Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 7, pp. 1481–1489
DOI : 10.47974/JIM-1731

Some sharp bounds for the Hilbert-Schmidt numerical radius of operators

Aici Soumia †
Frakis Abdelkader *
Department of Mathematics
Faculty of Exact Sciences
University of Mustapha Stambouli
Mascara 29000
Algeria

Abstract

Some sharp Hilbert-Schmidt numerical radius inequalities for an operator as well as for 2 × 2 operator matrices are presented. Also, refinements of certain earlier Hilbert-Schmidt numerical radius inequalities are provided.

Subject Classification: *(2010) 47A30, 47A63, 47B15, 47B10.*

Keywords: *Schatten p-norm, Hilbert-Schmidt norm, Inequality.*

1. Introduction

Let C_p, $(1 \le p \le \infty)$, be the Schatten p-class which is the set of all bounded linear compact operator T_1 satisfying $\| T_1 \|_p < \infty$, where $\| T_1 \|_p$ = $(tr\,|T_1|^p)^{\frac{1}{p}}$ is the Schatten p-norm, and $|T_1| = (T_1^* T_1)^{\frac{1}{2}}$. For $T_1, T_2 \in C_2$, we have, see [4], $T_1 T_2 \in C_1$ and

$$|trT_1T_2| \le \| T_1 \|_2 \| T_2 \|_2 . \tag{1.1}$$

The Hilbert-Schmidt numerical radius of $S_1 \in C_2$ is defined by

$$w_2(S_1) = \sup_{\phi \in \mathbb{R}} \| \Re(e^{i\phi} S_1) \|_2 ,$$

† *E-mail:* **soumia.aici@univ-mascara.dz**
* *E-mail:* **aekfrakis@yahoo.fr (Corresponding Author)**

where $\Re(e^{i\phi}S_1)$ is the real part of $e^{i\phi}S_1$. It is known that $w_2(\cdot)$ is a norm on C_2 and it is equivalent to $\|\cdot\|_2$, this says that

$$\frac{1}{\sqrt{2}}\|S_1\|_2 \leq w_2(S_1) \leq \|S_1\|_2 . \tag{1.2}$$

In [2], Aldalabih and Kittaneh proved that

$$\frac{\max\{w_2(A_1+A_2), w_2(A_1-A_2)\}}{\sqrt{2}} \leq w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) \leq \frac{w_2(A_1+A_2)+w_2(A_1-A_2)}{\sqrt{2}}. \tag{1.3}$$

where $A_1, A_2 \in C_2$. For more details, we refer the reader to [1], [2], [3], [5] and [6].

In this work, some new bounds for $w_2(\cdot)$ are derived. These bounds refine some existing ones.

2. Main results

Throughout this paper, $T_1 = A + iB$ denotes the *Cartesian decomposition* of $T_1 \in C_2$, where $A = \Re(T_1)$ and $B = \Im(T_1)$.

Theorem 2.1: *Let $T_1 \in C_2$ with $T_1 = A + iB$. Then*

$$w_2^2(T_1) = \frac{1}{2}[\|A\|_2^2 + \|B\|_2^2 + \sqrt{(\|A\|_2^2 - \|B\|_2^2)^2 + 4(trAB)^2}].$$

Proof: From $w_2(T_1) = \sup_{\phi\in\mathbb{R}} \|\Re(e^{i\phi}T_1)\|_2$, we have $w_2(T_1) = \sup_{\phi\in\mathbb{R}} \|A\cos\phi - B\sin\phi\|_2$. If we put $\alpha_1 = \cos\phi$ and $\alpha_2 = -\sin\phi$, then we obtain

$$\begin{aligned} w_2^2(T_1) &= \sup_{\alpha_1^2+\alpha_2^2=1} \|\alpha_1 A + \alpha_2 B\|_2^2 \\ &= \sup_{\alpha_1^2+\alpha_2^2=1} tr(\alpha_1 A^* + \alpha_2 B^*)(\alpha_1 A + \alpha_2 B) \\ &= \sup_{\alpha_1^2+\alpha_2^2=1} [\alpha_1^2\|A\|_2^2 + \alpha_2^2\|B\|_2^2 + 2\alpha_1\alpha_2 trAB] \\ &= \frac{1}{2}[\|A\|_2^2 + \|B\|_2^2 + \sqrt{(\|A\|_2^2 - \|B\|_2^2)^2 + 4(trAB)^2}]. \end{aligned}$$

Corollary 2.2: *Let $T_1 \in C_2$. Then*

$$w_2(T_1) \leq \sqrt{\|\Re(T_1)\|_2^2 + \|\Im(T_1)\|_2^2}.$$

Proof: The desired result can be deduced from (1.1) and Theorem 2.1.

Theorem 2.3: *Let* $T_1 \in C_2$ *with* $T_1 = A + iB$. *Then*

$$\frac{1}{4}\| T_1T_1^* + T_1^*T_1 \|_2 + \frac{1}{2}\sqrt{(\| A \|_2^2 - \| B \|_2^2)^2 + 4(trAB)^2} \leq w_2^2(T_1).$$

Proof: We have

$$\begin{aligned} w_2^2(T_1) &\geq \frac{1}{2}[\| A^2 \|_2 + \| B^2 \|_2 + \sqrt{(\| A \|_2^2 - \| B \|_2^2)^2 + 4(trAB)^2}\,] \\ &\geq \frac{1}{2}[\| A^2 + B^2 \|_2 + \sqrt{(\| A \|_2^2 - \| B \|_2^2)^2 + 4(trAB)^2}\,]. \end{aligned}$$

We have $A^2 + B^2 = \frac{1}{2}(T_1T_1^* + T_1^*T_1)$, so the required statement is obtained.

Theorem 2.4: *Let* $T_1 \in C_2$. *Then*

$$w_2^2(T_1) \geq \frac{1}{2}\| T_1 \|_2^2 + \frac{1}{2}\left|\Re(trT_1^2)\right|.$$

Proof: We have

$$\begin{aligned} w_2^2(T_1) &\geq \max\{\| \Re(T_1) \|_2^2, \| \Im(T_1) \|_2^2\} \\ &= \frac{1}{2}(\| \Re(T_1) \|_2^2 + \| \Im(T_1) \|_2^2 + \left|\| \Re(T_1) \|_2^2 - \| \Im(T_1) \|_2^2\right|) \\ &= \frac{1}{2}\| T_1 \|_2^2 + \frac{1}{2}\left|\Re(trT_1^2)\right|. \end{aligned}$$

Remark 2.5: The lower bound of $w_2(\cdot)$ in Theorem 2.4 is an improvement of the one in (1.2).

The following lemma is needed for the rest of this work.

Lemma 2.6: [2] *Let* $A_1, A_2 \in C_2$. *Then*

1. $w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}} \sup_{\theta \in \mathbb{R}} \| e^{i\theta}A_1 + e^{-i\theta}A_2^* \|_2$.
2. $w_2\left(\begin{bmatrix} 0 & A_1 \\ A_1 & 0 \end{bmatrix}\right) = \sqrt{2}w_2(A_1)$.
3. $w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & A_2 \\ A_1 & 0 \end{bmatrix}\right)$.
4. $w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) = w_2\left(\begin{bmatrix} 0 & A_1 \\ e^{i\phi}A_2 & 0 \end{bmatrix}\right)$ for every $\phi \in \mathbb{R}$.

Theorem 2.7: *Let* $A_1, A_2 \in C_2$. *Then*

$$\| A_1 \pm A_2^* \|_2 \leq \sqrt{2} w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) \leq (\| A_1 + A_2^* \|_2^2 + \| A_1 - A_2^* \|_2^2)^{\frac{1}{2}}. \tag{2.1}$$

Proof: Using Lemma 2.6 (1), we get

$$\sqrt{2} w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) = \sup_{\phi\in\mathbb{R}} \| \cos\phi(A_1 + A_2^*) + i \sin\phi(A_1 - A_2^*) \|_2$$
$$\leq (\| A_1 + A_2^* \|_2^2 + \| A_1 - A_2^* \|_2^2)^{\frac{1}{2}}.$$

To prove the first inequality, it is sufficient to take $\phi = 0$ and $\phi = \frac{\pi}{2}$.

Remark 2.8: The lower bound in (2.1) is a refinement of the inequality

$$\frac{1}{\sqrt{2}} \| A_1 + A_2 \|_2 \leq w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2^* & 0 \end{bmatrix}\right),$$

which was given in [2]. Moreover, if we take A_1 and A_2 to be self-adjoint in Theorem 2.7, then the inequalities in (2.1) refine the ones in (1.3).

Corollary 2.9: *Let* $A_1 \in C_2$. *Then*

$$\frac{1}{2} \| A_1 \pm A_1^* \|_2 \leq w_2(A_1) \leq (\| \Re(A_1) \|_2^2 + \| \Im(A_1) \|_2^2)^{\frac{1}{2}}.$$

Proof: The desired statement can be deduced by making $A_1 = A_2$ in (2.1) and using Lemma 2.6 (2).

Corollary 2.10: *Let* $A_1, A_2 \in C_2$. *Then*

$$w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) \leq \sqrt{\| A_1 \|_2^2 + \| A_2 \|_2^2}.$$

Proof: The required statement follows by using the *parallelogram law* in (2.1)

$$\| A_1 + A_2^* \|_2^2 + \| A_1 - A_2^* \|_2^2 = 2(\| A_1 \|_2^2 + \| A_2 \|_2^2).$$

Theorem 2.11: *Let* $A_1, A_2 \in C_2$. *Then*

$$\| A_1 \pm iA_2^* \|_2 \le \sqrt{2} w_2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) \le (\| A_1 + iA_2^* \|_2^2 + \| A_1 - iA_2^* \|_2^2)^{\frac{1}{2}}.$$

Proof: The assertion can be obtained directly by using Theorem 2.7 and also Lemma 2.6 (4).

Corollary 2.12: *Let* $T_1 \in C_2$ *with* $T_1 = A + iB.$ *Then*

$$\frac{1}{\sqrt{2}} \| T_1 \|_2 \le w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le \| T_1 \|_2 . \tag{2.2}$$

Clearly, the inequalities (2.2) are refinements of the following inequalities

$$\frac{w_2(T_1)}{2} \le \frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \le w_2(T_1),$$

which was already given in [2].

Corollary 2.13: *Let* $A_1 \in C_2$. *Then*

$$\| A_1 \pm iA_1^* \|_2 \le 2w_2(A_1) \le (\| A_1 + iA_1^* \|_2^2 + \| A_1 - iA_1^* \|_2^2)^{\frac{1}{2}}.$$

Theorem 2.14: *Let* $A_1, A_2 \in C_2$. *Then*

$$\max\{\| A_1^2 - (A_2^*)^2 \|_2, \| A_1^2 + (A_2^*)^2 \|_2\} \le 2w_2^2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right).$$

Proof: From the first inequality in (2.1), we have

$$\begin{aligned} 4w_2^2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) &\ge \| A_1 + A_2^* \|_2^2 + \| A_1 - A_2^* \|_2^2 \\ &\ge \| (A_1 + A_2^*)^2 \|_2 + \| (A_1 - A_2^*)^2 \|_2 \\ &\quad \text{(using the inequality } \| X^n \|_2 \le \| X \|_2^n, \text{for any operator } X) \\ &\ge \| (A_1 + A_2^*)^2 + (A_1 - A_2^*)^2 \|_2 \\ &\quad \text{(using the triangle inequality)} \\ &= \| 2A_1^2 + 2(A_2^*)^2 \|_2. \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
4w_2^2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right) &\geq \| A_1 + iA_2^* \|_2^2 + \| A_1 - iA_2^* \|_2^2 \\
&\geq \| (A_1 + iA_2^*)^2 \|_2 + \| (A_1 - iA_2^*)^2 \|_2 \\
&\geq \| (A_1 + iA_2^*)^2 + (A_1 - iA_2^*)^2 \|_2 \\
&= \| 2A_1^2 - 2(A_2^*)^2 \|_2.
\end{aligned}
$$

Corollary 2.15: *Let* $A_1 \in C_2$. *Then*

$$\frac{1}{4}\max\{\| A_1^2 - (A_1^*)^2 \|_2, \| A_1^2 + (A_1^*)^2 \|_2\} \leq w_2^2(A_1).$$

Theorem 2.16: *Let* $A_1, A_2 \in C_2$. *Then*

$$\frac{1}{2}\| A_1A_2^* + A_2^*A_1 \|_2 \leq w_2^2\left(\begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}\right).$$

Lemma 2.17: [7] *Let* $c, b \geq 0$ *and* $0 \leq \alpha \leq 1$. *Then*

1. $(\alpha c^r + (1-\alpha)b^r)^{\frac{1}{r}} \leq (\alpha c^s + (1-\alpha)b^s)^{\frac{1}{s}}$ for $0 < r \leq s$.
2. $(c^s + b^s)^{\frac{1}{s}} \leq (c^r + b^r)^{\frac{1}{r}}$ for $0 < r \leq s$.

Theorem 2.18: *Let* $T_1 \in C_2$ *with* $T_1 = A + iB$. *Then*

1. $w_2^r(T_1) \leq 2^{\frac{r}{2}-1}(\| A \|_2^r + \| B \|_2^r)$ *for* $r \geq 2$.
2. $w_2^r(T_1) \leq \| A \|_2^r + \| B \|_2^r$ *for* $0 < r \leq 2$.

Proof: From the inequality in Corollary (2.2), we have

$$w_2(T_1) \leq \sqrt{\| A \|_2^2 + \| B \|_2^2}.$$

Using Lemma 2.17 (1), we get

$$
\begin{aligned}
\frac{w_2(T_1)}{\sqrt{2}} &\leq \left(\frac{\|A\|_2^2 + \|B\|_2^2}{2}\right)^{\frac{1}{2}} \\
&\leq \left(\frac{\|A\|_2^r + \|B\|_2^r}{2}\right)^{\frac{1}{r}} \\
&= 2^{-\frac{1}{r}}(\| A \|_2^r + \| B \|_2^r)^{\frac{1}{r}}.
\end{aligned}
$$

Again, using Lemma 2.17 (2), we obtain

$$\begin{aligned} w_2(T_1) &\le (\|A\|_2^2 + \|B\|_2^2)^{\frac{1}{2}} \\ &\le (\|A\|_2^r + \|B\|_2^r)^{\frac{1}{r}}. \end{aligned}$$

Theorem 2.19: *Let $A_1, A_2 \in C_2$. Then*

$$\sqrt{2}\max\{w_2(A_1), w_2(A_2)\} \le w_2\left(\begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}\right).$$

Proof: Let $T_1 = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$ and $U_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, then

$$\begin{aligned} w_2(T_1) &= \frac{1}{2}(w_2(T_1) + w_2(U_1^* T_1 U_1)) \\ &\ge \frac{1}{2}\max\{w_2(T_1 + U_1^* T_1 U_1), w_2(T_1 - U_1^* T_1 U_1)\} \\ &= \sqrt{2}\max\{w_2(A_1), w_2(A_2)\}. \end{aligned}$$

Proposition 2.20: Let $T_1 = [T_{ij}]$, where $T_{ij} \in C_2$, for $i, j = 1, \ldots, n$. Then

$$\|T_1\|_2^2 = \|[\|T_{ij}\|_2]\|_2^2 = \sum_{i,j=1}^{n} \|T_{ij}\|_2^2 .$$

Theorem 2.21: *Let $T_1 = [T_{ij}]$, where $T_{ij} \in C_2$, for $i, j = 1, \ldots, n$. Then*

$$w_2(T_1) \le w_2([t_{ij}]),$$

where

$$t_{ij} = \begin{cases} w_2(T_{ii}) & \text{if} \quad i = j, \\ \frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} 0 & T_{ij} \\ T_{ji} & 0 \end{bmatrix}\right) & \text{if} \quad i \ne j. \end{cases}$$

Proof: We have

$$\|\Re(e^{i\theta}T_1)\|_2 = \left\| \begin{bmatrix} \Re(e^{i\theta}T_{11}) & \frac{1}{2}(e^{i\theta}T_{12} + e^{-i\theta}T_{21}^*) & \ldots & \frac{1}{2}(e^{i\theta}T_{1n} + e^{-i\theta}T_{n1}^*) \\ \frac{1}{2}(e^{i\theta}T_{21} + e^{-i\theta}T_{12}^*) & \Re(e^{i\theta}T_{22}) & \ldots & \frac{1}{2}(e^{i\theta}T_{2n} + e^{-i\theta}T_{n2}^*) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}(e^{i\theta}T_{n1} + e^{-i\theta}T_{1n}^*) & \frac{1}{2}(e^{i\theta}T_{n2} + e^{-i\theta}T_{2n}^*) & \ldots & \Re(e^{i\theta}T_{nn}) \end{bmatrix} \right\|_2$$

$$= \left\| \begin{bmatrix} \|\Re(e^{i\theta}T_{11})\|_2 & \frac{1}{2}\|e^{i\theta}T_{12}+e^{-i\theta}T_{21}^*\|_2 & \cdots & \frac{1}{2}\|e^{i\theta}T_{1n}+e^{-i\theta}T_{n1}^*\|_2 \\ \frac{1}{2}\|e^{i\theta}T_{21}+e^{-i\theta}T_{12}^*\|_2 & \|\Re(e^{i\theta}T_{22})\|_2 & \cdots & \frac{1}{2}\|e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^*\|_2 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\|e^{i\theta}T_{n1}+e^{-i\theta}T_{1n}^*\|_2 & \frac{1}{2}\|e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^*\|_2 & \cdots & \|\Re(e^{i\theta}T_{nn})\|_2 \end{bmatrix} \right\|_2$$

$$\leq \|[t_{ij}]\|_2$$

(by the norm monotonicity and by Lemma 2.6 (1)).

Since the matrix $[t_{ij}]$ is real symmetric, then we have $\|[t_{ij}]\|_2 = w_2([t_{ij}])$. Hence

$$w_2(T_1) \leq w_2([t_{ij}]).$$

Theorem 2.22: *Let* $A_1, A_2, A_3, A_4 \in C_2$. *Then*

$$w_2\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) \leq \sqrt{w_2^2(A_1) + w_2^2(A_4) + w_2^2\left(\begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}\right)}. \tag{2.3}$$

Proof: By Theorem 2.21 and Lemma 2.6 (3), it follows that

$$\begin{aligned} w_2\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right) &\leq w_2\left(\begin{bmatrix} w_2(A_1) & \frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}\right) \\ \frac{1}{\sqrt{2}} w_2\left(\begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix}\right) & w_2(A_4) \end{bmatrix}\right) \\ &= w_2(M). \end{aligned}$$

Since M is real symmetric, then $w_2(M) = \|M\|_2$. This completes the proof.

References

[1] Abdullah Abu-Omar and Fahd Kittaneh, "A generalization of the numerical radius," *Linear Algebra Appl.*, vol. 569, pp. 323-334 (2019).

[2] Abdullah Aldalabih and Fahd Kittaneh, "Hilbert-Schmidt numerical radius inequalities for operator matrices," *Linear Algebra Appl.*, vol. 581, pp. 72-84 (2019).

[3] William Audeh, "Hilbert-Schmidt numerical radius inequalities for 2 × 2 operator matrices," *Inter. J. Math. Comp. Sci.*, vol. 16, pp. 1161-1167 (2021).

[4] Ravi Bhatia, Matrix Analysis, Springer, New York (1997).

[5] Mouhamad Guesba, "On some numerical radius inequalities for normal operators in Hilbert spaces," *J. Interdiscip. Math.*, vol. 25, no. 2, pp. 463-470 (2022), doi: 10.1080/09720502.2021.1930658.

[6] Mohammad Hajmohamadi and Reza Lashkaripour, "Some inequalities involving Hilbert-Schmidt numerical radius on 2×2 operator matrices," *Filomat*, pp. 4649-4657 (2020).

[7] G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, 2nd ed., Cambridge, UK: Cambridge Univ. Press (1988).

Received December, 2022