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Soft topological soft polygroups

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Abstract

There are many uncertainties in the real world. Therefore, existence of some structures to support these uncertainties is necessary. To achieve this goal, soft sets are inserted into mathematical structures; in this regard, polygroups are equipped with variety of topologies and soft sets. In this article, firstly polygroups are combined with soft sets, then soft topological polygroups are introduced and a few examples have been examined.

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1. Introduction

Many scientists have tried to achieve the above goals, including Molodstov in [11] and this theory was extended and combined with fuzzy set and rough set by Feng et al. in [5, 6, 7, 8]. Soft groups were introduced by Aktas in [3] and soft rings were presented by Acar in [2]. The concept of fuzzy topological polygroups were presented by Abbasizadeh, Davvaz

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and proved interesting results in [1]. The topological polygroups improvements were completely done by Heidari and Davvaz in [9].

An attempt is to combine the uncertainty of the soft sets with the polygroups as the base set. For this purpose, the topics such as soft sets, polygroups, topological polygroups, soft polygroups, soft topological polygroups, soft dual-topological polygroups are discussed. It is tried that the definition which are related to polygroups do not have conflict with that are provided for the groups. For instance, the definition of the topological polygroup is the generalization of the topological group definition.

2. Preliminaries

2.1 *Algebraic Hyperstructures*

Hypergroups are a natural generalization of groups that appear in many natural phenomena.

Definition 2.1: If $K \neq \varnothing$ is a set and $\star$ is a hyper-operation on *K*, then $(K, \star)$ is a hypergroup when

(1) $(v_1 \star v_2) \star v_3 = v_1 \star (v_2 \star v_3)$, for every $v_1, v_2, v_3 \in K$;

(2) $\lambda \star K = K \star \lambda = K$, for every $\lambda \in K$.

In Definition 2.1, if K_1 and K_2 are non-empty subsets of *K*, then

$$\bigcup_{\substack{k_1 \in K_1 \\ k_2 \in K_2}} k_1 \star k_2.$$

If only the first condition of Definition 2.1 is satisfied, then *K* is a semihypergroup and if only the second condition is satisfied, then *K* is a quasihypergroup. For more details see [4].

2.2 *Polygroups*

An important category of hypergroups is polygroups, whose algebraic structure is very close to the algebraic structure of groups.

Definition 2.2: A system $U = (U, \star, e, ^{-1})$, where *e* is a member of *U*, is called a polygroup if

(1) $(v_1 \star v_2) \star v_3 = v_1 \star (v_2 \star v_3)$, for every $v_1, v_2, v_3 \in K$;

(2) $\lambda \star e = e \star \lambda = \lambda$, for every $\lambda \in U$;

(3) If $\lambda \in \mu \star v$, then $\mu \in \lambda \star \mu^{-1}$ and $v \in \mu^{-1} \star \lambda$, for every $\lambda, \mu, v \in U$.

Example 1: Suppose that $U = \{e, \lambda, \mu, \nu, \tau\}$ together with the following hyperoperation:

$\star$	e	λ	μ	ν	τ
e	e	λ	μ	ν	τ
λ	λ	e	μ	ν	τ
μ	μ	μ	$\{e, \lambda\}$	τ	ν
ν	ν	ν	τ	$\{e, \lambda\}$	μ
τ	τ	τ	ν	μ	$\{e, \tau\}$

Then U is a polygroup [4].

2.3 *Soft Polygroups*

Suppose that U is a polygroup and $(\mathcal{H}, A)$ is a non-null soft set (in short SS) over U.

If $\mathcal{H}(z)$ is a subpolygroup (respectively, normal subpolygroup) of U, for every z in $Supp(\mathcal{H}, A)$, then $(\mathcal{H}, A)$ is a soft polygroup, (in short SP) (respectively, normal SP) over U.

Example 2: Take the polygroup $U = \{e, \lambda, \mu, \nu\}$ with the following hyperoperation on U:

$\star$	e	λ	μ	ν
e	e	λ	μ	ν
λ	λ	$\{e, \lambda\}$	ν	$\{\mu, \nu\}$
μ	μ	ν	e	λ
ν	ν	$\{\mu, \nu\}$	λ	$\{e, \lambda\}$

Assume that $(\mathcal{H}, A)$ is a SS over U, so that $A = \mathcal{U}$ and $\mathcal{H}$ from A to $\mathcal{U}(U)$ is defined by $\mathcal{H}(a) = \{b \in \mathcal{U} \mid aRb \Leftrightarrow b \in a^2\}$, for every $a \in A$. Consequently, we obtain $\mathcal{H}(e) = \mathcal{H}(\mu) = \{e\}$ and $\mathcal{H}(\lambda) = \mathcal{H}(\nu) = \{e, \lambda\}$ are subpolygroups of U. This yields that $(\mathcal{H}, A)$ is a SP over U [14].

2.4 *Soft Topological Polygroups*

For details about topological groups we refer to [10].

Suppose that $(\mathcal{H}, A)$ is a SS over G. The $(\mathcal{H}, A, \Theta)$ is a soft topological group (in short STG) if for every a in A:

(1) $\mathcal{H}(a)$ is a subgroup of G,

(2) $\mathcal{H}(a)\times\mathcal{H}(a)\to\mathcal{H}(a)$, where $(x,y)\mapsto xy$ is continuous,

(3) $\mathcal{H}(a)\mapsto\mathcal{H}(a)$, where $x\mapsto x^{-1}$ is continuous.

Example 3: Suppose that

$$G=S_3=\{e,(1\,2),(2\,3),(1\,3),(1\,3\,2),(1\,2\,3)\},$$

the symmetric group of degree 3, and $A=\{e_1,e_2,e_3\}$ in this case the base for the topology Θ, is

$$\beta=\{\{e\},\{(1\,2)\},\{(1\,3\,2)\},\{(1\,2\,3)\}\}.$$

The SS $\mathcal{H}$ with $\mathcal{H}(e_1)=\{e\}$, $\mathcal{H}(e_2)=\{e,(12)\}$ and $\mathcal{H}(e_3)=\{e,(1\,3\,2),(1\,2\,3)\}$. $\mathcal{H}(a)$ is a subgroup of G, for every $a\in A$. Furthermore, the continuity condition is satisfied. This shows that $(\mathcal{H},A,\Theta)$ is a STG.

In 2006, Heidari, Davvaz and Modarres defined and investigated the concept of topological polygroups [9].

Definition 2.3: Assume Θ is a topology on a polygroup U and $(\mathcal{H},A)$ a SS on U. Then the system $(\mathcal{H},A,\Theta)$ is said to be a soft topological polygroup (in short STP) over U if for every a in A:

(1) $\mathcal{H}(a)$ is a subpolygroup of U,

(2) $\mathcal{H}(a)\times\mathcal{H}(a)\to U(\mathcal{H}(a))$, where $(x,y)\mapsto x\star(y^{-1})$ is continuous.

Example 4: Suppose that (U,Θ) is a topological polygroup and $H_1,\ldots,H_n$ are all of the subpolygroups of U. Let A be a set and $\omega_1,\omega_2,\ldots,\omega_n$ are elements of A. We define $\mathcal{H}$ as follows:

$$\mathcal{H}(z)=\begin{cases} H_1 & \text{if } z=\omega_1\\ H_2 & \text{if } z=\omega_2\\ H_3 & \text{if } z=\omega_3\\ \vdots & \vdots\\ H_n & \text{if } z=\omega_n\\ \varnothing & \text{otherwise}\end{cases}$$

Then $(\mathcal{H},A,\Theta)$ is a STP.

Example 5: Assume that U is the set $\{e,\lambda,\mu\}$ together with the following hyperoperation:

$\star$	e	λ	μ
e	e	λ	μ
λ	λ	e	μ
μ	μ	μ	$\{e,\lambda\}$

Then the hyperoperation $\star$ is not continuous with the following topologies:

$$\Theta_1 = \{\varnothing, U, \{e\}\},$$

$$\Theta_2 = \{\varnothing, U, \{\lambda\}\},$$

$$\Theta_3 = \{\varnothing, U, \{\mu\}\},$$

$$\Theta_4 = \{\varnothing, U, \{e,\lambda\}\},$$

$$\Theta_5 = \{\varnothing, U, \{e,\mu\}\},$$

$$\Theta_6 = \{\varnothing, U, \{\lambda,\mu\}\}.$$

Obviously, $\varnothing, U, \{e\}, \{e,\lambda\}$ are subpolygroups of U. Suppose that A is a set such that ω_1, ω_2 are elements of A. Take

$$\mathcal{H}(z) = \begin{cases} \{e\} & \text{if } z = \omega_1 \\ \{e,\lambda\} & \text{if } z = \omega_2 \\ \varnothing & \text{otherwise} \end{cases}$$

This yields that $(\mathcal{H}, A, \Theta_3)$ and $(\mathcal{H}, A, \Theta_4)$ are STP, since the restriction of topologies Θ_3 and Θ_4 to the subspaces $\{e\},\{e,p\}$ are discrete or anti-discrete topologies.

Example 6: Suppose that U is $\{e,\lambda,\mu,\nu\}$ with the following hyperoperation:

$\star$	e	λ	μ	ν
e	e	λ	μ	ν
λ	λ	$\{e,\lambda\}$	ν	$\{\mu,\nu\}$
μ	μ	ν	e	λ
ν	ν	$\{\mu,\nu\}$	λ	$\{e,\lambda\}$

Hyperoperation $\star$ is continuous respect to the following topologies:

$$\Theta_{dis},$$

$$\Theta_{ndis},$$

$$\Theta_1 = \{\varnothing, U, \{e, \mu\}\},$$

$$\Theta_2 = \{\varnothing, U, \{e\}, \{\mu\}\}.$$

Then U respect to the topologies Θ_1 and Θ_2 is a TP. Hyperoperation $\star$ is not continuous with respect to the below topologies:

$$\Theta_3 = \{\varnothing, U, \{e\}\},$$

$$\Theta_4 = \{\varnothing, U, \{\lambda\}\},$$

$$\Theta_5 = \{\varnothing, U, \{\mu\}\},$$

$$\Theta_6 = \{\varnothing, U, \{\nu\}\},$$

$$\Theta_7 = \{\varnothing, U, \{e, \lambda\}\},$$

$$\Theta_8 = \{\varnothing, U, \{e, \nu\}\},$$

$$\Theta_9 = \{\varnothing, U, \{a, \mu\}\},$$

$$\Theta_{10} = \{\varnothing, U, \{\lambda, \nu\}\},$$

$$\Theta_{11} = \{\varnothing, U, \{\mu, \nu\}\},$$

$$\Theta_{12} = \{\varnothing, U, \{e, \lambda, \nu\}\},$$

$$\Theta_{13} = \{\varnothing, U, \{e, \lambda, \nu\}\},$$

$$\Theta_{14} = \{\varnothing, U, \{e, \mu, \nu\}\},$$

$$\Theta_{15} = \{\varnothing, U, \{\lambda, \mu, \nu\}\},$$

$$\Theta_{16} = \{\varnothing, U, \{e\}, \{\lambda\}\}.$$

Subpolygroups of U are $\varnothing, U, \{e\}, \{e, \lambda\}, \{e, \mu\}$. Let A be an arbitrary set and $\omega_1, \omega_2, \omega_3, \omega_4 \in A$ and define:

$$\mathcal{H}(z)=\begin{cases}\{e\} & \text{if } z=\omega_1\\ \{e,\lambda\} & \text{if } z=\omega_2\\ \{e,\mu\} & \text{if } z=\omega_3\\ \lambda & \text{if } z=\omega_4\\ \varnothing & \text{otherwise}\end{cases}$$

Consequently, $(\mathcal{H},A,\Theta_1)$ and $(\mathcal{H},A,\Theta_2)$ are STPs.

Next, take $\Theta_5=\{\varnothing,U,\{q\}\}$ and let:

$$\mathcal{H}(z)=\begin{cases}\{e\} & \text{if } z=\omega_1\\ \{e,\lambda\} & \text{if } z=\omega_2\\ \varnothing & \text{otherwise}\end{cases}$$

Hyperoperation $\star:\mathcal{H}(z)\times\mathcal{H}(z)\mapsto U(\mathcal{H}(z))$ is continuous respect to Θ_5. Hence, $(\mathcal{H},A,\Theta_5)$ is a STP.

Finally, suppose that $\Theta_6=\{\varnothing,U,\{\nu\}\}$ and consider:

$$\mathcal{H}(z)=\begin{cases}\{e\} & \text{if } z=\omega_1\\ \{e,\lambda\} & \text{if } z=\omega_2\\ \{e,\mu\} & \text{if } z=\omega_3\\ \varnothing & \text{otherwise}\end{cases}$$

Then, the hyperoperation $\star:\mathcal{H}(z)\times\mathcal{H}(z)\mapsto U(\mathcal{H}(z))$ is continuous with respect to Θ_6. This implies that $(\mathcal{H},A,\Theta_6)$ is a STP.

Example 7: Consider the non-commutative polygroup $U=\{e,\lambda,\mu,\nu\}$ with the hyperoperation:

$\star$	e	λ	μ	ν
e	e	λ	μ	ν
λ	λ	λ	U	ν
μ	μ	$\{e,\lambda,\mu\}$	μ	$\{\mu,\nu\}$
ν	ν	$\{\lambda,\nu\}$	ν	U

Then $\star$ is continuous with respect to the below topologies:

$$\Theta_{dis},$$

$$\Theta_{ndis},$$

$$\Theta_1=\{\varnothing,U,\{e\}\},$$

$$\Theta_2 = \{\varnothing, U, \{e, \lambda\}\},$$

$$\Theta_3 = \{\varnothing, U, \{e, \mu\}\},$$

$$\Theta_4 = \{\varnothing, U, \{e\}, \{\lambda\}\},$$

$$\Theta_5 = \{\varnothing, U, \{e\}, \{\mu\}\}.$$

-1 is not continuous according to $\Theta_2, \Theta_3, \Theta_4, \Theta_5$ and it is continuous with respect to Θ_1. This shows U with respect to the topologies Θ_1, Θ_{dis}, Θ_{ndis} is a TP.

The hyperoperation $\star$ with under the following topologies is not continuous.

$$\Theta_6 = \{\varnothing, U, \{\lambda\}\},$$

$$\Theta_7 = \{\varnothing, U, \{\mu\}\},$$

$$\Theta_8 = \{\varnothing, U, \{\nu\}\},$$

$$\Theta_9 = \{\varnothing, U, \{e, \nu\}\},$$

$$\Theta_{10} = \{\varnothing, U, \{\lambda, \nu\}\},$$

$$\Theta_{11} = \{\varnothing, U, \{\lambda, \nu\}\},$$

$$\Theta_{12} = \{\varnothing, U, \{\mu, \nu\}\},$$

$$\Theta_{13} = \{\varnothing, U, \{e, \lambda, \nu\}\},$$

$$\Theta_{14} = \{\varnothing, U, \{e, \lambda, \nu\}\},$$

$$\Theta_{15} = \{\varnothing, U, \{e, \mu, \nu\}\},$$

$$\Theta_{16} = \{\varnothing, U, \{\lambda, \mu, \nu\}\}.$$

$\varnothing, U, \{e\}$ are subpolygroups of U. Assume A is an arbitrary set, $\omega_1, \omega_2 \in A$ and consider:

$$\mathcal{H}(z)=\begin{cases}\{e\} & \text{if } z=\omega_1\\ U & \text{if } z=\omega_2\\ \varnothing & \text{otherwise}\end{cases}$$

Thus, $(\mathcal{H}, A, \Theta_1)$, $(\mathcal{H}, A, \Theta_{dis})$ and $(\mathcal{H}, A, \Theta_{ndis})$ are STPs.

Finally, put:

$$\mathcal{H}(z)=\begin{cases}\{e\} & \text{if } z=\omega_1\\ \varnothing & \text{otherwise}\end{cases}$$

As $\star : \mathcal{H}(x)\times\mathcal{H}(x)\mapsto U(\mathcal{H}(x))$ and $^{-1}$ are continuous according to (Θ_i) $(i=1,\ldots,16)$. Thus, $(\mathcal{H}, A, \Theta_i)$ is a STP.

3. Soft Topological Soft Polygroups

In this section, we suppose that U is a polygroup and we show yje hyperoperation by "+" and $SS(U)$ is the family of all SSs over parameters E.

Definition 3.1: Suppose Θ is a ST on polygroup U and $\mathcal{H}\in SS(U)$ is a SP. Then $(\mathcal{H},\Theta)$ is a soft topological soft polygroup over U (STSP) if the below maps are SC:

(1) $f:(\mathcal{H}\hat{\times}\mathcal{H},\Theta\hat{\times}\Theta)\to(\mathcal{H},\Theta_{\mathcal{H}})$, where $f(x,y)=x+y$,

(2) $j:(\mathcal{H},\Theta_{\mathcal{H}})\to(\mathcal{H},\Theta_{\mathcal{H}})$, where $j(x)=-x$.

Theorem 3.2: *Let $\mathcal{H}$ be a SP over a polygroup U and Θ is a ST on U. Then $(\mathcal{H},\Theta)$ is an STSP over U iff we have*

(1) *For all $a,b\hat{\in}\mathcal{H}$ and every SONhd $\mathcal{H}_{a+b}$, exists an SONhd $\mathcal{H}_a$ and an SONhd $\mathcal{H}_b$, so that $\mathcal{H}_a\hat{+}\mathcal{H}_b\hat{\subseteq}\mathcal{H}_{a+b}$.*

(2) *For all $a\hat{\in}\mathcal{H}$ and every SONhd $\mathcal{H}_{-a}$, there must be an SONhd $\mathcal{H}_a$ such that $-\mathcal{H}_a\hat{\subseteq}\mathcal{H}_{-a}$.*

Proof: It is straightforward by using the definitions.

Theorem 3.3: *The mapping*

$$f:(\mathcal{H}\hat{\times}\mathcal{H},\Theta_{\mathcal{H}}\hat{\times}\Theta_{\mathcal{H}})\mapsto(\mathcal{H}\hat{\times}\mathcal{H},\Theta_{\mathcal{H}}\hat{\times}\Theta_{\mathcal{H}}),$$

$f(a,b)=(a,-b)$ *is SC, where $(\mathcal{H},\Theta)$ is an STSP over U.*

Proof: By [13], the projection mappings $proj_i : (\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times} \Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$, $proj_1\,(a,b) = a$ and $proj_2\,(a,b) = b$ are SC. In order to show $f(a,b) = (a,-b)$ is SC. According to [13] we have $proj_i \star f(a,b)$, $i = 1,2$ is SC. Also, $proj_1 \star f(a,b) : (\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$ is defined by $proj_1 \star f(a,b) = proj_1(a,-b) = a = proj_1(a,b)$, and so $proj_1 \star f = proj_1$. So, $proj_1 \star f$ is SC. Now $(\mathcal{H},\Theta)$ is an STSP, follows that $(\mathcal{H},\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$ defined by $j(a) = -a$ is SC. Then $j \star proj_2(a,b) : (\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$ is defined by $j(b) = -b$ is SC, since $proj_2 \star f(a,b) = proj_2(a,-b) = -b = j \star proj_2(a,b)$. This concludes that $proj_2 \star f = j \star proj_2$ is SC. Thus, $f(a,b) = (a,-b)$ is SC from $(\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}})$ to $(\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}})$.

Theorem 3.4: *The STS $(\mathcal{H},\Theta)$ is an STSP iff the subtraction mapping $k(a,b) = a-b$ is SC from $(\mathcal{H}\hat{\times}\mathcal{H},\Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}})$ to $(\mathcal{H},\Theta_{\mathcal{H}})$, where that Θ is a ST on* U *and $\mathcal{H} \in SS(U)$ is a SP.*

Proof: $[\Rightarrow]$ Assume that $(\mathcal{H},\Theta)$ is an STSP. In this case,

$$f : (\mathcal{H}\hat{\times}\mathcal{H},\Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$$

by $f(a,b) = a+b$ is SC, for any $a,b \hat{\in} \mathcal{H}$. Now, $j : (\mathcal{H},\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$ which is defined by $j(b) = -b$ is SC $\forall b \hat{\in} \mathcal{H}$. Furthermore, the mapping $f : (\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$ which is defined by $f(a,b) = (a,-b)$ is SC $\forall a,b \hat{\in} \mathcal{H}$. Since $k(a,b) = f(a,b) \star f(a,b) : (\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$, it follows that the mapping $k(a,b) = a-b$ is SC.

$[\Leftarrow]$ Consider $k : (\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H},\Theta_{\mathcal{H}})$ by $k(a,b) = a-b$ is SC for all $a,b \hat{\in} \mathcal{H}$, since $\mathcal{H}$ is a SP, we have *e* is a soft element in $\mathcal{H}$. Specially, we have $a = e_a$ and $b \in \mathcal{H}$. So $k(a,b)$ is SC at (e,b). In this case, by definition of SC, for any SONhd $\mathcal{H}_{e-b} = \mathcal{H}_{-b}$ of $e-b = -b$, there exists an SONhd $\mathcal{H}_e$ and SONhd $\mathcal{H}_b$ of *b*. Such that $\mathcal{H}_e \hat{-} \mathcal{H}_b \hat{\subseteq} \mathcal{H}_{-b}$. .

In particular, we have $-\mathcal{H}_b \hat{\subseteq} \mathcal{H}_{-b}$, Thus we have the inverse mapping $j(b) = -b$ is SC for all $b \hat{\in} \mathcal{H}$. Also, from Theorem 3.3,

$$f : (\mathcal{H}\hat{\times}\mathcal{H},\Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}}) \mapsto (\mathcal{H}\hat{\times}\mathcal{H},\Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}})$$

by $f(a,b) = (a,-b)$, for all $a,b \hat{\in} \mathcal{H}$, is SC. Thus, $f(a,b) = k(a,b) \star f(a,b)$ is SC from $(\mathcal{H}\hat{\times}\mathcal{H},\ \Theta_{\mathcal{H}} \hat{\times}\Theta_{\mathcal{H}})$ to $(\mathcal{H},\Theta_{\mathcal{H}})$, $\forall a,b \hat{\in} \mathcal{H}$

Theorem 3.5: *The system $(\mathcal{H},\Theta)$ is an STSP over U, if (U,Θ) be an $\widehat{STP}$ and $\mathcal{H} \in SS(U)$ a SP.*

Proof: Assume that (U,Θ) is an $\widehat{STP}$ on *U*. Thus $\forall a,b \in U$, and arbitrary SONhd $\mathcal{H}_{a-b}$ there must be an SONhd $\mathcal{H}_a \in \Theta$ and an SONhd $\mathcal{H}_b \in \Theta$ such that $a-b \hat{\in} \mathcal{H}_a - \mathcal{H}_b \hat{\subseteq} \mathcal{H}_{a-b}$. Since $\mathcal{H}$ is a SP over U, $\mathcal{H}(e)$ is a subpolygroup of U for all $e \in E$. In this case, for any $a,b \hat{\in} \mathcal{H}$ then $a,b \in U$. Consider $a,b \hat{\in} \mathcal{H}$. Then for all arbitrary SONhd $\mathcal{H}_{a-b}$ of $a-b$ there is an SONhd $\mathcal{H}_a \in \Theta$ and an SONhd $\mathcal{H}_b \in \Theta$, such that $a-b \hat{\in} \mathcal{H}_a - \mathcal{H}_b \hat{\subseteq} \mathcal{H}_{a-b}$. Conclude $a-b \hat{\in} \mathcal{H} \hat{\cap} \mathcal{H}_a - \mathcal{H} \hat{\cap} \mathcal{H}_b \hat{\subseteq} \mathcal{H} \hat{\cap} \mathcal{H}_{a-b}$. Because the soft open sets of $\Theta_{\mathcal{H}}$ are in the form of $\mathcal{H} \hat{\cap} \mathcal{H}$, with $\mathcal{H} \in \Theta$ thus we conclude $\mathcal{H} \hat{\cap} \mathcal{H}_a$, $\mathcal{H} \hat{\cap} \mathcal{H}_b$ and $\mathcal{H} \hat{\cap} \mathcal{H}_{a-b}$ are soft open sets in $\Theta_{\mathcal{H}}$. Therefore, $(\mathcal{H},\Theta)$ is an STSP over U.

The converse of Theorem 3.5 does not hold.

Example 8: Suppose that $U = \mathbb{Z}_6$, $E = \{\alpha_1, \alpha_2\}$ and $\mathcal{H}$ is an SS in the form of $\mathcal{H}(\alpha_1) = \{0,2,4\}$ and $\mathcal{H}(\alpha_2) = \{0\}$. Assume that Θ is defined by

$$\begin{aligned}\Theta &= \{\hat{\varnothing}, \{(\alpha_1,\{0\}),(\alpha_2,\{0\})\}, \{(\alpha_1,\{2\}),(\alpha_2,\{0,1\})\}, \\ &\quad \{(\alpha_1,\{0,2\}),(\alpha_2,\{0,1,2\})\}, \widehat{\mathbb{Z}_4}\}.\end{aligned}$$

Since $\mathcal{H}(\alpha_1), \mathcal{H}(\alpha_2)$ are subgroups of U, it follows that $\mathcal{H}$ is an SP on U.

Consider the only soft element belongs to $\mathcal{H}$ is 0. Thus, the mapping $(a,b) \mapsto a-b$ is SC So, $(\mathcal{H},\Theta)$ is an STSP on U. Furthermore, we assert that (U,Θ) is not $\widehat{STP}$. To prove it suppose that $a=3$ and $b=1$. In this case $2 \hat{\in} \{(\alpha_1, \{0,2\}), (\alpha_2, \{0,1,2\})\}$. (U,Θ) to be an $\widehat{STP}$ there exist an SONhd $\mathcal{H}_3$ of 3 and an SONhd $\mathcal{H}_1$ of 1, such that $a-b \hat{\in} \mathcal{H}_3 - \mathcal{H}_1 \hat{\subseteq} \mathcal{H}_2$. But the only SONhd of 3 and 1 is $\widehat{\mathbb{Z}_4}$. Furthermore, $\widehat{\mathbb{Z}_4} \hat{-} \widehat{\mathbb{Z}_4} = \widehat{\mathbb{Z}_4} \hat{\nsubseteq} \{(\alpha_1,\{0,2\}), (\alpha_2,\{0,1,2\})\}$.

Thus, (U,Θ) is not an $\widehat{STP}$.

If we generalize Nazmul and Samanta (2012)'s definition of STSG [12] to polygroups, it will be as follows.

Definition 3.6: Let $\mathcal{H}$ be a SP over U and Θ be a ST on U, then $(\mathcal{H},\Theta)$ is called a SP over U if for each $e \in E$, $(\mathcal{H}(e),(\Theta^e)_{\mathcal{H}(e)})$ is a TP on $(\mathcal{H}(e))$.

Next, we show Definition 3.6 is entirely different from Definition 3.1 for an STSP.

Theorem 3.7: *Assume Θ is an ST on U and $\mathcal{H} \in SS(U)$ is a SP and $(\mathcal{H},\Theta)$ is an STSP on U, then $(\mathcal{H}(e),(\Theta^e)_{\mathcal{H}(e)})$ is a TP for every $e \in E$.*

Proof: Let $(\mathcal{H},\Theta)$ be an STSP on U. Hence, Theorem 3.4 shows that for every $a,b \hat{\in} \mathcal{H}$, and an arbitrary SONhd $\mathcal{H}_{a-b}$ there exist an SONhd $\mathcal{H}_a$ and an SONhd $\mathcal{H}_b$, such that $a-b \hat{\in} \mathcal{H}_a - \mathcal{H}_b \hat{\subseteq} \mathcal{H}_{a-b}$, for every $e \in E$.

Since $(\Theta^e)_{\mathcal{H}(e)} = \{U(e) | U \in \Theta_{\mathcal{H}}\}$, it proves that $\mathcal{H}_a(e), \mathcal{H}_b(e)$ and $\mathcal{H}_{a-b}(e)$ are open sets in $(\Theta^e)_{\mathcal{H}(e)}$. Thus, by reference to definition of TP conclude that the system $(\mathcal{H}(e),(\Theta^e)_{\mathcal{H}(e)})$ is a TP.

4. Future work

We can equip polygroups with two topologies, a typical topology τ and a ST Θ. In this case, we will be dealing with dual-topological. This can be another horizontal illuminator.

Definition 4.1: Let Θ be a ST on U and $\mathcal{H}$ be a SP over U and τ be a general topology on U. Then $(\mathcal{H},\Theta,\tau)$ is called a soft dual-topological polygroup over U if

(1) for every $a,b \hat{\in} \mathcal{H}$ and every SONhd $\mathcal{H}_{a+b}$ of $a+b$, there exist an SONhd $\mathcal{H}_a$ of a and an SONhd $\mathcal{H}_b$ of b, such that $\mathcal{H}_a \hat{+} \mathcal{H}_b \hat{\subseteq} \mathcal{H}_{a+b}$,

(2) for every $a \hat{\in} \mathcal{H}$ and every SONhd $\mathcal{H}_{-a}$ of $-a$, there must be an SONhd $\mathcal{H}_a$ of a such that $-\mathcal{H}_a \hat{\subseteq} \mathcal{H}_{-a}$,

(3) for every $e \in E$ and for all $a,b \in \mathcal{H}(e)$ the mapping $(a,b) \mapsto a-b$ from $\mathcal{H}(e) \times \mathcal{H}(e)$ to $U(\mathcal{H}(e))$ is continuous. Where the $\mathcal{H}(e)$, $\mathcal{H}(e) \times \mathcal{H}(e)$ are equipped with induced topographies $\tau_{\mathcal{H}(e)}$, $\tau_{\mathcal{H}(e)} \times \tau_{\mathcal{H}(e)}$.

Instead of (1), (2) we can write $(\mathcal{H},\Theta)$ be $\widehat{STP}$ and instead of (3) write $(\mathcal{H}(e),\tau)$ be TP for all $e \in E$.

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