Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 7, pp. 1663–1677
DOI : 10.47974/JIM-2006

Bridging economic insights through fixed point theory and game theory applications

Khalid El Hajioui [†]
Faculté d'Economie et de Gestion
Ibn Tofaï University
B.P. 133
Kenitra 14000
Morocco

Radouane Azennar *
Department of Mathematics
Faculty of Sciences
Ibn Tofaï University
B.P. 133
Kenitra 14000
Morocco

Abstract

The purpose of this paper is to contribute to research on fixed point theory and its applications to the game theory. Indeed, we show, under qualification conditions, the existence of a common fixed point for tow multivalued mappings as well as the existence of the solution of a game.

Subject Classification: *[2010] 47H10, 47H08.*

Keywords: *Ordered topological vector spaces, Measure of noncompactness, Condensing and weakly isotone multivalued mappings, Fixed point, Game theory.*

1. Introduction

Minimax theory is applicable in analyzing the bargaining and negotiation situations. It provides insights into strategic decision making

[†] *E-mail:* **elhajkhalid@hotmail.com**
* *E-mail:* **radouane.azennar@uit.ac.ma (Corrsponding Author)**

and optimal bargaining strategies to maximize individual gains while accounting for the possible outcomes if negotiations fail.

Bargaining and negotiation are essential components of human interactions, whether in personal relationships or business transactions. These processes involve reaching a mutually satisfactory agreement between two or more parties who have different preferences, goals, or resources. The minimax theory is a mathematical approach that can be applied to decision making in bargaining and negotiation scenarios.

Bargaining and negotiation involve a series of strategic moves and counter moves by the parties involved. Each party aims to maximize her own benefits while considering the interests and positions of the other party/parties. The goal is to find a solution that satisfies all parties to the greatest extent possible. The minimax theory involves maximizing gains while minimizing losses. In the context of bargaining and negotiation, the minimax theory helps to identify the optimal strategy for each party by considering the potential outcomes and their associated values or utilities. The minimax strategy focuses on finding the best course of action for a party, assuming that the opposing party will act in a way that minimizes the first party's potential gains or maximizes her potential losses. This strategy takes into account the worst case scenario and aims to minimize the potential harm or maximize the potential benefit. In bargaining and negotiation, the minimax theory can be applied in various ways. For example, it can help to determine the reservation point or the lowest acceptable outcome for each party. By considering the worst case scenario, parties can set their reservation points to ensure they do not agree to an outcome that is worse than their alternatives. Additionally, the minimax theory can be used to analyze and predict the behavior of the opposing party. By understanding the potential moves and strategies that the other party might employ, a negotiator can adjust the approach maximizing the potentiel gains while minimizing the potential losses. It is important to note that while the minimax theory provides a valuable framework for decision making in bargaining and negotiation, real life negotiations are often more complex and involve various factors beyond pure mathematical calculations. Psychological factors, emotions, relationships, and power dynamics also play significant roles in shaping negotiation outcomes. Therefore, the minimax theory should be used as one tool among many to support effective bargaining and negotiation strategies.

The game theory is to study strategic interactions between different actors [21], aware of the goals of the protagonists [11]. It constitutes the mathematical theory of strategic behaviors [15]. It is based on extensive mathematical research in the field of fixed point and then applied to a multitude fields and scientific disciplines such as economics, international relations and psychology. For more details on fixed point and applications we refer to [2, 5, 6, 8, 12, 18, 19, 23, 25]. Gaming research accelerated after the establishment of Nash equilibrium by the mathematician and economist John Forbes Nash [23, 28, 9, 27]. This theory has been developed by other researchers using theoretical mathematical concepts [2, 3, 5, 13, 20]. The game theory gradually spreads in the second half of the twentieth century to a multitude of domains, transforming itself into a practical tool at the same time as it was integrated in the analysis of many diverse phenomena [17]. Recently, an important application in newsvendor's supply chain model was studied in [6]. The adoption of modeled game theory has broadened by reaching several scientific fields including those related to competition issues and strategic decisions making. The work of Robert Ysraël Aumann has linked the concepts of game theory to the problems of economic analysis of the market and contributed to the development of the Nash equilibrium by defining the correlated equilibrium [26, 24].

Our work is a contribution to scientific research on minimax theorem and application of the outcoming results to the game theory.

2. Notations and preliminaries

Let X be a real vector space, its positive cone is denoted X^+ and satisfies $X^+ + X^+ \subset X^+$, $\lambda X^+ \subset X^+$ for all $\lambda \geq 0$ and $X^+ \cap (-X^+) = \{0\}$. As usual X will be ordered by the partial order relation $x \leq y \Leftrightarrow y - x \in X^+$. The usual partial order induced in 2^X, the family of all nonempty closed subsets of X, is defined by the relation $Q \leq Q'$ means $q \leq q'$ for all $q \in Q$ and $q' \in Q'$. In addition, we consider a second partial order $\preceq$ introduced in [13] and defined for all $Q, Q' \in 2^X$ by

$$Q \preceq Q' \Leftrightarrow \begin{cases} \forall x \in Q, \exists x' \in Q' & \text{such that } x \leq x', \\ , \forall x' \in Q', \exists x \in Q & \text{such that } x \leq x'. \end{cases}$$

It is clear that the usual order implies the second one, but the reverse fails. Indeeed : Consider in 2^X with $X = \mathbb{R}^2$, the subsets $Q = [0, x_1] \times [0, y_1]$ and $Q' = [0, x_2] \times [0, y_2]$ with $0 \leq x_1 \leq x_2$ and $0 \leq y_1 \leq y_2$; we have $Q \preceq Q'$ but $Q \nleq Q'$.

For more details on order topology, see [1].

Let us recall te following lemma which will be useful in the sequel.

Lemma 2.1: [*1, Lemma 2.3, Lemma 2.22, Theorem 2.23*]

Let (X,τ) be an ordered topological vector space with positive cone X^+. Then the following holds:

(1) *X is Hausdorff and its ordered intervals are closed;*

(2) *If moreover X^+ is normal then we have:*

 (a) *Every ordered interval is $\tau-$bounded;*

 (b) *If $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are two nets of X satisfying $0 \le x_i \le y_i$ for each i and $y_i \xrightarrow{\tau} 0$, then $x_i \xrightarrow{\tau} 0$.*

In [22], Lothar Kaniok presented some examples of measures of noncompactness and some fixed point theorems for multivalued mappings in general topological vector space. Let *X* be an ordered topological vector space which is assumed to be Hausdorff and real, and let $K \subset X$, by $\mathcal{U}(X)$ we denote a fundamental system of circled, closed neighbourhoods of zero in *X*, and we denote by $\overline{K}$, $co(K)$, $\overline{co}(K)$ the closed hull, the convex hull, the closed convex hull of *K*. For every $K \in 2^X$, $M \in \overline{co}(K)$, and $U \in \mathcal{U}(X)$ The measure of noncompactness on X is defined by:

$$J(M,U) := \inf\{d > 0 : \exists D_1,....,D_n \subset X \,/\, M \subset \bigcup_i D_i \text{ and } D_i - D_i \subset dU, i = 1,...,n\}.$$

In short, we will write $J(M)$ instead of $J(M,U)$. The properties of such a measure of noncompactness in general topological vector spaces are presented in [22, Proposition 1].

An operator $T : Q \subseteq X \to X$ is called to be countably condensing if $T(Q)$ is bounded and if for any countably bounded subset Q' of *Q* with $J(Q') > 0$ we have $J(T(Q')) < J(Q')$.

Finally, two mapings $S,T : X \to X$ are said to be weakly isotone increasing if $Sx \le TSx$ and $Tx \le STx$ for all $x \in X$; similarly *S* and *T* are said to be weakly isotone decreasing if $Sx \ge TSx$ and $Tx \ge STx$ for all $x \in X$. We say that *S* and *T* are weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

3. Fixed point results

We generalize here,in general topological vector spaces, the results established in [14] in the case of Banach spaces. We prove also another result whith low condtions. In all the following, except express mention, *X* is a general complete topological vector space with normal cone X^+ and *Q* is a nonempty closed subset of *X*.

Definition 3.1: Let $f : X \to X$ and $S, T : Q \to Q$ any mappings.

(1) f is said to be monotone-continuous in x if $f(x_i) \to f(x)$ for each increasing or decreasing net $\{x_i\}$ that converges to x.

(2) S and T are said to satisfy the condition D_Q if for any countable subset A of Q and for any fixed element $a \in Q$ the condition $A \subseteq \{a\} \cup S(A) \cup T(A)$ implies the compactness of $\overline{A}$.

(3) S and T are said to satisfy the weak-condition D_Q if for any monotone net $\{x_i\}$ of Q and for any fixed $a \in Q$ the condition $\{x_i\} \subseteq \{a\} \cup S(\{x_i\}) \cup T(\{x_i\})$ implies the convergence of $\{x_i\}$.

Lemma 3.2: *Let (X, τ) be an ordered topological vector space with normal cone X^+. Then a monotone net $(x_i)_{i \in I} \subset X$ is τ-convergent if and only if it has a weakly convergent subnet.*

Definition 3.3: Let $T : X \to 2^X$ be a multivalued mapping.

(1) An element $x \in X$ is said to be a fixed point of T if $x \in Tx$;

(2) T is said to be isotone increasing (resp. decreasing) if for all $x, y \in X$ such that $x \le y$ we have $Tx \le Ty$ (resp $Tx \ge Ty$).

Definition 3.4: Let $S, T : X \to 2^X$ be two multivalued mappings. S and T are said to be weakly isotone increasing if for any $x \in X$ we have $Sx \le Ty$ for all $y \in Sx$ and $Tx \le Sy$ for all $y \in Tx$. S and T are weakly isotone decreasing if for any $x \in X$ we have $Sx \ge Ty$ for all $y \in Sx$ and $Tx \ge Sy$ for all $y \in Tx$. They are called weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

We note here that we have analogous definitions of isotone and weakly isotone multivalued mappings when we consider on 2^X the second partial order $\preceq$ instead of the usual one $\le$.

Definition 3.5: A multivalued mapping $T : Q \to 2^Q$ is said to be countably condensing if $T(Q)$ is bounded and if for any countably bounded subset A of Q with $J(A) > 0$ we have $J(T(A)) < J(A)$, where $T(A) = \bigcup_{x \in A} Tx$.

Definition 3.6: *Let $S, T : Q \to 2^Q$ be two multivalued mappings.*

(1) We say that S and T satisfy the condition D_Q if for any countable subset A of Q and for any fixed element $a \in Q$ the condition $A \subseteq \{a\} \cup S(A) \cup T(A)$ implies the compactness of $\overline{A}$.

(2) S and T are said to satisfy the weak condition D_Q if for any monotone sequence $\{x_n\} \subset Q$ and for any fixed element $a \in Q$ the condition $\{x_n\} \subseteq \{a\} \cup S(\{x_n\}) \cup T(\{x_n\})$ implies the convergence of $\{x_n\}$.

We remark at this level that the condition D_Q implies the weak-condition D_Q in the case of multivalued mappings. Indeed, let $(x_n) \subset Q$ be a monotone sequence sucht that $\{x_n\} \subseteq \{a\} \cup S(\{x_n\}) \cup T(\{x_n\})$ for a fixed element $a \in Q$, and pose $Q' = \{x_n\}$; Q' is obviousely countable and $Q' \subseteq \{a\} \cup S(Q') \cup T(Q')$. By the condition D_Q, $\overline{Q'}$ is compact; and by [10, p 89], (x_n) has a convergent subsequence. The original sequence (x_n) is then convergente by lemma 3.2. Note also that a Hausdorff ordered topological vector space is regular [16, Chapter VI]..

Theorem 3.7: *Let $S, T : Q \to 2^Q$ be two closed and weakly isotone multivalued mappings satisfying the weak-condition D_Q. Then, S and T have a common fixed point.*

Proof: Let $x \in Q$ be fixed and consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by its first terme $x_0 = x$ and the recurente relations $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$. Suppose that S and T are weakly isotone increasing on Q. For both partial orders on 2^Q, we can build the sequence (x_n) in any increasing way. Moreover we have

$$\{x_n\} = \{x_0\} \cup \{x_1, x_3, ...\} \cup \{x_2, x_4, ...\} \subseteq \{x_0\} \cup S(\{x_n\}) \cup T(\{x_n\}) .$$

By the weak-condition D_Q, the sequence (x_n) converges to $x^\star \in Q$. Now we get

$$x^\star = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} x_{2n} x_{2n+1} \in Sx_{2n}$$

and

$$x^\star = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} x_{2n+1} x_{2n+2} \in Tx_{2n+1}.$$

The result $x^\star \in Sx^\star$ and $x^\star \in Tx^\star$ is then required because S and T are closed. The other case, S and T are weakly isotone decreasing , is done in the same way.

Theorem 3.8: *Let* $S, T : Q \to 2^Q$ *be two closed countably condensing and weakly isotone multivalued mappings. Then, S and T have a common fixed point.*

Proof: We show that S and T satisfy the condition D_Q. Let Q' be a countable subset of Q such that $Q' \subseteq \{a\} \cup S(Q') \cup T(Q')$ for a fixed element $a \in Q$. In addition Q' is bounded since S and T are condensing, and so Q' is relatively compact. In fact, in the contrary case we will have $J(Q') > 0$, but by [22, Properties 1 (6)] and the fact that S and T are condensing we obtain

$$J(Q') \leq \max\{J(\{a\}), J(S(Q')), J(T(Q'))\} < J(Q')$$

which is a contradiction.Thus, $J(Q') = 0$ and $\overline{Q'}$ is compact by [22, Properties 1 (6)]. Finally, S and T have a common fixed point by theorem 3.7 since the condition D_Q implies the weak-condition D_Q.

4. Application to game theory

The minimax theory, also known as game theory, is a mathematical framework that analyzes strategic interactions between multiple decision makers. This mathematical approach can be applied to model and analyze bargaining and negotiation situations where the different parties have special concessions and conflicting interests. It helps in determining the efficient outcomes in negotiation and so the optimal stratigy for each party. By incorporating various factors such as power dynamics, information asymmetry, concession patterns and the presence of multiple issues, the game-theoretic models help in understanding the dynamics of different strategies and in identifying the reservation point beyond which the negotiators are willing to walk away from the negotiation. Here are some specific and significant applications of the minmax theory in the field of bargaining and negotiation :

(1) Nash Bargaining Solution: It is a concept in the game theory introduced by J.F.Nash and wich aims to find a mutually agreable outcome where the diffirent negotiating parties receive their minimum acceptable payoff.

(2) ZOPA Analysis: The Zone of Possible Agreement refers to the range of options and potential agreements that are acceptable by the parties. By identifying the boundaries of the ZOPA, negotiators can then focus on finding mutually beneficial solutions within this zone.

(3) BATNA Analysis: The Best Alternative to a Negotiated Agreement refers to the alternative to take if negotiations fail. By assessing the potential outcomes and payoffs associated with pursuing alternative options, the BATNA is evaluated and the negotiators are guiden towards achieving better outcomes.

(4) Pareto Efficiency: It refers to a state where it is impossible for a party to make a better off unless another party makes worse off. By applying the minmax theory, the negotiators can strive to reach Pareto-efficient agreements that maximize the joint gains for all parties involved.

(5) Game-theoretic Strategies: It provides a set of strategies that can be employed during the negotiation, and suggests the concept of tit-for-tat where the parties start with cooperation and mutual clarification. This strategy aims to promote trust and encourages mutually beneficial outcomes.

(6) Risk Assessment: By evaluating the potential outcomes and their associated probabilities, this analysis helps negotiators in analyzing and managing the risks involved and making informed decisions that minimize potential losses.

In summary, the minmax theory offers mathematical methodes and valuable tools and concepts that can facilitate and enhance the understanding and analysis of the decision making processes and stratigic intercations of the others in bargaining and negotiations. Indeed, it enables negotiators to analyze potential outcomes, determine optimal solutions, assess alternatives, and manage the negotiation process to achieve mutually beneficial agreements.

Let now Y be a second complete ordered locally convex space with normal positive cone Y^+. It is knowen that the product space $X \times Y$ is also a complete ordered locally convex space with normal positive cone $X^+ \times Y^+$. In the sequel and without any confusion, the tow partial orders are denoted by the same notations $\leq$ and $\preceq$ in 2^X, 2^Y and $2^{X\times Y}$ and are defined in $2^{X\times Y}$ by :

For any $Q_1 \times Q_2, Q_1' \times Q_2' \in 2^{X\times Y}$,

$$Q_1 \times Q_2 \leq Q_1' \times Q_2' \text{ iff } Q_1 \leq Q_1' \text{ and } Q_2 \leq Q_2';$$

and

$$Q_1 \times Q_2 \preceq Q_1' \times Q_2' \text{ iff } Q_1 \preceq Q_1' \text{ and } Q_2 \preceq Q_2'.$$

The natural projections of the product space $X \times Y$ into X or Y defined respectively by $\pi_X : (x, y) \in X \times Y \to x \in X$ and $\pi_Y : (x, y) \in X \times Y \to y \in Y$ are continuous [7. p 81]. For arbitrary elements $a \in X$ and $b \in Y$, the mappings σ_X and σ_Y defined respetively by $\sigma_X : x \in X \to (x, b) \in X \times Y$ and $\sigma_Y : y \in Y \to (a, y) \in X \times Y$ are continuous and $\sigma_X(X) = X \times \{b\}$, $\sigma_Y(Y) = \{a\} \times Y$ [7, P 83]. In a product space, we consider the mesure of noncompactness $J^{\times}$ established in [4] and stated in the following theorem:

Theorem 4.1: **[4]** *Let* X_i *be a ordered topological vector spaces with measure of noncompactness* J_i*, for each* $i = 1, ..., n$.

Then,

$$J^{\times}(D) := F(J_1(D_1), J_2(D_2), ..., J_n(D_n))$$

for each $D \in \mathcal{B}(X_1 \times X_2 \times ... \times X_n)$.

Definition 4.2: The mapping σ_Y is said to be $(J_2, J^{\times})$-condensing on Y if for all bounded subset Q of Y we have

$$J^{\times}(\sigma_Y(Q)) < J_2(Q);$$

and the mapping σ_X is said to be $(J_1, J^{\times})$-condensing on X if for all bounded subset Q of X we have

$$J^{\times}(\sigma_X(Q)) < J_1(Q).$$

The mapping π_X is said to be $(J^{\times}, J_1)$-condensing on X if

$$J_1(\pi_X(Q)) < J^{\times}(Q),$$

and the mapping π_Y is said to be $(J^{\times}, J_2)$-condensing on Y if

$$J_2(\pi_Y(Q)) < J^{\times}(Q),$$

for all $Q \in \mathcal{B}(X \times Y)$.

In the context of bargaining and negotiation, the economic gain is a decisive element in taking choices, especially in zero-sum games. In this case, each actor should have a closed and bounded set of choises. The gain function K is then characterized by its values $K(x_i, y_j)$ and the game is represented by a matrix where the bargaining stratigies are identified whith rows and the negotiation stratigies whith columns.

The target is to maximize the gain of the first player and to minimize the loss of the second player. Thus, the first player must choose $x_0 \in A$ such that

$$\min_{y\in B} K(x_0, y) = \max_{x\in A} \min_{y\in B} K(x, y), \tag{4.1}$$

and the second player must choose $y_0 \in B$ such that

$$\max_{x\in A} K(x, y_0) = \min_{y\in B} \max_{x\in A} K(x, y). \tag{4.2}$$

Since, $\min_{y\in B} K(x_0, y) \le K(x_0, y_0) \le \max_{x\in A} K(x, y_0)$, it follows that

$$\max_{x\in A} \min_{y\in B} K(x, y) \le \min_{y\in B} \max_{x\in A} K(x, y). \tag{4.3}$$

When equality holds in (4.3), we obtain

$$\max_{x\in A} \min_{y\in B} K(x, y) = K(x_0, y_0) = \min_{y\in B} \max_{x\in A} K(x, y). \tag{4.4}$$

The common value $K(x_0 . y_0)$ is called the value or solution of the game, and x_0, y_0 are the winning strategies.

Before giving the main results of this section, let us consider the function ϕ defined for all $x \in A$ by

$$\phi(x) := \min_{y\in B} K(x, y) = \min K(x \times B),$$

and the function ψ defined for all $y \in B$ by

$$\psi(y) := \max_{x\in A} K(x, y) = \max K(A \times y);$$

and assume that for each $(x, y) \in A \times B$, the subsets

$$N_y := \{x' \in A : K(x', y) = \psi(y)\} \text{ and } M_x := \{y' \in B : K(x, y') = \phi(x)\}$$

are nonempty. Obviousely, for any subset $A' \times B'$ of $A \times B$, , we have

$$N_{B'} = \bigcup_{y\in B'} N_y \quad \text{and} \quad M_{A'} = \bigcup_{x\in A'} M_x$$

Theorem 4.3: *Let (A, B, K) be a game where $A \subset X$ and $B \subset Y$ are nonempty closed subsets. Suppose that :*

(1) *The functions K, ψ and ϕ are continuous;*

(2) *σ_A is $(J_1, J^\times)$-condensing on A and σ_B is $(J_2, J^\times)$-condensing on B, π_A is $(J^\times, J_1)$-condensing on A and π_B is $(J^\times, J_2)$-condensing on B;*

(3) $B \preceq M_{x'} \forall x' \in N_y, \forall y \in B$ and $A \preceq N_{y'} \forall y' \in M_x, \forall x \in A$.

Then, the game (A, B, K) has a solution.

Proof: Let us consider the closed product subset $C = A \times B$ and the following two multivalued mappings :

$$S : C \to 2^C \qquad \text{and} \qquad T : C \to 2^C$$
$$c \mapsto N_y \times \{y\} \qquad\qquad c \mapsto \{x\} \times M_x$$

whith $c = (x, y) \in A \times B$.

T is closed. Indeed, let $\{(x_i, y_i)\}$ be a net in C such that $(x_i, y_i) \to (x, y) \in C$ and let $(u_i, v_i) \in T(x_i, y_i)$ such that $(u_i, v_i) \to (u, v)$; we have :

$$\begin{aligned} (u_i, v_i) \in T(x_i, y_i) \quad &\Leftrightarrow \quad (u_i, v_i) \in \{x_i\} \times M_{x_i} \\ &\Leftrightarrow \quad u_i = x_i \text{ and } K(x_i, v_i) = \phi(x_i). \end{aligned}$$

By continuity of K and ϕ we obtain $u = x$ and $K(x, v) = \phi(x)$, and so $(u, v) \in T(x, y)$.

T is countably condensing. Indeed, let $Q \subseteq C$ be a countable and bounded subset, by application of (2) we get :

$$\begin{aligned} J^{\times}(T(Q)) &= J^{\times}(\bigcup_{c \in Q} Tc) \\ &= J^{\times}(\bigcup_{(x,y) \in Q} \{x\} \times M_x) \\ &= J^{\times}(\bigcup_{(x,y) \in Q} \sigma_B(M_x)) \\ &= J^{\times}(\bigcup_{(x,y) \in Q} \sigma_B(M_{\pi_A(x,y)})) \\ &= J^{\times}(\sigma_B(M_{\pi_A(Q)})) \\ &< J_2(M_{\pi_A(Q)}) \\ &\le J_2(\pi_B(Q)) \\ &< J^{\times}(Q). \end{aligned}$$

Similarly, we prove that S is closed and countably condensing.

Moreover, S and T are weakly isotone incresing relatively to the partial order $\preceq$. In fact, let $c = (x, y) \in C$ and $c' = (x', y') \in Sc$, we have $(x', y') \in Sc \Leftrightarrow (x', y') \in N_y \times \{y\} \Leftrightarrow x' \in N_y$ and $y = y'$; since the partial order $\preceq$ is reflexive, then by (3) we obtain :

$N_y \times \{y\} \preceq \{x'\} \times M_{x'}, \forall (x', y') \in N_y \times \{y\}, \forall (x, y) \in C,$ wich is equivalent to $N_y \preceq N_y$ and $B \preceq M_{x'}$, in other words $Sc \preceq Tc' \ \forall c' \in Sc$. By the same arguments we prove that $Tc \preceq Sc' \ \forall c' \in Tc$.

Thus, by theorem 3.8, S and T have a common fixed point $c^{\star} = (x^{\star}, y^{\star})$.

Now we have

$$(x^\star, y^\star) \in N_{y^\star} \times \{y^\star\} \Leftrightarrow K(x^\star, y^\star) = \max_{x\in A} K(x, y^\star) \geq \min_{y\in B} \max_{x\in A} K(x, y),$$

$$(x^\star, y^\star) \in \{x^\star\} \times M_{x^\star} \Leftrightarrow K(x^\star, y^\star) = \min_{y\in B} K(x^\star, y) \leq \max_{x\in A} \min_{y\in B} K(x, y).$$

By combining these last two inequalities with (4.3, we get

$$K(x^\star, y^\star) \leq \max_{x\in A} \min_{y\in B} K(x, y) \leq \min_{y\in B} \max_{x\in A} K(x, y) \leq K(x^\star, y^\star).$$

This implies that

$$\max_{x\in A} \min_{y\in B} K(x, y) = K(x^\star, y^\star) = \min_{y\in B} \max_{x\in A} K(x, y),$$

and the proof is complete.

Since X^+ and Y^+ are two closed subsets in X and Y respectively, we obtain the following corollary:

Corollary 4.4: *Under the assumptions* (1),(2) *and* (3) *of theorem 4.3, the game* (X^+, Y^+, K) *has a solution.*

We end this paper by generalizing a result of [28].

Theorem 4.5: *Let* $A \subset X$ *and* $B \subset Y$ *be nonempty and closed, and let* Q *and* Q' *be tow nonempty closed subsets in* $A \times B$.

Suppose that :

(1) *For every* $(x, y) \in A \times B$, *The subsets* $M_x := \{y' \in B : (x, y') \in Q'\}$ *and* $N_y := \{x' \in A : (x', y) \in Q\}$ *are nonempty and closed;*

(2) σ_A *is* $(J_1, J^\times)$*-condensing on A and* σ_B *is* $(J_2, J^\times)$*-condensing on B,* π_A *is* $(J^\times, J_1)$*-condensing on A and* π_B *is* $(J^\times, J_2)$*-condensing on B;*

(3) $B \preceq M_{x'}\, \forall x' \in N_y, \forall y \in B$ *and* $A \preceq N_{y'}\, \forall y' \in M_x, \forall x \in A$.

Then, the subset $Q \cap Q'$ *is nonempty.*

Proof: By the same notations used in the proof of theorem 4.3 and with slight modifications, we prove that S and T are closed countably condensing and weakly isotone increasing. Indeed,

$$\begin{aligned}(u_i, v_i) \in T(x_i, y_i) &\Leftrightarrow (u_i, v_i) \in \{x_i\} \times M_{x_i} \\ &\Leftrightarrow u_i = x_i \text{ and } (x_i, v_i) \in Q';\end{aligned}$$

and so $u = x$ and $(u, v) \in Q'$ since Q' is closed. Thus, by theorem 3.8, S and T have a common fixed point $(x^\star, y^\star)$. In other words we get

$$(x^\star, y^\star) \in N_{y^\star} \times \{y^\star\} \text{ and } (x^\star, y^\star) \in \{x^\star\} \times M_{x^\star},$$

that is $(x^\star, y^\star) \in Q \cap Q'$.

References

[1] C. D. Aliprantis and R. Tourky, Cones and Duality, Graduate Studies in Mathematics, vol. 84, Providence, RI, USA: American Mathematical Society, xiv+279 (2007).

[2] J. P. Aubin, Mathematical Methods of Games and Economic Theory, Amsterdam, Netherlands: North-Holland (1982).

[3] A. Radouane and D. Mentagui, "A solution for the minimax problem via fixed point theory in complete ordered locally convex spaces," *Commun. Optim. Theory,* vol. 2020, no. 11, pp. 1–10 (2020).

[4] J. Banas and K. Goebel, Measure of Noncompactness in Banach Spaces, New York, NY, USA: Dekker (1980).

[5] C. S. Heikkilä, Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory, New York, NY, USA: Springer (2010).

[6] C. H. Chen, "Applying game theory in Newsvendor's supply chain model," *J. Inf. Optim. Sci.,* vol. 42, no. 6, pp. 1367–1382 (2021).

[7] C. Berge, Espaces Topologiques-Fonctions Multivoques, Paris, France: Dunod (1959).

[8] S. Cobzas, "Fixed point theorems in locally convex spaces: The Schauder mapping method," Fixed Point Theory Appl., vol. 2006, Article ID 57950, pp. 1–13 (2006). DOI: 10.1155/FPTA/2006/57950.

[9] A. Cournot, 1st ed., p. 11 (1982). eBook ISBN 9781003051091.

[10] K. Deimling, Nonlinear Functional Analysis, New York, NY, USA: Springer-Verlag 91984).

[11] G. Demange Delta and J. P. Ponssard, Théorie des jeux et analyse économique, Économie Series, Paris, France: Presses Universitaires de France, p. 9 (1994).

[12] E. Denécé, "Diplomatie économique et compétition des États," Géoéconomie, no. 56, pp. 71–78 (2011).

[13] B. C. Dhage, “A fixed point theorem for multivalued mappings in ordered Banach spaces with applications I,” *Nonlinear Anal. Forum,* vol. 10, no. 1, pp. 105–126 (2005).

[14] B. C. Dhage, D. O’Regan, and R. P. Agarwal, “Common fixed point theorems for a pair of countably condensing mappings in ordered Banach spaces,” *J. Appl. Math. Stoch. Anal.*, vol. 16, no. 3, pp. 243–248(2003).

[15] E. Nicolas, Théorie des Jeux, Paris, France: Dunod, p. 6 (2013).

[16] R. Engelking, General Topology, 2nd ed., Sigma Series in Pure Mathematics, vol. 6, Berlin, Germany: Heldermann (1989).

[17] G. Amory, “Apports et limites de la théorie des jeux,” *Regards croisés sur l’économie,* vol. 22, no. 1, pp. 68–71 (2018).

[18] H. Günerhan, M. Yiğider, J. Manafian, and O. Alp Ilhan, “Numerical solution of fractional order logistic equations via conformable fractional differential transform method,” *J. Interdiscip. Math.*, vol. 24, no. 5, pp. 1207–1220 (2021). DOI: 10.1080/09720502.2021.1918319.

[19] A. Hajji and E. Hanebaly, “Commutating mappings and α-compact type fixed point theorems in locally convex spaces,” *Int. J. Math. Anal.,* vol. 1, pp. 661–680 (2007).

[20] K. Fan, “Fixed-point and minimax theorems in locally convex topological linear spaces,” Department of Mathematics, University of Notre Dame, communicated by J. von Neumann, Dec. 13 (1951).

[21] K. Leyton-Brown and Y. Shoham, Essentials of Game Theory: A Concise, Multidisciplinary Introduction, San Rafael, CA, USA: Morgan & Claypool Publishers (2008).

[22] L. Kaniok, “On measures of noncompactness in general topological vector spaces,” *Comment. Math. Univ. Carolinae,* vol. 31, no. 3, pp. 479–487 (1990).

[23] J. F. Nash, “Equilibrium points in n-person games,” *Proc. Natl. Acad. Sci.*, vol. 36, pp. 48–49 (1950).

[24] Nobel Prize in Economics, “J. F. Nash 1994, Reinhard Selten and John Harsanyi 2005, Thomas Schelling and Robert Aumann 2007, Leonid Hurwicz, Eric Maskin and Roger Myerson 2012, Alvin Roth and Lloyd Shapley 2014, Jean Tirole.”

[25] S. Erden and M. Z. Sarikaya, “Pre-Grüss inequality involving conformable fractional integrals and its applications for random vari-

ables," *J. Interdiscip. Math.,* vol. 22, no. 5, pp. 757–771 (2019). DOI: 10.1080/09720502.2019.1675289.

[26] T. C. Schelling, The Strategy of Conflict, Cambridge, MA, USA: Harvard University Press (1960).

[27] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior, Princeton, NJ, USA: Princeton University Press (1944). ISBN: 978-0691003627.

[28] J. von Neumann, "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eines Mathematischen Kolloquiums,* vol. 8, pp. 73–83 (1937).

Received January, 2024