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Terminal value problem for impulsive fractional differential equation via Kuratowski measure of non-compactness

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Abstract

In this paper, we study the existence of solutions for terminal value problem for impulsive differential equation via Θ-Riemann-Liouville fractional derivative in a appropriate Banach space. Our approach is based on Mönch fixed point theorem via Kuratowski measure of non-compactness, an example is given to illustrate our approach.

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1. Introduction

Differential equations with fractional derivatives are distinguished from those with ordinary derivatives as being better at describing the genetic properties of different processes and materials. Applications of fractional calculus in many fields of science are of great interest by researchers and are developing rapidly, see, [14, 15, 18]. There are many

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findings regarding the existence of solutions to various classes of fractional differential equations, see for example [10, 11, 13, 17].

Impulsive fractional differential equations are of great importance in the fields of medicine, physics and biology. Due to the various factors that affect the process dynamics in medicine and biology and the dynamics in physics leading to distortions, the latter can be modeled into different impulsive fractional differential equations with divers conditions, see for instance [2].

Recently, many interesting works have appeared in the study fractional differential equations on Banach spaces, which resides in the existence and uniqueness results by using fixed point theorems and some basic tools from functional analysis, we refer the reader to [4, 5, 6, 7, 8, 9, 16].

We consider, in this work, the following fractional differential equation with impulses

$$ {}^{RL}\mathcal{D}^{\sigma,\theta}_{\zeta_k^+} y(t) = \hbar(t, y(t)),\ t \in (\zeta_k, \zeta_{k+1}], k = 0, \cdots, m, \tag{1.1} $$

$$ y(L) = M \text{ and } \Delta_{\sigma,\theta} y\,|_{\zeta_k} = J_k(y_{\sigma,\theta}(\zeta_k^+)), k = 1, \cdots, m, \tag{1.2} $$

where ${}^{RL}\mathcal{D}^{\sigma,\theta}$ denotes the $\theta-$Riemann-Liouville fractional derivative with $0 < \sigma < 1$. Ξ is a Banach space with the norm $\|.\|$, $M \in \Xi$, $a = \zeta_0 < \zeta_1 < \ldots < \zeta_m < \zeta_{m+1} = L$, $\hbar : (a, L] \times \Xi \to \Xi$ a function satisfying some specified conditions (see, section 3), $J_k \in \mathcal{C}(\Xi, \Xi)$, $\theta \in \mathcal{C}^1([a, L], \mathbb{R}^+)$ satisfies $\theta'(t) > 0$, for all $t \in [a, L]$ and $\Delta_{\sigma,\theta} y\,|_{\zeta_k} = \frac{\mathcal{I}^{1-\sigma,\theta}_{\zeta_k^+} y(\zeta_k^+)}{(\theta(\zeta_{k+1}) - \theta(\zeta_k))^{1-\sigma}\Gamma(\sigma)} - y(\zeta_k^-)$, where $\mathcal{I}^{1-\sigma,\theta}_{\zeta_k^+} y(\zeta_k^+) = \lim_{t \to \zeta_k^+} \mathcal{I}^{1-\sigma,\theta}_{\zeta_k^+} y(t)$, $y(\zeta_k^-) = \lim_{t \to \zeta_k^-} y(t)$ and $y_{\sigma,\theta}(\zeta_k^+) = \lim_{t \to \zeta_k^+} (\theta(t) - \theta(\zeta_k))^{1-\sigma} y(t)$.

The present work is ordered as follows: to make the reader understand our problem, we give in section 2 some definitions, lemmas and basic results. Next, in the section 3, we present our main results by applying the fixed point theorem combined with the technique of measure of non-compactness to show the existence of solutions of the problem (1.1)-(1.2). Finally an example to reinforce our work in the section 4.

2. Background and basic results

We introduce in this section some notation and technical results which are used throughout this paper. Let $\mathcal{C}([c, d], \Xi)$ be the space of

Ξ-valued continuous functions on the interval $[c,d] \subset \mathbb{R}$ associated with the following norm:

$$\| u \|_{\infty} = \sup_{t \in [c,d]} \| u(t) \|.$$

Let $\mathcal{C}_{1-\sigma}([c,d],\Xi)$ be the Banach space of functions from the interval $(c,d]$ into Ξ which is defined as:

$$\mathcal{C}_{1-\sigma,\theta}([c,d],\Xi) = \{u \in \mathcal{C}((c,d],\Xi) : \lim_{t \to c^+} (\theta(t) - \theta(c))^{1-\sigma} u(t) \text{ finite and exists}\}.$$

A norm in this space is given by

$$\| u \|_{\sigma,\theta} = \sup_{t \in (c,d]} \frac{(\theta(t)-\theta(c))^{1-\sigma}}{(\theta(d)-\theta(c))^{1-\sigma}} \| u(t) \|.$$

In the following, for all $\eta > -1$, we pose $\Theta_{\eta}(r,s) = (\theta(r) - \theta(s))^{\eta}$, for all $s, r \in [a,L]$ with $r > s$ and $\Theta_{\eta}^{*} = \max_{k=0,\ldots,m} \{\Theta_{\eta}(\zeta_{k+1}, \zeta_k)\}$.

We consider the following Banach space

$$\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi) = \{y : [a,L] \to \Xi : y_k \in \mathcal{C}_{1-\sigma,\theta}([\zeta_k, \zeta_{k+1}],\Xi), k = 0,\ldots,m \text{ such that } y(\zeta_k) = y(\zeta_k^-)\},$$

with

$$\| y \|_{\mathcal{PC}_{\sigma,\theta}} = \max_{k=0,\ldots m.} \| y_k \|_{\sigma,\theta},$$

where y_k is the restriction of y on $(\zeta_k, \zeta_{k+1}]$.

For any subset N of $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, we put $N_{\sigma,\theta} =: \{u_{\sigma,\theta}, u \in N\}$, where

$$(u_{\sigma,\theta})_k(t) = \begin{cases} \Theta_{1-\sigma}(t,\zeta_k) u(t), & \text{if } t \in (\zeta_k, \zeta_{k+1}] \\ \lim_{t \to \zeta_k} \Theta_{1-\sigma}(t,\zeta_k) u(t), & \text{if } t = \zeta_k, \end{cases}$$

where $(u_{\sigma,\theta})_k$ is the restriction of $u_{\sigma,\theta}$ on $[\zeta_k, \zeta_{k+1}]$,

Let us now give the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq \Xi$, we denote by $S_b(G)$ the set of all bounded subsets of G.

Definition 2.1: [3, 12] Let $D \in S_b(\Xi)$. The Kuratowski measure of non-compactness ϑ of the subset D is defined as follows:

$\vartheta(D) = \inf\{e > 0 : D$ admits a finite cover by sets of diameter $\leq e\}$.

Lemma 2.2: [3, 12] *Let* $D_1, D_2 \in S_b(\Xi)$. *The following properties are valid:*

(i_1) $\vartheta(D_1) = 0$ if and only if D_1 is relatively compact,

(i_2) $\vartheta(D_1) = \vartheta(\overline{D_1})$, where $\overline{D_1}$ denotes the closure of D_1,

(i_3) $\vartheta(D_1 + D_2) \leq \vartheta(D_1) + \vartheta(D_2)$,

(i_4) $D_1 \subset D_2$ implies $\vartheta(D_1) \leq \vartheta(D_2)$,

(i_5) $\vartheta(r.D_1) = |r|.\vartheta(D_1)$ for all $r \in \mathbb{R}$,

(i_6) $\vartheta(\{e\} \cup D_1) = \vartheta(D_1)$ for all $e \in \Xi$,

(i_7) $\vartheta(D_1) = \vartheta(Conv(D_1))$, where $Conv(A)$ is the smallest convex that contains D_1.

Lemma 2.3: [12] *If* Δ *is a equicontinuous and bounded subset of* $\mathcal{C}([a,b],\Xi)$, *then* $\vartheta(\Delta(.)) : [a,b] \to \mathbb{R}^+$, *is continuous function and*

$$\vartheta_{\mathcal{C}}(\Delta) = \max_{t\in[a,b]} \vartheta(\Delta(t)), \vartheta(\{\int_a^b \varpi(t)dt : \varpi \in \Delta\}) \leq \int_a^b \vartheta(\Delta(t))dt,$$

such that $\Delta(t) = \{\varpi(t) : \varpi \in \Delta\}$ *and* $\vartheta_{\mathcal{C}}$ *is the non-compactness measure on the space* $\mathcal{C}([a,b],\Xi)$.

We denote by $\vartheta^k_{\sigma,\theta}$ and $\vartheta_{\sigma,\theta}$ the Kuratowski measures of non-compactness defines respictively on $\mathcal{C}_{1-\sigma,\theta}([\zeta_k, \zeta_{k+1}],\Xi)$ and $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, $k = 0,\ldots,m$.

Lemma 2.4: [12] *For all bounded subset D of* $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, *we have*

$$\vartheta_{\sigma,\theta}(D) = \max_{k=0,\ldots,m} \vartheta^k_{\sigma,\theta}(D_k),$$

where D_k *is the restriction of D on* $(\zeta_k, \zeta_{k+1}]$.

Theorem 2.5: [1] *Let* ρ *the Kuratowski measure of non-compactness on Banach space* E *and* G *a closed, bounded and convex subset of* E *which contains the* 0. *Let* Υ *be an operator from* G *to* G, *assume that* Υ *is continuous and satisfied, for every subset* Δ *of* G, *we have the following implication:*

$$\Delta = \Upsilon(\Delta) \cup \{0\} \text{ or } \Delta = \overline{conv}\Upsilon(\Delta) \Rightarrow \rho(\Delta) = 0.$$

Then the set $\{w \in G : \Upsilon(w) = w\}$ is nonempty.

We begin with some definitions from the theory of fractional calculus.

Definition 2.6: [13, 19]. Let Γ be the gamma function, ξ a non-negative real number and $\delta \in \mathcal{C}_{1-\sigma}([a,L],\Xi)$.

(i) θ-Riemann- Liouville fractional integral of order $\xi > 0$ of δ is given by

$$\mathcal{I}_{0^+}^{\xi,\theta}\delta(t) = \frac{1}{\Gamma(\xi)}\int_0^t \theta'(s)\Theta_{\xi-1}(t,s)\delta(s)ds,$$

(ii) Let $0 < \sigma < 1$. The θ-Riemann- Liouville fractional derivative of order ξ of the function δ is given as

$${}^{RL}\mathcal{D}_{a^+}^{\xi,\theta}\delta(t) = \frac{1}{\theta'(t)\Gamma(1-\xi)}\frac{d}{dt}\left(\int_0^t \theta'(s)\Theta_{-\xi}(t,s)\delta(s)ds\right).$$

Lemma 2.7: [13, 19] *Let* $\xi, \kappa \in \mathbb{R}_+^*$. *We have then*

1. $\mathcal{I}_{a^+}^{\xi,\theta}\Theta_{\kappa-1}(t,a) = \frac{\Gamma(\kappa)}{\Gamma(\xi+\kappa)}\Theta_{\xi+\kappa-1}(t,a),\ t \geq a.$
2. If $0 < \xi < 1$, then ${}^{RL}\mathcal{D}_{a^+}^{\xi,\theta}\Theta_{\xi-1}(t,a) = 0.$

Lemma 2.8: [20]. *Let* $\sigma > 0$ *and* $0 \leq \gamma \leq \sigma$. *Then,* $\mathcal{I}_{a^+}^{\sigma,\theta}$ *is bounded from* $\mathcal{C}_\gamma([a,L],\Xi)$ *into* $\mathcal{C}([a,L],\Xi)$.

Lemma 2.9: [20]. *Let* $0 < \sigma < 1,\ 0 \leq \gamma < 1$. *If* $\delta \in \mathcal{C}_\gamma([a,L],\Xi)$ *and* $\mathcal{I}_{a^+}^{1-\sigma,\theta}\ \delta \in \mathcal{C}_\gamma^1([a,L],\Xi)$, *then*

$$\mathcal{I}_{a^+}^{\sigma,\theta\,RL}\mathcal{D}_{a^+}^{\sigma}\delta(t) = \delta(t) - \frac{\mathcal{I}_{a^+}^{1-\sigma,\theta}\delta(a^+)}{\Gamma(\sigma)}\Theta_{\sigma-1}(t,a), \text{ for all } t \in (a,L].$$

Lemma 2.10: *Let* $0 < \sigma < 1$ *and let* $\ell \in \mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$. *A function* $y \in \mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$ *is a solution of the fractional integral equation*

$$y(t) = \begin{cases} (M - \mathcal{I}_{\zeta_m}^{\sigma,\theta}\ell(L))\dfrac{\Theta_{\sigma-1}(t,\zeta_m)}{\Theta_{\sigma-1}(\zeta_{m+1},\zeta_m)} + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_{\zeta_m}^t \theta'(s)\Theta_{\sigma-1}(s,\zeta_m)\ell(s)ds, & \text{if } t \in I_m, \\ \left(M - \displaystyle\sum_{i=0}^{k}\mathcal{I}_{\zeta_{m-i}}^{\sigma,\theta}\ell(\zeta_{m-i+1}) - \sum_{i=0}^{k-1}J_{m-i}(y_\sigma(\zeta_{m-i}^+))\right) \\ \times\dfrac{\Theta_{\sigma-1}(t,\zeta_{m-2})}{\Theta_{\sigma-1}(\zeta_{m-1},\zeta_{m-2})} + \dfrac{1}{\Gamma(\sigma)}\displaystyle\int_{\zeta_{m-k}}^t \theta'(s)\Theta_{\sigma-1}(t,s)\ell(s)ds, & \text{if } t \in I_{m-k}, \\ k = 1,\ldots,m. \end{cases} \tag{2.1}$$

if and only if y is a solution of the following problem

$${}^{RL}\mathcal{D}_{\zeta_{m-k}^+}^{\sigma,\theta}y(t) = \ell(t),\ t \in (\zeta_{m-k},\zeta_{m-k+1}], k = 0,\ldots,m, \tag{2.2}$$

$$y(L)=M \text{ and } \Delta_{\sigma,\theta}y\,|_{\zeta_{m-k}}=J_{m-k}(y_{\sigma,\theta}(\zeta^{+}_{m-k})),\ k=1,\dots,m, \tag{2.3}$$

Proof: Assume y satisfies the problem (2.2)-(2.3). We want to prove that y verified (2.1).

If $t\in(\zeta_m,\zeta_{m+1}]$, we have ${}^{RL}\mathcal{D}^{\sigma,\theta}y(t)=\ell(t)$, from Lemma 2.8, we obtain $\mathcal{I}^{1-\sigma,\theta}_{\zeta^{+}_{m}}\ y\in\mathcal{C}\ \ ([\zeta_m,L],\Xi)$ and $\left(\frac{1}{\theta'(t)}\frac{d}{dt}\right)\ (\mathcal{I}^{1-\sigma,\theta}_{\zeta^{+}_{m}}y(t))={}^{RL}\mathcal{D}^{\sigma,\theta}_{\zeta^{+}_{m}}y(t)\in\mathcal{C}_{1-\sigma,\theta}([\zeta_m,L],\Xi)$.

By utilizing Lemma 2.9, we get

$$y(t)=\frac{\mathcal{I}^{1-\sigma,\theta}_{\zeta^{+}_{m}}y(\zeta^{+}_{m})}{\Gamma(\sigma)}\Theta_{\sigma-1}(t,\zeta_m)+\frac{1}{\Gamma(\sigma)}\int_{\zeta_m}^{t}\theta'(s)\Theta_{\sigma-1}(t,s)\ell(s)ds.$$

Since $y(L)=M$, we obtain

$$\frac{\mathcal{I}^{1-\sigma,\theta}_{\zeta^{+}_{m}}y(\zeta^{+}_{m})}{\Theta_{1-\sigma}(\zeta_{m+1},\zeta_m)\Gamma(\sigma)}=M-\mathcal{I}^{\sigma,\theta}_{\zeta_m}\ell(L). \tag{2.4}$$

So,

$$y(t)=(M-\mathcal{I}^{\sigma,\theta}_{\zeta_m}\ell(L))\frac{\Theta_{\sigma-1}(t,\zeta_m)}{\Theta_{\sigma-1}(\zeta_{m+1},\zeta_m)}+\frac{1}{\Gamma(\sigma)}\int_{\zeta_m}^{t}\theta'(s)\Theta_{\sigma-1}(s,\zeta_m)\ell(s)ds.$$

If $t\in(\zeta_{m-1},\zeta_m]$, then, from Lemma 2.8 and Lemma 2.9, we get

$$y(t)=\frac{\mathcal{I}^{1-\sigma,\theta}_{\zeta^{+}_{m-1}}y(\zeta^{+}_{m-1})}{\Gamma(\sigma)}\Theta_{\sigma-1}(t,\zeta_{m-1})+\frac{1}{\Gamma(\sigma)}\int_{\zeta_{m-1}}^{t}\theta'(s)\Theta_{\sigma-1}(t,s)\ell(s)ds.$$

By Equations (2.3) and (2.4), we have

$$\frac{\mathcal{I}^{1-\sigma,\theta}_{\zeta^{+}_{m-1}}y(\zeta^{+}_{m-1})}{\Theta_{1-\sigma}(\zeta_m,\zeta_{m-1})\Gamma(\sigma)}=M-\mathcal{I}^{\sigma,\theta}_{\zeta_m}\ell(L)-\mathcal{I}^{\sigma,\theta}_{\zeta_{m-1}}\ell(\zeta_m)-J_m(y_\sigma(\zeta^{+}_{m})). \tag{2.5}$$

So,

$$\begin{aligned}y(t)=&(M-\mathcal{I}^{\sigma,\theta}_{\zeta_m}\ell(L)-\mathcal{I}^{\sigma,\theta}_{\zeta_{m-1}}\ell(\zeta_m)-J_m(y_\sigma(\zeta^{+}_{m})))\frac{\Theta_{\sigma-1}(t,\zeta_{m-1})}{\Theta_{\sigma-1}(\zeta_m,\zeta_{m-1})}\\&+\frac{1}{\Gamma(\sigma)}\int_{\zeta_{m-1}}^{t}\theta'(s)\Theta_{\sigma-1}(s,\zeta_m)\ell(s)ds.\end{aligned}$$

If $t\in(\zeta_{m-2},\zeta_{m-1}]$, from Lemma 2.8 and Lemma 2.9, we obtain

$$y(t)=(y(\zeta_{m-1})-\mathcal{I}^{\sigma,\theta}_{\zeta_{m-2}}\ell(\zeta_{m-1}))\frac{\Theta_{\sigma-1}(t,\zeta_{m-2})}{\Theta_{\sigma-1}(\zeta_{m-1},\zeta_{m-2})}+\frac{1}{\Gamma(\sigma)}\int_{\zeta_{m-2}}^{t}\theta'(s)\Theta_{\sigma-1}(s,\zeta_{m-1})\ell(s)ds.$$

Using (2.3) and (2.5), we get

$$y(t)=\left(M-\sum_{i=0}^{2}\mathcal{I}_{\zeta_{m-i}}^{\sigma,\theta}\ell(\zeta_{m-i+1})-\sum_{i=0}^{1}J_{m-i}(y_{\sigma}(\zeta_{m-i}^{+}))\right)\frac{\Theta_{\sigma-1}(t,\zeta_{m-2})}{\Theta_{\sigma-1}(\zeta_{m-1},\zeta_{m-2})}$$
$$+\frac{1}{\Gamma(\sigma)}\int_{\zeta_{m-2}}^{t}\theta'(s)\Theta_{\sigma-1}(t,s)\ell(s)ds.$$

If $t\in(\zeta_{m-k},\zeta_{m-k+1}]$, by using Lemmas 2.8 and 2.9 again, we find

$$y(t)=\left(M-\sum_{i=0}^{k}\mathcal{I}_{\zeta_{m-i}}^{\sigma,\theta}\ell(\zeta_{m-i+1})-\sum_{i=0}^{k-1}J_{m-i}(y_{\sigma}(\zeta_{m-i}^{+}))\right)\frac{\Theta_{\sigma-1}(t,\zeta_{m-2})}{\Theta_{\sigma-1}(\zeta_{m-1},\zeta_{m-2})}$$
$$+\frac{1}{\Gamma(\sigma)}\int_{\zeta_{m-k}}^{t}\theta'(s)\Theta_{\sigma-1}(t,s)\ell(s)ds.$$

Conversely, assume that y satisfies the impulsive equation (2.1).

If $t\in(\zeta_m,\zeta_{m+1}]$ then $y(L)=M$ and using the fact that ${}^{RL}\mathcal{D}_{\zeta_m}^{\sigma,\theta}$ is the left inverse of $\mathcal{I}_{\zeta_m}^{\sigma,\theta}$, we get

$${}^{RL}\mathcal{D}_{\zeta_m}^{\sigma,\theta}y(t)=\ell(t),\quad \text{for each } t\in(\zeta_m,\zeta_{m+1}].$$

Next. If $t\in(\zeta_{m-k},\zeta_{m-k+1}]$, $k=1,\dots,m$ and according to the properties of ${}^{RL}\mathcal{D}_{\zeta_{m-k}}^{\sigma,\theta}$, we get

$${}^{RL}\mathcal{D}_{\zeta_{m-k}}^{\sigma,\theta}y(t)=\ell(t),\quad \text{for each }\ t\in(\zeta_{m-k},\zeta_{m-k+1}].$$

Also, we can easily show that

$$\Delta_{\sigma,\theta}y\,|_{t=\zeta_{m-k}}=J_{m-k}(y_{\sigma}(\zeta_{m-k}^{+})),\quad k=0,\dots,m-1.$$

3. Main result

Suppose that the function $\hbar:(\zeta_k,\zeta_{k+1}]\times\Xi\to\Xi$ verifies $\hbar(t,0)\in\mathcal{C}([\zeta_k,\ \zeta_{k+1}],\ \Xi)$, $k=0,\ \dots,\ m$ and there exists $A\in\mathbb{R}^{+}$ and $\lambda\geq 1-\sigma$ such that

(H_1) For any $\mu,\nu\in\Xi$:

$$\|\hbar(t,\mu)-\hbar(t,\nu)\|\leq A\Theta_{\lambda}(t,\zeta_k)\,\|\mu-\nu\|,\ t\in(\zeta_k,\zeta_{k+1}],k=0,\dots,m.$$

(H_2) For each nonempty, bounded set $\Delta\subset\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, we have

$$\vartheta(\hbar(t,\Delta(t))\leq A\Theta_{\lambda}(t,\zeta_k)\vartheta(\Delta(t)),\quad t\in(\zeta_k,\zeta_{k+1}],k=0,\dots,m,$$

Suppose that the functions $J_k:\Xi\to\Xi,k=1,\dots,m,$ are continuous and there exists $B\in\mathbb{R}^{+}$ such that

(H_3) For any $u\in\Xi$:

$$\| J_k(u) \| \leq B \| u \|, k = 1, \ldots, m.$$

(H_4) For each nonempty, bounded set $\Delta \subset \mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, we have

$$\vartheta(J_k(\Delta_{\sigma,\theta}(t))) \leq B\vartheta(\Delta_{\sigma,\theta}(t)), k = 1, \ldots, m,$$

where $\Delta_{\sigma,\theta}(t) = \{y_{\sigma,\theta} / \ y \in \mathcal{PC}_{1-\sigma,\theta}((a,L],\Xi)\}$.

(H_5) $\frac{(m+2)A\Theta^*_{\lambda+\sigma} + mB\Theta^*_{1-\sigma}\Gamma(\sigma+1)}{\Gamma(\sigma+1)} < 1.$

Consider the operator $N : \mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi) \to \mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$ defined by

$$Ny(t) = \begin{cases} (M - \mathcal{I}^{\sigma,\theta}_{\zeta_m}\hbar(L, y(L)))\frac{\Theta_{\sigma-1}(t,\zeta_m)}{\Theta_{\sigma-1}(\zeta_{m+1},\zeta_m)} \\ + \frac{1}{\Gamma(\sigma)}\int_{\zeta_m}^{t} \theta'(s)\Theta_{\sigma-1}(s,\zeta_m)\hbar(s, y(s))ds, & \text{if } t \in I_m, \\ (M - \sum_{i=0}^{k} \mathcal{I}^{\sigma,\theta}_{\zeta_{m-i}}\hbar(\zeta_{m-i+1}, y(\zeta_{m-i+1})) - \sum_{i=0}^{k-1} J_{m-i}(y_\sigma(\zeta^+_{m-i}))) \\ \times \frac{\Theta_{\sigma-1}(t,\zeta_{m-k})}{\Theta_{\sigma-1}(\zeta_{m-k+1},\zeta_{m-k})} + \frac{1}{\Gamma(\sigma)}\int_{\zeta_{m-k}}^{t} \theta'(s)\Theta_{\sigma-1}(t,s)\hbar(s, y(s))ds, & \text{if } t \in I_{m-k}, \end{cases}$$

where $I_{m-k} = (\zeta_{m-k}, \zeta_{m-k+1}]\ k = 0, \ldots, m.$

The theorem below is the main result

Theorem 3.1: *Suppose that the conditions* $(\mathbf{H_1}) - (\mathbf{H_5})$ *are valid. Then the solution set of the problem* (1.1) – (1.2) *is nonempty.*

Proof: According to Lemma 2.10, the solutions of (1.1) – (1.2) and the fixed points of *N* are identical. For this reason, it suffices to verify the axioms of Theorem 2.5, it is done in four steps.

First step. We start to prove that *N* is continuous. Let $\varepsilon > 0$ and $\{y_n\}_{n\in\mathbb{N}} \to y$ in $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$. The hypothesis $(\mathbf{H_1})$ and $(\mathbf{H_3})$ confirm the existence of an integer $m \in \mathbb{N}$ such that, for all $n \geq m$ and $t \in (\zeta_{m-k}, \zeta_{m-k+1}]$, $k = 1, \ldots, m$, we have

$$\| \hbar(t, y_n(t)) - \hbar(t, y(t)) \| < \frac{\Gamma(\sigma+1)\varepsilon}{2(m+2)\Theta^*_\sigma} \tag{3.1}$$

and

$$\| J_{m-k}((y_\sigma)_n(\zeta^+_{m-k}) - J_{m-k}(y_\sigma(\zeta^+_{m-k})) \| < \frac{\varepsilon}{2m}. \tag{3.2}$$

Thus, for all $t \in (\zeta_{m-k}, \zeta_{m-k+1}], k = 1, \ldots, m,$ we have

$$\frac{\Theta_{1-\sigma}(t,\zeta_{m-k})}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\| Ny_n(t)-Ny(t)\| \le \sum_{i=0}^{k-1}\| J_{m-i}(y_{\sigma,n}(\zeta_{m-i}^{+}))-J_{m-i}(y_{\sigma}(\zeta_{m-i}^{+}))\|$$
$$+\sum_{i=0}^{k}\mathcal{I}_{\zeta_{m-i}}^{\sigma,\theta}\| \hbar(\zeta_{m-i+1},y_n(\zeta_{m-i+1}))-\hbar(\zeta_{m-i+1},y(\zeta_{m-i+1}))\|$$
$$+\frac{\Theta_{1-\sigma}(t,\zeta_{m-k})}{\Gamma(\sigma)\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\int_{m-k}^{t}\theta'(s)\Theta_{1-\sigma}(t,s)\| \hbar(s,y_n(s)-\hbar(s,y(s))\| ds.$$

By Equations (3.1) and (3.2), we get

$$\| Ny_n - Ny\|_{\mathcal{PC}_{\sigma,\theta}} < \frac{\varepsilon}{2}+\frac{(m+2)\Theta_{\sigma}^{*}}{\Gamma(\sigma+1)}\times\frac{\Gamma(\sigma+1)\varepsilon}{2(m+2)\Theta_{\sigma}^{*}}=\varepsilon.$$

Thus, N is continuous on $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$).

Second step. Now we will prove that N is bounded. Let $y\in\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, from $(\mathbf{H_1})$ it is easy to deduce that $Ny\in\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$. Using $(\mathbf{H_1})$, for all $y\in D_{\kappa}=\{y\in\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi):\| y\|_{\mathcal{PC}_{\sigma,\theta}}<\kappa\}$ and $t\in(\zeta_{m-k},\zeta_{m-k+1}]$, we get

$$\| \frac{\Theta_{1-\sigma}(t,\zeta_{m-k})Ny(t)}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\| \le \| M\|+\sum_{i=0}^{k}\mathcal{I}_{\zeta_{m-i}}^{\sigma,\theta}\| \hbar(\zeta_{m-i+1},y(\zeta_{m-i+1}))\|+\sum_{i=0}^{k-1}\| J_{m-i}(y_{\sigma}(\zeta_{m-i}^{+}))\|$$
$$+\frac{\Theta_{1-\sigma}(t,\zeta_{m-k})}{\Gamma(\sigma)\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\int_{\zeta_{m-k}}^{t}\theta'(s)\Theta_{\sigma-1}(t,s)\| \hbar(s,y(s))\| ds$$
$$\le\| M\|+mB\Theta_{1-\sigma}^{*}\kappa+\frac{(m+2)\hbar^{*}\Theta_{\sigma}^{*}}{\Gamma(\sigma+1)}+\frac{(m+2)A\Theta_{\sigma+\lambda}^{*}\kappa}{\Gamma(\sigma+1)}.$$

So,

$$\| Ny\|_{\mathcal{PC}_{\sigma,\theta}}\le\| M\|+mB\Theta_{1-\sigma}^{*}\kappa+\frac{(m+2)\hbar^{*}\Theta_{\sigma}^{*}}{\Gamma(\sigma+1)}+\frac{A(m+2)\Theta_{\sigma+\lambda}^{*}\kappa}{\Gamma(\sigma+1)}=\ell,$$

where $\hbar^{*}=\max\{\sup_{t\in[\zeta_k,\zeta_{k+1}]}\| \hbar(t,0)\|,k=1,\ldots,m\}$.

Third step. we will demonstrate that $(ND)_k$ is equicontinuous for all bounded subset D of $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$, such that $(ND)_k$ is the restriction of ND on the interval $(\zeta_k,\zeta_{k+1}]$, let D_{κ} be the subset which was previously defined. It suffices to prove that $(ND_{\kappa})_k$ is equicontinuous in $\mathcal{C}_{\sigma,\theta}([\zeta_{m-k},\zeta_{m-k+1}],\Xi)$. Let $y\in(D_{\kappa})_{m-k}$ and $\zeta_1,\zeta_2\in(\zeta_{m-k},\zeta_{m-k+1}]$ with $\zeta_1<\zeta_2$, we have

$$\| \frac{\Theta_{1-\sigma}(\zeta_2,\zeta_{m-k})Ny(\zeta_2)}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}-\frac{\Theta_{1-\sigma}(\zeta_1,\zeta_{m-k})Ny(\zeta_1)}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\|$$
$$\le\frac{\Theta_{1-\sigma}(\zeta_1,\zeta_{m-k})}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})\Gamma(\sigma)}\int_{\zeta_{m-k}}^{\zeta_1}\theta'(s)[\Theta_{\sigma-1}(\zeta_1,s)-\Theta_{\sigma-1}(\zeta_2,s)]\| \hbar(s,y(s))\| ds$$
$$+\frac{[\Theta_{1-\sigma}(\zeta_2,\zeta_{m-k})-\Theta_{1-\sigma}(\zeta_1,\zeta_{m-k})]}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})\Gamma(\sigma)}\int_{\zeta_{m-k}}^{\zeta_1}\theta'(s)\Theta_{\sigma-1}(\zeta_2,s)\| \hbar(s,y(s))\| ds$$

$$+\frac{\Theta_{1-\sigma}(\zeta_2,\zeta_{m-k})}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})\Gamma(\sigma)}\int_{\zeta_1}^{\zeta_2}\theta'(s)\Theta_{\sigma-1}(\zeta_2,s)\,\|\,\hbar(s,y(s))\,\|\,ds$$
$$\leq\frac{\hbar^*+A\kappa\Theta_\lambda^*}{\Gamma(\sigma)}\int_{\zeta_{m-k}}^{\zeta_1}\theta'(s)[\Theta_{\sigma-1}(\zeta_1,s)-\Theta_{\sigma-1}(\zeta_2,s)]ds$$
$$+\frac{\hbar^*[\Theta_{1-\sigma}(\zeta_2,\zeta_{m-k})-\Theta_{1-\sigma}(\zeta_1,\zeta_{m-k})]}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})\Gamma(\sigma)}\int_{\zeta_{m-k}}^{\zeta_1}\theta'(s)\Theta_{\sigma-1}(\zeta_2,s)ds$$
$$+\frac{A\kappa[\Theta_{1-\sigma}(\zeta_2,\zeta_{m-k})-\Theta_{1-\sigma}(\zeta_1,\zeta_{m-k})]}{\Gamma(\sigma)}\int_{\zeta_{m-k}}^{\zeta_1}\theta'(s)\Theta_{\sigma-1}(\zeta_2,s)\Theta_{\lambda+\sigma-1}(s,\zeta_{m-k})ds$$
$$+\frac{\hbar^*+A\kappa\Theta_\lambda^*}{\Gamma(\sigma)}\int_{\zeta_1}^{\zeta_2}\theta'(s)\Theta_{\sigma-1}(\zeta_2,s)ds$$
$$\leq\frac{\hbar^*+A\kappa\Theta_\lambda^*}{\Gamma(\sigma+1)}[\Theta_\sigma(\zeta_2,\zeta_1)+\Theta_\sigma(\zeta_1,\zeta_{m-k})-\Theta_\sigma(\zeta_2,\zeta_{m-k})]$$
$$+\frac{\hbar^*\Theta_{2\sigma-1}^*+A\kappa\Theta_{2\sigma+\lambda-1}^*}{\Gamma(\sigma+1)}[\Theta_{1-\sigma}(\zeta_2,\zeta_{m-k})-\Theta_{1-\sigma}(\zeta_1,\zeta_{m-k})]$$
$$+\frac{\hbar^*+A\kappa\Theta_\lambda^*}{\Gamma(\sigma+1)}\Theta_\sigma(\zeta_2,\zeta_1).$$

Taking ζ_2 tends towards ζ_1, we get that, the last inequality tends to zero. Then $(ND_\kappa)_k$ is equicontinuous in $\mathcal{C}_{\sigma,\theta}$ ($[\zeta_{m-k},\ \zeta_{m-k+1}]$, Ξ), $k=0,\ldots,m$.

Final step. We verify that N satisfies the assumptions of theorem 2.5. We pose

$$D=\{y\in\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi):\|\,y\,\|_{\mathcal{PC}_{\sigma,\theta}}\leq R\},$$

where R is a real number verifies the following equality

$$R>\frac{\|M\|\Gamma(\sigma+1)+(m+2)\hbar^*\Theta_\sigma^*}{\Gamma(\sigma+1)-mB\Theta_{1-\sigma}^*\Gamma(\sigma+1)+A(m+2)\Theta_{\sigma+\lambda}^*}. \tag{3.3}$$

First, we now show that N is defined from D to D, Indeed, for any $y\in D$, by above conditions $(\mathbf{H_2}),(\mathbf{H_5})$ and by according to a little calculation, for all $t\in(\zeta_{m-k},\ \zeta_{m-k+1}]$, we have

$$\|\frac{\Theta_{1-\sigma}(t,\zeta_{m-k})Ny(t)}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\leq\|\,M\,\|+\frac{(m+2)\hbar^*\Theta_\sigma^*}{\Gamma(\sigma+1)}+[mB\Theta_{1-\sigma}^*+\frac{A(m+2)\Theta_{\sigma+\lambda}^*}{\Gamma(\sigma+1)}]R$$
$$\leq\frac{\|M\|\Gamma(\sigma+1)+(m+2)\hbar^*\Theta_\sigma^*}{\Gamma(\sigma+1)}+\left(\frac{mB\Theta_{1-\sigma}^*\Gamma(\sigma+1)+A(m+2)\Theta_{\sigma+\lambda}^*}{\Gamma(\sigma+1)}\right)R$$

From the inequality (3.3), we obtain

$$\forall y\in D:\|\,Ny\,\|_{\mathcal{PC}_{\sigma,\theta}}<R.$$

Then N remains defined from D to D. Note that D is bounded, convex and closed subset of $\mathcal{PC}_{1-\sigma,\theta}([a,L],\Xi)$ and N is continuous on D, we can easily show the following equality

$$\vartheta^k_{\sigma,\theta}((N\Delta)_{m-k}) = \sup\left\{\vartheta\left(\frac{\Theta_{1-\sigma}(t,\zeta_{m-k})N\Delta(t)}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\right), t\in(\zeta_{m-k},\zeta_{m-k+1}]\right\},$$

for all $\Delta\subset D, k=0,\dots,m.$ Next, we need to prove the following implication

$$\Delta\subset\overline{conv}\{N(\Delta)\cup\{0\}\}\Rightarrow\vartheta_{\sigma,\theta}(\Delta)=0, \text{ for any } \Delta\subset D.$$

Let $\Delta\subset D$ such that $\Delta\subset\overline{conv}\{N(\Delta)\cup\{0\}\}$. From $(\mathbf{H_2})$, $(\mathbf{H_4})$, Lemmas 2.2, 2.3, 2.4 and the previous steps, for all $t\in(\zeta_{m-k},\zeta_{m-k+1}]$, we have

$$\begin{aligned}\vartheta\left(\frac{\Theta_{1-\sigma}(t,\zeta_{m-k})(N\Delta)(t)}{\Theta_{1-\sigma}(\zeta_{m-k+1},\zeta_{m-k})}\right) &\le \frac{A\Theta^*_\lambda}{\Gamma(\sigma)}\sum_{i=0}^{k}\int_{\zeta_{m-i}}^{\zeta_{m-i+1}}\theta'(s)\Theta_{\sigma-1}(\zeta_{m-i+1},s)\vartheta^{m-i}_{\sigma,\theta}(N\Delta_{m-i})ds\\ &+B\Theta^*_{1-\sigma}\sum_{i=0}^{k-1}\vartheta^{m-i}_{\sigma,\theta}(N\Delta_{m-i})\\ &+\frac{A\Theta^*_\lambda}{\Gamma(\sigma)}\int_{\zeta_{m-k}}^{t}\theta'(s)\Theta_{\sigma-1}(t,s)\vartheta^{m-k}_{\sigma,\theta}(N\Delta_{m-k})ds.\end{aligned}$$

Thus,

$$\vartheta_{\sigma,\theta}(N\Delta)\le\frac{(m+2)A\Theta^*_{\lambda+\sigma}+mB\Theta^*_{1-\sigma}\Gamma(\sigma+1)}{\Gamma(\sigma+1)}\vartheta_{\sigma,\theta}(N\Delta).$$

By condition $(\mathbf{H_3})$, we get $\vartheta_{\sigma,\theta}(N\Delta)=0$, that is $\vartheta_{\sigma,\theta}(\Delta)=0$. From the theorem 2.5, N has a fixed point $y\in D$ which is a solution of Problem (1.1)-(1.2).

4. Example

As an example to use of the main result , we shall now consider the following fractional differential equation

$$^{RL}\mathcal{D}^{\frac{1}{2},\theta}_{k}y(t)=\left(\frac{1}{10+nt^2}+\frac{\sqrt{e^t-e^{\zeta_i}}}{1+t+t^2}y_n(t)\right)_{n\in\mathbb{N}^*}, t\in(\zeta_k,\zeta_{k+1}]\subset(0,1], k=0,1 \tag{4.1}$$

$$y(1)=(1,0,\dots,0,\dots). \tag{4.2}$$

$$\Delta_{\sigma,\theta}y\,|_{t=\frac{1}{2}}=\frac{1}{20}y_{\sigma,\theta}\left(\frac{1}{2}^+\right), \tag{4.3}$$

where $\theta(t)=e^t$, $\zeta_0=0,\zeta_1=0.5,\zeta_2=1,\sigma=\lambda=0.5$ and

$$\Xi = \{(y_n)_{n\in\mathbb{N}^*} : \sup_n |y_n| < \infty\},$$

with $\|y\| = \sup_n |y_n|$, then $(\Xi, \|.\|)$ consists a Banach space, by comparing with the (1.1)-(1.2), we notice that

$$\hbar(t, y(t)) = (\hbar_n(t, y_n(t)))_{n\in\mathbb{N}^*},$$

where

$$\hbar_n(t, y_n(t)) = \frac{1}{1+nt^2} + \frac{\sqrt{e^t - e^{\zeta_k}}}{10+t+t^2} y_n(t), t \in (\zeta_k, \zeta_{k+1}], k = 0,1, n \in \mathbb{N}^* \text{ and}$$

$$J_1(u) = \frac{1}{20}u, \text{ for all } u \in \Xi.$$

Cleary that $\hbar : (\zeta_k, \zeta_{k+1}] \times \Xi \to \Xi, k = 0,1$ and $J_1 : \Xi \to \Xi$ are continuous and there exists $A = \frac{1}{10}$, $B = \frac{1}{20}$ such that

$$\|\hbar(t,\mu) - \hbar(t,\nu)\| \le \frac{1}{10}\sqrt{e^t - e^{\zeta_k}} \|\mu - \nu\|, \text{ for all } t \in (\zeta_k, \zeta_{k+1}] \text{ and } \mu, \nu \in \Xi \text{ and}$$

$$\|J_1(u)\| = \frac{1}{20}\|\mu\|, \text{for all } \mu \in \Xi.$$

Hence, $(\mathbf{H_2})$ and $(\mathbf{H_4})$ are satisfied. Next, For any bounded set $\Delta \subset \mathcal{PC}_{1-\sigma,\theta}([0,1], \Xi)$, we have

$$\vartheta(\hbar(t, \Delta(t))) \le \frac{\sqrt{e^t - e^{\zeta_k}}}{10} \vartheta(\Delta(t)), t \in (\zeta_k, \zeta_{k+1}], k = 0,1 \text{ and}$$

$$\vartheta(J_1(\Delta_{\sigma,\theta}(t))) = \frac{1}{20}\vartheta(\Delta_{\sigma,\theta}(t)).$$

Thus, $(\mathbf{H_1})$ and $(\mathbf{H_3})$ are satisfied. Since

$$\frac{(m+2)A\Theta^*_{\lambda+\sigma} + mB\Theta^*_{1-\sigma}\Gamma(\sigma+1)}{\Gamma(\sigma+1)} < 1.$$

So, $(\mathbf{H_5})$ holds. Therefore, Theorem 3.1 ensures that the problem (4.1)-(4.3) has a solution.

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