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Convergent piecewise series solutions of ordinary differential equations

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Abstract

The series approximations of ordinary differential equations usually possess a limited validity range. Outside the convergence interval, the solution does not comply with the real solution. A technique is proposed in this work to expand the validity interval of the series approximations. Solutions are constructed in the vicinity of several points and are joined to each other with satisfaction of the continuity conditions. By employing the piecewise series solutions, the validity range of the approximations can be made to cover the whole domain of interest. In constructing a single analytical expression from the piecewise series, the Gamma Interval function is used. Sample problems of first, second and third order are attacked with the proposed method and comparisons with the numerical and exact solutions are given. It is shown that the method can be effectively used in constructing approximate solutions of ordinary differential equations.

Subject Classification: *34A12, 34A25, 34A34, 34A45.*

Keywords: *Ordinary differential equations, Series solutions, Convergence, Gamma interval functions.*

1. Introduction

Determining series solutions of ordinary differential equations is a very common and widely used technique as an approximation of the real solution. The assumed polynomial form of the series is inserted into the equation with unknown coefficients which are determined from the resulting recursive relations and initial conditions. As more terms are employed in the series, the

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solution approaches to the real solution within the so called radius of convergence. The deficiencies of the method are: 1) A high number of terms may be required for convergent solutions, 2) The radius of convergence may be small and the solutions may possess limited range of applicability.

Series solution approach is well established and discussed in books on differential equations [11,27,35]. Contrary to the perturbation methods [23-25] which are mainly asymptotic in nature (converges for a few terms and diverges as more terms are added), the series solutions are convergent and inclusion of additional terms increases the precision of the solutions. Some analytical techniques such as Taylor Series Solutions [1,8,16,28,29,31], Numerical Taylor Series Method [7,19], Taylor Matrix Method [12,18,22,32], Homotopy Analysis Method [2,20], Adomian Decomposition Method [3,4,9,17,34], Perturbation-Iteration Method [5,6,10], Differential Transform Method [13,26,30], Homotopy Perturbation Method [15] and Variational Iteration Method [14] may also produce polynomial type series solutions as well as functional type series solutions. In the mentioned literature [1-10,12-20,22,26,28-32,34], detailed comparisons between the methods were outlined. Apart from the semi-analytical techniques, an exact analytical method in the form of a general complex integral transform has been proposed very recently [21]. The new method has been applied to problems such as concentration of drug in the blood, radioactive decay and static deflection of beams. For a first order differential equation system arising in Neural Network theory, special activation functions were taken [33]. For these special functions, exponential stabilization was proven.

In this study, an algorithm is proposed to expand the validity interval of the basic series solutions. A step size sufficiently small is taken and the piecewise convergent series solutions in the vicinity of each step are joined to each other to form a convergent solution over the whole domain of interest. It is shown that there is no need to take lots of terms for convergence, only a few terms may be adequate to represent the exact solution. This is done by decreasing the step size. There are two mechanisms to ensure convergence: 1) The number of terms in the series may be increased 2) The step size may be decreased. The optimum choice depends on the nature of the differential equations. Especially, for nonlinear differential equations, convergence over the whole domain with a single series expansion might be a real problem. First and second order sample equations which possess analytical solutions are treated. The solutions presented are expressed as single continuous solutions with the aid of Gamma Interval functions. Although the method is applicable to problems with initial values, it can also be implemented for problems with boundary values if combined with shooting techniques. The Blasius equation is treated finally as a boundary value problem and the solution is contrasted with the Runge-Kutta solution.

2. Theory of the proposed method

The nonlinear equation considered has m'th order highest derivatives

$$F\left(x, y, y', y'', \dots, y^{(m)}\right) = 0 \, , \, y^{(m)} = \frac{d^m y}{dx^m} \, , \, y^{(0)} = y \, . \tag{2.1}$$

The initial conditions are

$$y^{(j)}(0) = \alpha_j \; j = 0,1,2, \dots m-1, \tag{2.2}$$

for which a series solution is constructed in the vicinity of $x = x_i$

$$y^i(x) = \sum_{k=0}^{K} a_k^i (x - x_i)^k \;, x_i = ih, i = 0,1,2, \dots n-1, \tag{2.3}$$

where h is the constant step size for simplicity albeit it can be taken as variable for stiff problems and the series is truncated at a specific K value. Substitution of (2.3) into (2.1) yields the recursive relations for the coefficients in its most general form

$$a_{k+1}^i = f(a_k^i, a_{k-1}^i, \dots, a_1^i, a_0^i). \tag{2.4}$$

With the aid of initial conditions and recursive relation for $i = 0$, a_k^0 can be calculated in the vicinity of $x_0 = 0$ and hence the first piecewise series solution would then be y^0. The initial conditions at each further step would be

$$(y^i)^{(j)}(x_{i+1}) = (y^{i+1})^{(j)}(x_{i+1}), i = 0,1,2, \dots, n-1, j = 0,1,2, \dots, m-1. \tag{2.5}$$

These conditions can be named as continuity conditions and the number of the conditions is equal to the order of equation. Employing (2.4) and (2.5), each coefficient a_k^i can be computed. It would be good to express the piecewise solutions as a single expression valid over the whole domain of interest. In order to achieve this goal, a new function is proposed

$$\gamma[a,b)(x) = \begin{cases} 1 & a \le x < b \\ 0 & x < a, x \ge b \end{cases}, \tag{2.6}$$

where the left limit is a (included) and the right limit is b (excluded). The function is called Gamma Interval Function which is a unit function within the interval and zero outside it. The argument of the function is the independent variable x. Depending on the inclusion of the limits, indeed there are four possibilities one of which is shown above. As usual, the brackets represent inclusion of the limiting value to the domain and parentheses represent exclusion of the limiting value.

The final solution valid throughout the whole domain of interest can be expressed as a single expression

$$y(x) = \sum_{i=0}^{n-1} \sum_{k=0}^{K} a_k^i (x - x_i)^k \gamma[x_i, x_{i+1})(x). \tag{2.7}$$

Instead of a single series to represent the solution in the whole domain of interest, $x \epsilon [0, x_f]$, in the piecewise series solution method, the domain is divided into parts with $h = x_f/n$, $n \epsilon N, n > 1$. Invoking the ratio test, the convergence requires

$$\lim_{n\to\infty} \frac{a_{n+1}^i}{a_n^i} h < 1. \tag{2.8}$$

If a single expression had been used, then the ratio test would yield

$$\lim_{n\to\infty} \frac{a_{n+1}^0}{a_n^0} x < 1, \tag{2.9}$$

which is then hard to be satisfied for large values of x. By limiting h to smaller values, the convergence can be assured for divergent solutions of large x. It may happen that the single series may be convergent over the whole domain but requires many terms to be added for convergence. In that case, taking a smaller step size and dividing the domain into smaller intervals would make it possible to take only few terms in the series expansions.

3. Equations that do not explicity depend on the independent variable

When there is no explicit x-dependency, the differential equation would be

$$F\left(y, y', y'', \dots, y^{(m)}\right) = 0 \tag{3.1}$$

In that case there is no need to approximate the coefficients as well as the non-homogenous functions in a series. Sample problems will be treated below.

Example Problem 1

The differential equation which is nonlinear and first order

$$y' + y^2 = 0\,, y(0) = 1, \tag{3.2}$$

possesses an exact solution

$$y_e = \frac{1}{1+x}\,. \tag{3.3}$$

The series solution

$$y^i(x) = \sum_{k=0}^{K} a_k^i (x - x_i)^k\,, x_i = ih, i = 0,1,2, \dots n-1, \tag{3.4}$$

yields the recursive relation

$$a_{k+1}^i = -\frac{1}{k+1}\sum_{l=0}^{k} a_l^i\, a_{k-l}^i, k = 0,1,2, \dots K-1\,, i = 0,1,2, \dots n-1. \tag{3.5}$$

The above recursive relation can be solved in terms of the leading coefficients in each series

$$a_k^i = (-1)^k \left(a_0^i\right)^{k+1}\,, k = 1,2, \dots K\,, i = 0,1,2, \dots n-1. \tag{3.6}$$

The condition $y^0(0) = 1$ requires

$$a_0^0 = 1 \tag{3.7}$$

The continuity equations from (2.5)

$$y^i(x_{i+1}) = y^{i+1}(x_{i+1})\,, i = 0,1,2, \dots n-1, \tag{3.8}$$

produce the leading coefficients of the series solution at each step size

$$a_0^{i+1} = a_0^i \sum_{l=0}^{K} (-1)^l \left(a_0^i\right)^l h^l, i = 0,1,2, \dots n-2. \tag{3.9}$$

The piecewise series solution by employing the Gamma interval function is

$$y(x) = \sum_{i=0}^{n-1} \sum_{k=0}^{K} a_k^i (x - x_i)^k \gamma[x_i, x_{i+1})(x). \tag{3.10}$$

In open form, the result for cubic polynomials and n step sizes is

$$\begin{aligned}
&y(x) = (1 - x + x^2 - x^3)\gamma[0, h)(x) \\
&+a_0^1(1 - a_0^1(x-h) + (a_0^1)^2(x-h)^2 - (a_0^1)^3(x-h)^3)\gamma[h, 2h)(x) \\
&+a_0^2(1 - a_0^2(x-2h) + (a_0^2)^2(x-2h)^2 - (a_0^2)^3(x-2h)^3)\gamma[2h, 3h)(x) + \cdots \\
&+a_0^{n-1}(1 - a_0^{n-1}(x-(n-1)h) + (a_0^{n-1})^2(x-(n-1)h)^2
\end{aligned}$$

$$-(a_0^{n-1})^3(x-(n-1)h)^3)\gamma[(n-1)h, nh)(x), \tag{3.11}$$

where

$$a_0^{i+1} = a_0^i(1 - a_0^i h + (a_0^i)^2h^2 - (a_0^i)^3h^3),\ i = 0,1,2,\dots n-2. \tag{3.12}$$

The convergence criterion (Eq. 2.8) for a single series solution

$$x < 1, \tag{3.13}$$

is rather restrictive. Choosing a suitable step size

$$h < 1, \tag{3.14}$$

the convergence is assured in the piecewise series solutions (See the convergence criteria 2.8 and 2.9).

The third-power piecewise series solution (Eq. 3.11) and the seventh power single series solution

$$y = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7\ , \tag{3.15}$$

are contrasted with the exact one in Figure 1 for $h = 0.4$ ($n = 25$).

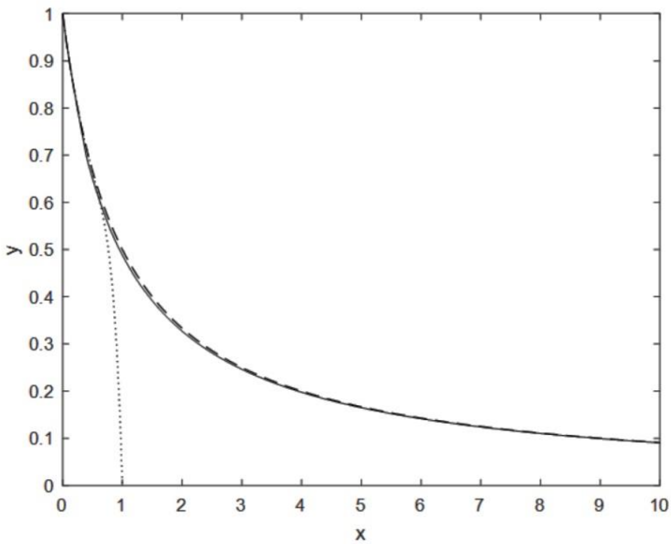

Figure 1
Comparison of the piecewise series solution (solid) and the single series solution (dotted) with the exact solution (dashed) (Problem 1, $h = 0.4$)

From the figure, the 7th power single series solution diverges immediately after the starting point. The single series is not expected to converge for $x \geq 1$. However, the 3rd power piecewise series solution perfectly matches with the exact function in the interval [0, 10] since h satisfies the convergence criterion given in Eq. (3.14).

The absolute errors at the end point $x = 10$ are given in Table 1. As the step size decreases, the absolute errors decrease also.

Table 1
Absolute errors at $x = 10$ for various step sizes (Problem 1)

h	Exact	Piecewise Series	Absolute Errors
0.4	0.090909	0.090408	$5.0155\ 10^{-4}$
0.2	0.090909	0.090863	$4.5781\ 10^{-5}$
0.1	0.090909	0.090904	$4.8487\ 10^{-6}$

Example Problem 2

The differential equation of second order

$$y'' + y'^2 = 0\ ,\ y(0) = 0\ ,\ y'(0) = 1 \tag{3.16}$$

has an exact solution

$$y_e = \ln(1 + x). \tag{3.17}$$

The series solution

$$y^i(x) = \sum_{k=0}^{K} a_k^i (x - x_i)^k\ ,\ x_i = ih, i = 0,1,2, \dots n - 1, \tag{3.18}$$

yields the recursive relation

$$a_k^i = -\frac{1}{k(k-1)} \sum_{l=0}^{k} (k - l) l a_{k-l}^i\, a_l^i,\ k = 2, \dots K\ , i = 0,1,2, \dots n - 1. \tag{3.19}$$

The above recursive relation can be solved for a general k'th term

$$a_k^i = \frac{(-1)^{k+1}}{k} \left(a_1^i\right)^k\ ,\ k = 2, \dots K\ , i = 0,1,2, \dots n - 1. \tag{3.20}$$

The initial conditions require

$$a_0^0 = 0\ ,\ a_1^0 = 1 \tag{3.21}$$

The continuity conditions

$$y^i(x_{i+1}) = y^{i+1}(x_{i+1})\ ,\ (y^i)'(x_{i+1}) = (y^{i+1})'(x_{i+1})\ ,\ i = 0,1,2, \dots n - 1\ , \tag{3.22}$$

are employed to calculate the first two coefficients in each of the piecewise series solutions

$$a_0^{i+1} = \sum_{l=0}^{K} a_l^i h^l,\ i = 0,1,2, \dots n - 1 \tag{3.23}$$

$$a_1^{i+1} = \sum_{l=1}^{K} l a_l^i h^{l-1},\ i = 0,1,2, \dots n - 1. \tag{3.24}$$

The final result by using the Gamma interval functions is

$$y(x) = \sum_{i=0}^{n-1} \sum_{k=0}^{K} a_k^i (x - x_i)^k \gamma[x_i, x_{i+1})(x), \tag{3.25}$$

where the coefficients are calculated using Equations (3.20), (3.21), (3.23) and (3.24). The convergence criterion (Eq. 2.8) for a single series solution

$$x < 1, \tag{3.26}$$

is confined to a narrow region. Choosing a suitable step size

$$h < 1, \tag{3.27}$$

the convergence is assured in the piecewise series solutions.

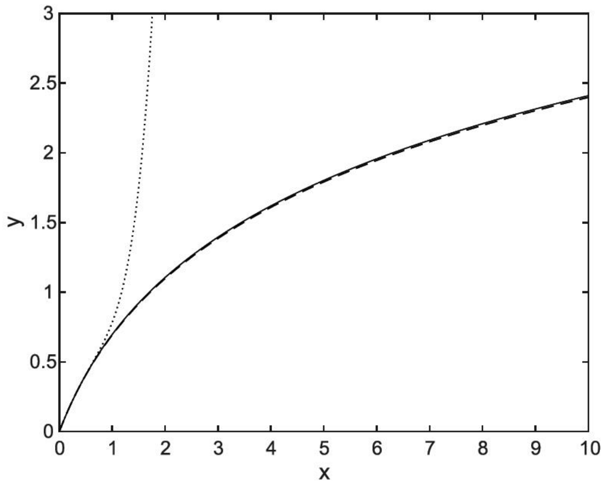

Figure 2
Comparison of the piecewise series solution (solid) and the single series solution (dotted) with the exact solution (dashed) (Problem 2, $h = 0.4$)

The fifth-power piecewise series solution and the fifth power single series solution

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5, \tag{3.28}$$

are contrasted with the exact function for $h = 0.4$ ($n = 25$) (Figure 2).

In the single series solution, addition of extra terms would not work, since the convergence criteria is not met for $x \geq 1$. Figure 2 depicts that the 5^{th} power single series solution has a very narrow range of validity. However, the 5^{th} power piecewise series solution produces excellent results for *h*=0.4.

The absolute errors at the end point $x = 10$ are given in Table 2. As the step size decreases, the absolute errors substantially decrease.

Table 2
Absolute errors at $x = 10$ for various step sizes (Problem 2)

h	Exact	Piecewise Series	Absolute Errors
0.4	2.397895	2.409828	$1.1933\ 10^{-2}$
0.2	2.397895	2.398421	$5.2552\ 10^{-4}$
0.1	2.397895	2.397922	$2.7020\ 10^{-5}$

The step sizes are the ranges for each piecewise analytical solution and should not be confused with the smaller step sizes of purely numerical algorithms. Hence, larger step sizes compared to purely numerical calculations can be taken in this series approach.

4. Equations that explicity depend on the independent variable

When the differential equation is variable coefficient and/or non-homogenous, the functions should also be expressed via a power series. Linear variable coefficient first and second order equations are solved below.

Theorem 1: *For the linear equation which is homogenous, first order and variable coefficient*

$$y' + b(x)y = 0 \, , \, y(0) = \alpha, \tag{4.1}$$

with $b(x)$ being an analytical function

$$b^i(x) = \sum_{k=0}^{K} b_k^i (x - x_i)^k \, , \tag{4.2}$$

the piecewise series solution is

$$y(x) = \sum_{i=0}^{n-1} \sum_{k=0}^{K} a_k^i (x - x_i)^k \gamma[x_i, x_{i+1})(x), \tag{4.3}$$

where

$$a_0^0 = \alpha \, , \, a_0^i = \sum_{k=0}^{K} a_k^{i-1} h^k, \, i = 1,2, \dots n-1, \tag{4.4}$$

$$a_{k+1}^i = -\frac{1}{k+1} \sum_{l=0}^{k} a_l^i \, b_{k-l}^i, \, k = 0,1,2, \dots K-1 \, , i = 0,1,2, \dots n-1 \tag{4.5}$$

An application of the theorem is given next.

Example Problem 3

Consider the variable coefficient linear first order equation

$$y' + e^x y = 0 \, , \, y(0) = 1, \tag{4.6}$$

for which the function

$$y_e = e^{(1-e^x)}. \tag{4.7}$$

satisfies exactly the equation. The coefficient is expanded first in a power series in the vicinity of $x = x_i$

$$b^i(x) = e^x = \sum_{k=0}^{K} b_k^i (x - x_i)^k \, , \tag{4.8}$$

where

$$b_k^i = \frac{1}{k!} e^{ih}. \tag{4.9}$$

For a series up to cubic powers, the coefficients are

$$a_0^{i+1} = a_0^i + a_1^i h + a_2^i h^2 + a_3^i h^3, \, i = 0,1,2, \dots n-2 \tag{4.10}$$

$$a_1^i = -a_0^i b_0^i, \, i = 1,2, \dots n-1 \tag{4.11}$$

$$a_2^i = -\frac{1}{2} a_0^i (b_1^i - (b_0^i)^2) \; i = 1,2, \dots n-1 \tag{4.12}$$

$$a_3^i = -\frac{1}{6} a_0^i (2b_2^i - 3b_0^i b_1^i + (b_0^i)^3) \; i = 1,2, \dots n-1. \tag{4.13}$$

The piecewise series solution is

$$y(x) = \sum_{i=0}^{n-1} a_0^i \left\{1 - e^{ih}(x - x_i) - \frac{1}{2} e^{ih}(1 - e^{ih})(x - x_i)^2 - \frac{1}{6} e^{ih}(1 - 3e^{ih} + e^{2ih})(x - x_i)^3\right\} \gamma[x_i, x_{i+1})(x) \, . \tag{4.14}$$

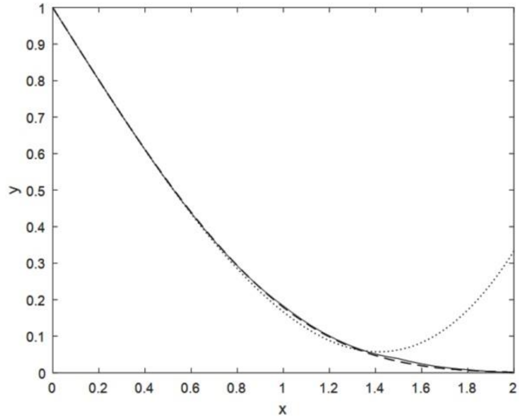

Figure 3
Comparison of the piecewise series (solid) and the single series (dotted) solutions with the exact solution (dashed) (Problem 3, $\boldsymbol{h = 0.5}$)

The above third-power piecewise series solution and the third-power single series solution

$$y = 1 - x + \frac{1}{6}x^3, \tag{4.15}$$

are contrasted with equation (4.7) for $h = 0.5$ ($n = 4$) (Figure 3).

While the single series solution blows up at approximately $x = 1.4$, the piecewise series solution displays a good match with the exact one.

The absolute errors at $x = 2$ are written in Table 3. As the step size decreases, the absolute errors decrease also. Four digit precision is achieved for $h = 0.1$ despite the consideration of only few terms in the series.

Table 3
Absolute errors at $\boldsymbol{x = 2}$ for various step sizes (Problem 3)

h	Exact	Piecewise Series	Absolute Errors
0.5	0.001680	-0.000145	0.0018
0.25	0.001680	0.001426	$2.5399\ 10^{-4}$
0.1	0.001680	0.001668	$1.1526\ 10^{-5}$

Theorem 2: *For the linear homogenous equation of second order with variable coefficients*

$$y'' + b(x)y' + d(x)y = 0\ ,\ y(0) = \alpha\ ,\ y'(0) = \beta, \tag{4.16}$$

with $b(x)$ and $d(x)$ being analytical functions

$$b^i(x) = \sum_{k=0}^{K} b_k^i (x - x_i)^k\ ,\ d^i(x) = \sum_{k=0}^{K} d_k^i (x - x_i)^k\ , \tag{4.17}$$

the piecewise series solution is

$$y(x) = \sum_{i=0}^{n-1} \sum_{k=0}^{K} a_k^i (x - x_i)^k \gamma[x_i, x_{i+1})(x), \tag{4.18}$$

where

$$a_0^0 = \alpha \ , a_1^0 = \beta \tag{4.19}$$

$$a_0^i = \sum_{k=0}^{K} a_k^{i-1} h^k, \ a_1^i = \sum_{k=1}^{K} k a_k^{i-1} h^{k-1}, \ i = 1,2, \dots n-1 \tag{4.20}$$

$$a_{k+2}^i = -\frac{1}{(k+2)(k+1)} \left\{ \sum_{l=0}^{k+1} l a_l^i \, b_{k-l+1}^i + \sum_{l=0}^{k} a_l^i \, d_{k-l}^i \right\},$$

$$k = 0,1, .. K-2 \ , i = 0,1, \dots n-1 \tag{4.21}$$

An application of the theorem is given next.

Example Problem 4

The equation which is second order and variable coefficient

$$y'' + sinx \ y' + cosx \ y = 0 \ , y(0) = 0 \ , y'(0) = 1, \tag{4.22}$$

does not possess an exact solution. Therefore, the series solutions will be contrasted with the numerical Runge-Kutta solution (MATLAB, ODE45). A series solution with cubic powers is considered.

First the $sinx$ and $cosx$ coefficients are calculated in the vicinity of $x = x_i = ih$

$$b_0^i = sin(ih) \ , b_1^i = cos(ih) \ , b_2^i = -\frac{1}{2} sin(ih) \ , b_3^i = -\frac{1}{6} cos(ih),$$

$$i = 0,1, \dots n-1 \tag{4.23}$$

$$d_0^i = cos(ih) \ , d_1^i = -sin(ih) \ , d_2^i = -\frac{1}{2} cos(ih), \ d_3^i = \frac{1}{6} sin(ih),$$

$$i = 0,1, \dots n-1. \tag{4.24}$$

The coefficients in the series solution are

$$a_0^0 = 0, \ a_1^0 = 1, \ a_2^0 = 0 \ , a_3^0 = -\frac{1}{3} \tag{4.25}$$

$$a_0^i = a_0^{i-1} + a_1^{i-1} h + a_2^{i-1} h^2 + a_3^{i-1} h^3, \ i = 1,2, \dots n-1 \tag{4.26}$$

$$a_1^i = a_1^{i-1} + 2a_2^{i-1} h + 3a_3^{i-1} h^2, \ i = 1,2, \dots n-1 \tag{4.27}$$

$$a_2^i = -\frac{1}{2} (a_1^i \sin(ih) + a_0^i \cos(ih)) \ , i = 1,2, \dots n-1 \tag{4.28}$$

$$a_3^i = -\frac{1}{6} [a_1^i \left(2 \, cos(ih) - sin^2(ih)\right) - a_0^i \, sin(ih)(1 + \cos(ih))],$$

$$i = 1,2, \dots n-1. \tag{4.29}$$

The piecewise series solution is

$$y(x) = \sum_{i=0}^{n-1} \left\{ a_0^i + a_1^i (x - x_i) + a_2^i (x - x_i)^2 + a_3^i (x - x_i)^3 \right\} \gamma[x_i, x_{i+1})(x), \tag{4.30}$$

with the single series solution being

$$y = x - \frac{1}{3} x^3. \tag{4.31}$$

Both approximations are contrasted with the numerical solution for $h = 0.2$ $(n = 25)$ (Figure 4)

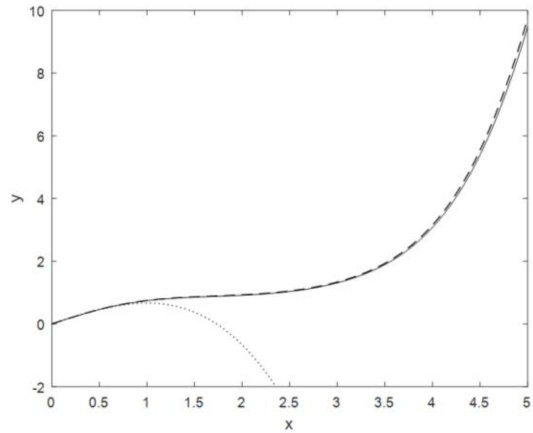

Figure 4
Comparison of the piecewise series (solid) and the single series (dotted) solutions with the numerical solution (dashed) (Problem 4, $h = 0.2$)

The single series solution has a limited range of validity. However, the piecewise series solution agrees well with the numerical one in the whole domain. If h is lowered, the agreement would be better and the curves are indistinguishable from each other.

5. A boundary value problem, the blasius equation

All problems treated so far are initial value problems. The method is essentially an initial value method. However, problems with boundary conditions can also be solved if the proposed technique is combined with the shooting technique. The Blasius equation is treated in this section. Consider the third order nonlinear equation

$$y''' + y\,y'' = 0\,,\ y(0) = 0\,,\ y'(0) = 0\,,\ y'(\infty) = 1. \tag{5.1}$$

The last condition at infinity is replaced by the initial one

$$y''(0) = \alpha, \tag{5.2}$$

where α is chosen such that the condition at infinity is satisfied (Shooting technique). A power series with up to 8[th] order is considered. Implementing the algorithm, the piecewise series solution is

$$y(x) = \sum_{i=0}^{n-1} \sum_{k=0}^{8} a_k^i (x - x_i)^k \gamma[x_i, x_{i+1})(x), \tag{5.3}$$

where the coefficients are

$$a_0^0 = 0,\ a_1^0 = 0,\ a_2^0 = \frac{\alpha}{2}\,,\ a_3^0 = 0\,,\ a_4^0 = 0\,,\ a_5^0 = -\frac{\alpha^2}{120} \tag{5.4}$$

$$a_6^0 = 0\,,\ a_7^0 = 0\,,\ a_8^0 = \frac{11\alpha^3}{40320} \tag{5.5}$$

$$a_0^{i+1} = \sum_{k=0}^{8} a_k^i h^k, \tag{5.6}$$

$$a_1^{i+1} = \sum_{k=1}^{8} k a_k^i h^{k-1} \tag{5.7}$$

$$a_2^{i+1} = \frac{1}{2}\sum_{k=2}^{8} k(k-1)a_k^i h^{k-2} \tag{5.8}$$

$$a_3^{i+1} = -\frac{1}{3}a_0^i a_2^i \tag{5.9}$$

$$a_4^{i+1} = -\frac{1}{4}a_0^i a_3^i - \frac{1}{12}a_1^i a_2^i \tag{5.10}$$

$$a_5^{i+1} = -\frac{1}{5}a_0^i a_4^i - \frac{1}{10}a_1^i a_3^i - \frac{1}{30}(a_2^i)^2 \tag{5.11}$$

$$a_6^{i+1} = -\frac{1}{6}a_0^i a_5^i - \frac{1}{10}a_1^i a_4^i - \frac{1}{15}a_2^i a_3^i \tag{5.12}$$

$$a_7^{i+1} = -\frac{1}{7}a_0^i a_6^i - \frac{2}{21}a_1^i a_5^i - \frac{2}{35}a_2^i a_4^i - \frac{1}{35}(a_3^i)^2 - \frac{1}{105}a_2^i a_4^i \tag{5.13}$$

$$a_8^{i+1} = -\frac{1}{8}a_0^i a_7^i - \frac{5}{56}a_1^i a_6^i - \frac{3}{56}a_3^i a_4^i - \frac{11}{168}a_2^i a_5^i, i = 0,1,2,\dots n-2. \tag{5.14}$$

with the single series solution being

$$y = \frac{\alpha}{2}x^2 - \frac{\alpha^2}{120}x^5 + \frac{11\alpha^3}{40320}x^8 \;. \tag{5.15}$$

Numerical simulations reveal that $\alpha = 0.47$ so that the right hand side condition can be satisfied. Both derivatives of solutions are contrasted with the numerical one for $h = 0.5$ ($n = 20$) (Figure 5).

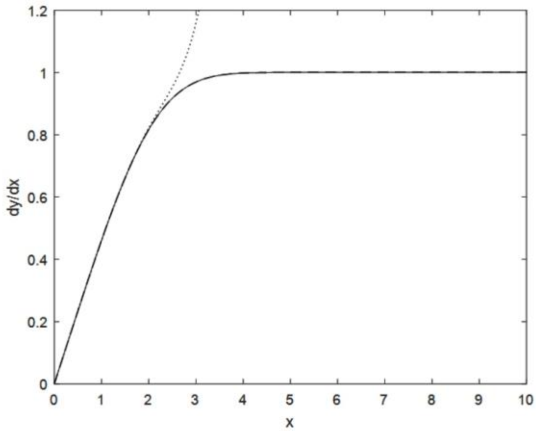

Figure 5

Comparison of the derivative of piecewise series solution (solid) and the single series solution (dotted) with the numerical solution (dashed) (Problem 5, $h = 0.5$)

The single series solution blows up within the boundary layer solution whereas the piecewise series solution correctly determines the solution (y') both in the boundary layer and outside it.

6. Concluding remarks

A new method is developed to validate the series solutions in the whole domain of interest. While the single series solutions usually have narrow validity ranges, it is shown that the ranges can be increased substantially by employing piecewise continuous series solutions. The solutions are not purely numerical discrete solutions, rather, they are expressed as continuous solutions with the aid of Gamma Interval functions. It is shown that a few terms in the piecewise series solutions are adequate in representing the real solution and the error can be decreased by decreasing the step size. The successful implementations of the technique to linear and nonlinear equations up to third order are outlined.

Although the method is an initial value method, it may be applied successfully to problems with boundary conditions also if incorporated with shooting methods. The famous Blasius equation which models the fluid flow over a flat plate is solved by the method. It is expected that many other physical problems can be solved by implementation of the method. Applications of the method to partial differential, differential-difference and fractional derivative equations remain a further topic of interest. The expression of the results in terms of Gamma Interval functions requires some modification in case of partial differential equations, since, instead of one-dimensional intervals, one has to consider regions of two and higher order dimensions.

7. Declarations

Data Availability: No additional data is reported with this work.
Competing Interests: Author declares no competing interests.

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