alpha f (D_\epsilon^\alpha Y)(D_t^\alpha Y)(D_t^\alpha Y)^{\alpha-1}\right] \quad (7.6)$$

$$= I_{0|t_0}^\alpha \left[D_Y^\alpha f Y^{\alpha-1}\eta + D_{D_t^\alpha Y}^\alpha f (D_t^\alpha Y)^{\alpha-1}(D_t^\alpha \eta)\right] = 0 \quad (7.7)$$

By employing integration by parts and Eq. (3.5), we obtain:

$$I_{0|t_0}^\alpha \left[D_Y^\alpha f Y^{\alpha-1}\eta - D_t^\alpha \left(D_{D_t^\alpha Y}^\alpha f (D_t^\alpha Y)^{\alpha-1}\right)(\eta)\right] \quad (7.8)$$

This leads to the fractional Euler-Lagrange equation.

$$D_Y^\alpha f y^{\alpha-1} = D_t^\alpha \left(D_{D_t^\alpha Y}^\alpha f (D_t^\alpha y)^{\alpha-1}\right) \quad (7.9)$$

From Eq. (2.7), we know

$$D_y^\alpha f(y)^{\alpha-1} = \partial_y f, D_{D_t^\alpha Y}^\alpha f (D_t^\alpha y)^{\alpha-1} = \partial_{D_t^\alpha Y} f \quad (7.10)$$

Using Equation (4.5), we can rephrase the fractional Euler-Lagrange equation as follows:

$$\frac{\partial f}{\partial Y} - \partial_x \left(\frac{\partial f}{\partial D_t^\alpha Y}\right) = 0 \quad (7.11)$$

In equation (3.1), we can substitute $D_\epsilon^\alpha J = 0$ with $\partial_\epsilon J = 0$ as the condition for an extreme value. We will show in subsequent steps that $\partial_\epsilon J = 0$ results in the same fractional Euler-Lagrange equation. When we expand the expression (f) in terms of (L) and (Y) in form $((\phi^+, \phi))$, the resulting equation has the following form.

$$\frac{\partial \mathcal{L}}{\partial \phi^+} - \partial_x \left(\frac{\partial \mathcal{L}}{\partial D_t^\alpha \phi^+}\right) = 0 \quad (7.12)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_x \left(\frac{\partial \mathcal{L}}{\partial D_t^\alpha \phi}\right) = 0 \quad (7.13)$$

By applying Equations (7.12 and 7.13) of the Euler-Lagrange equation, we arrive at the following conclusion:

$$\frac{\partial L}{\partial (D_\tau^\alpha \phi)} = D_\tau^\alpha \phi^+ \quad (7.14)$$

$$\frac{\partial L}{\partial (D_\tau^\alpha \phi^+)} = D_\tau^\alpha \phi \quad (7.15)$$

Eqs. (7.14) and (7.15) go to the standard equations if α goes to 1. It's important to note that the outcome obtained agrees with the results published in Reference [21].

7. The Application of Hamilton-Jacobi Theory in Constrained Systems

Let's consider the study of Hamilton-Jacobi theory applied to complex fields, treating them as constrained systems with composite configurations. Due to the intricacy of the subject, let's examine a simplified mathematical example highlighting the fundamental concepts in this area. Imagine a mechanical system comprising a ball moving on a complex inclined surface. We can represent the ball's motion using the Hamilton-Jacobi theory, treating it as a system constrained by composite configurations. These configurations allow us to analyze the complex trajectory of the ball on the inclined surface. For instance, the theory aids in tracking the ball's movement over time, comprehending the constraints, and assessing the forces influencing its motion. Applying composite configurations and the Hamilton-Jacobi theory, researchers can delve into the details of velocity, acceleration, and the geometric constraints governing the motion. These findings can be further employed in the design of control systems or in enhancing the performance of similar intricate mechanisms. This example serves as a demonstration of how the Hamilton-Jacobi theory can be effectively employed to analyze the motion of constrained complex systems within the realm of applied mathematics.

8. Conclusion

Theories that are reparametrization invariant have constrained dynamics and no canonical Hamiltonian. Without imposing gauge fixing of the type t - f (t) = 0 via Dirac's approach, it is not possible to obtain the correct physical Hamiltonian and equations of motion. Usually, it is hard to set this kind of gauge. However, when this system is addressed via the Hamilton-Jacobi approach, the Hamiltonian H is independent of two gauge fixing requirements and Lagrange multipliers. Another method is to compute the route integral of the system as an integration over the canonical phase space coordinates ϕ, ϕ^{+}, π and π^{+}. Gauge fixation conditions are not necessary to perform this. Lastly, the outcomes here motivate us to Subsequent research and development efforts could concentrate on a deeper examination of the Hamilton-Jacobi treatment of complex fields as limited fractional systems, with a particular focus on developing an advanced theoretical framework that would enable more precise and efficient analysis of these fields. Future research can also look into the relationships between quantum theories and classical physics to improve our understanding of complex and constrained physical systems.

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