Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 7, pp. 1701–1717
DOI : 10.47974/JIM-2031

A novel approach to initial boundary value problems using Laplace transform Adomian decomposition method

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Abstract

This document contains, a novel approach to dealing with initial boundary value concerns that combines boundary and initial conditions to provide a new solution during each repetition utilising Adomian Decomposition Method . A new series solution structure can deliver a most accurate result.

Subject Classification: *(2010) 44A10 , 34A08 , 35R11 , 35K52.*

Keywords: *Initial boundary value problem, Adomian decomposition method, Laplace transform.*

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1. Introduction

Most of the academics debated the start and boundary value concerns [6, 7]. Wazwaz described the Adomian decomposition approach to dealing with higher-dimensional boundary value problems [2]. Solving initial boundary value problems with larger dimensions using the variational iteration decomposition approach [6, 12]. Biazar J. and Aminikhah H. used the variational iteration approach to solve Burger's problem [12]. Wu G. and Lee E. W. used Modified Riemann-Liouville derivatives to solve fractional differential equations with initial boundary conditions [4]. It is worth noting that all of these academics solved boundary value and initial problems only by employing boundary or initial conditions [6–10]. As a result, we propose a dependable framework by employing a novel strategy for treating boundary value and initial problems by combining boundary and initial conditions to get a Modified initial solution during each iteration using the Adomian decomposition method [2, 6, 8]. These techniques for constructing fresh initial solutions that are consecutive can provide a more accurate answer; several problems are provided in the given work to demonstrate the usefulness and this method's simplicity [1, 3, 5, 11–13].

2. Adomian decomposition method

Adomian designed and constructed method of decomposition for addressing nonlinear or linear problems like ODE. It entails separating the supplied equation into linear and nonlinear components, In the initial approximation,the derivative operator of highest order included on both sides of the linear operator is inverted, boundary and initial conditions are identified, as well as the terms influencing the independent variable alone are identified. Collapsing the the unidentified function into a sequence whose elements must be found, dividing the nonlinear function into distinct polynomials known as Adomian's polynomials, and identifying the series solution's successive terms via recurrent relation with the help of Adomian's polynomials. Consider the following equation

$$\Phi(\kappa(\gamma)) = \xi(\gamma) \tag{2.1}$$

here Φ denotes non-linear differential operator and ξ is predefined. In $\Phi(\kappa(\gamma)))$ are broken down into $L\kappa + R\kappa$, where L is operator that can be readily inverted and R is the linear operator's remainder. Therefore, above equation may be written as

$$L\kappa + R\kappa + N\kappa = g, \tag{2.2}$$

where $N\kappa$ is a representation of the nonlinear terms in $\Phi(\kappa(\gamma))$. On both sides, use the inverse operator L^{-1} results in

$$\kappa = \Psi + L^{-1}(\xi) - L^{-1}(R\kappa) - L^{-1}(N\kappa) \tag{2.3}$$

where Ψ is the integration constant that fulfils the requirement $L\Psi = 0$. Let suppose solution κ can be written as

$$\kappa = \sum_{n=0}^{\infty} \kappa_n \tag{2.4}$$

Assume $N\kappa$ may be expressed as

$$N\kappa = \sum_{n=0}^{\infty} A_n \tag{2.5}$$

Using the formula Adomian polynomials A_n of $N\kappa$ are examined.

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left(\sum_{i=0}^{\infty}(\lambda^i u_i)\right)\Bigg|_{\lambda=0}, \quad n = 0,1,2,\ldots \tag{2.6}$$

After that, replacing (2.4) and (2.5) in (2.3) gives

$$\sum_{n=0}^{\infty} \kappa_n = \varphi + L^{-1}(\xi) - L^{-1}\left(R\sum_{n=0}^{\infty}\kappa_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{2.7}$$

The recurrent connection determines each term of the series (2.4).

$$\left.\begin{array}{ll} \kappa_0 = \varphi + L^{-1}(\xi), & n = 0 \\ \kappa_{n+1} = L^{-1}(R\kappa_n) - L^{-1}(A_n). & n \geq 0 \end{array}\right\} \tag{2.8}$$

Several writers have explored the convergence of the decomposition series.

3. Initial boundary Value Problems Solution Using ADM Method

To teach the fundamental concept of treating initial and boundary conditions using the Adomian decomposition. We investigate the following one-dimensional differential equation to solve Problems.

$$D^{\alpha}(\kappa(\gamma,\sigma)) = \xi(\gamma,\sigma), \quad 0 < \gamma < 1, \sigma > 0 \tag{3.1}$$

with the condition

$$\kappa(\gamma,0) = \phi_0(\gamma), \frac{\partial\kappa(\gamma,0)}{\partial\sigma} = \phi_1(\gamma), \quad 0 \leq \gamma \leq 1, \tag{3.2}$$

and

$$\kappa(0,\sigma)=\xi_0(\sigma), \quad \kappa(1,\sigma)=\xi_1(\sigma), \quad \sigma>0 \tag{3.3}$$

where $\phi_0(\gamma),\phi_1(\gamma),\xi_0(\sigma)$ and $\xi_1(\sigma)$ are assigned functions.

Apply Laplace Transform to equation (3.1) we get,

$$L(D^{\alpha}(\kappa(\gamma,\sigma)))=L(\xi(\gamma,\sigma)) \tag{3.4}$$

$$L[\kappa(\gamma,\sigma)]=S^{-1}\kappa(\gamma,0)+S^{-\alpha}L[(\xi(\gamma,\sigma)] \tag{3.5}$$

$$\kappa(\gamma,\sigma)=S^{-1}\phi_0(\gamma)+L^{-1}[S^{-\alpha}L[(\xi(\gamma,\sigma)] \tag{3.6}$$

The initial solution may be expressed as

$$\kappa_0(\gamma,\sigma)=\phi_0(\gamma)+\sigma\phi_1(\gamma) \tag{3.7}$$

The consecutive first solutions that we create are new κ_n^* by using a new method for equation (3.1) at each iteration

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[\xi_0(\sigma)-\kappa_n(0,\sigma)]+\gamma[\xi_1(\sigma)-\kappa_n(1,\sigma)] \tag{3.8}$$

where $n=0,1,2,\ldots,$ Without a doubt, the new iterative initial solutions κ_n^* of Eq. (3.1) meet boundary and the initial requirements jointly as shown in the following.

- if $\sigma=0$ then $\kappa_n^*(\sigma,0)=\kappa_n(\gamma,0)$,
- if $\gamma=0$ then $\kappa_n^*(0,\sigma)=\xi_0(\sigma)$,
- if $\gamma=1$ then $\kappa_n^*(1,\sigma)=\xi_1(\sigma)$.

In general, a new formula (3.5) approach may be used for higher dimensional initial boundary value problems and combining boundary and initial conditions for providing new consecutive initial solutions. κ_n^* as stated below, consider the k dimensional equation

$$\Phi(\kappa(\gamma_1,\gamma_2,\ldots,\gamma,\sigma))=\xi(\gamma_1,\gamma_2,\ldots,\gamma_k,\sigma), \quad 0<\gamma_1,\gamma_2,\ldots,\gamma_k<1,\sigma>0 \tag{3.9}$$

Then we have

$$\begin{aligned}
&\kappa_n^*(\gamma_1,\gamma_2,\ldots,\gamma_k,\sigma)\\
&\quad=\kappa_n(\gamma_1,\gamma_2,\ldots,\gamma_k,\sigma)\\
&\quad+(1-\gamma_1)[\xi_{01}(\gamma_2\gamma_3,\ldots,\gamma_k,\sigma)-\kappa_n(0,\gamma_2,\ldots,\gamma_k,\sigma)]\\
&\quad+\gamma_1[\xi_{11}(\gamma_2\gamma_3,\ldots,\gamma_k,\sigma)-\kappa_n(1,\gamma_2,\ldots,\gamma_k,\sigma)]+\cdots\\
&\quad+(1-\gamma_k)[\xi_{0k}(\gamma_1\gamma_2,\ldots,\gamma_{k-1},\sigma)-\kappa_n(\gamma_1,\gamma_2,\ldots,\gamma_{k-1},0,\sigma)]\\
&\quad+\gamma_k[\xi_{1k}(\gamma_1,\gamma_2,\ldots,\gamma_{k-1},\sigma)-\kappa_n(\gamma_1,\gamma_2,\ldots,\gamma_{k-1},1,\sigma)]
\end{aligned} \tag{3.10}$$

where $n=0,1,2,\ldots,$ and boundary conditions linked to equation (3.6) take the form

$$\kappa(0,\gamma_2,\dots,\gamma_k,\sigma)=\xi_{01}(\gamma_2,\gamma_3,\dots,\gamma_k,\sigma),\quad \kappa(1,\gamma_2,\dots,\gamma_k,\sigma)=\xi_{11}(\gamma_2,\gamma_3,\dots,\gamma_k,\sigma),$$
$$\kappa(\gamma_1,0,\dots,\gamma_k,\sigma)=\xi_{02}(\gamma_1,\gamma_3,\dots,\gamma_k,\sigma),\quad \kappa(\gamma_1,1,\dots,\gamma_k,\sigma)=\xi_{12}(\gamma_1,\gamma_3,\dots,\gamma_k,\sigma),$$
$$\kappa(\gamma_1,\dots,\gamma_{k-1},0,\sigma)=\xi_{0k}(\gamma_1,\dots,\gamma_{k-1},\sigma),\quad \kappa(\gamma_1,\dots,\gamma_{k-1},1,\sigma)=g_{1k}(\gamma_1,\dots,\gamma_{k-1},\sigma),$$

(3.11)

and the initial conditions are given by

$$\kappa(\gamma_1,\gamma_2,\dots,\gamma_k,0)=\phi_0(\gamma),\quad \frac{\partial\kappa(\gamma_1,\gamma_2,\dots,\gamma_k,0)}{\partial\sigma}=\phi_1(\gamma) \tag{3.12}$$

Such as treatment of formula (3.7) when $k=2$ according to this research, is a highly effective.

Using adomian decomposition method we have

$$\kappa(\gamma,0)=\phi_0(\gamma) \tag{3.13}$$

$$\kappa_{n+1}(\gamma,\sigma)=L^{-1}[S^{-\alpha}L[(\xi(\gamma,\sigma)] \tag{3.14}$$

put n = 1 in equation (3.5) we get

$$\kappa_1^*(\gamma,\sigma)=\kappa_1(\gamma,\sigma)+(1-\gamma)[\kappa(0,\sigma)-\kappa_1(0,\sigma)]+\gamma[\kappa(1,\sigma)-\kappa_1(1,\sigma)] \tag{3.15}$$

$$\kappa_2(\gamma,\sigma)=L^{-1}[S^{-\alpha}L((\kappa_1^*)_{\gamma\gamma})] \tag{3.16}$$

In similar manner we find κ_3,κ_4 and so on ...

$$\kappa=\kappa_0+\kappa_1+\kappa_2+\kappa_3+\dots \tag{3.17}$$

4. Applications

Example 1: Consider heat-like equation

$$\frac{\partial^\alpha\kappa}{\partial\sigma^\alpha}-\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}=0, 0<\gamma<1, \sigma>0 \tag{4.1}$$

subject to the conditions

$$\kappa(\gamma,0)=\gamma^2,\quad 0<\gamma<1 \tag{4.2}$$

and

$$\kappa(1,\sigma)=e^\sigma,\kappa(0,\sigma)=0,\quad \sigma>0 \tag{4.3}$$

Now by employing a novel approximation κ_n^* in Eq. (3.5) we get

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[\kappa(0,\sigma)-\kappa_n(0,\sigma)]+\gamma[\kappa_n(1,\sigma)-\kappa_n(1,\sigma)]$$

substituting the given condition we get ,

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[0-\kappa_n(0,\sigma)]+\gamma[\mathrm{e}^{\sigma}-\kappa_n(1,\sigma)] \tag{4.4}$$

where $n=0,1,2,\dots$ consider initial approximation is $\kappa_0(\gamma,\sigma)=\kappa(\gamma,0)=\gamma^2$.

Now, Let new initial approximation κ_0^* (when $n = 0$)

$$\kappa_0^*(\gamma,\sigma)=\gamma^2+\gamma(\mathrm{e}^{\sigma}-1) \tag{4.5}$$

From Eq.(4.1) We have

$$L[D_\sigma^\alpha\kappa]=L\left[\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}\right],\quad 0<\gamma<1,\sigma>0 \tag{4.6}$$

$$L[\kappa(\gamma,\sigma)]=S^{-1}\kappa(\gamma,0)+S^{-\alpha}L\left[\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}\right] \tag{4.7}$$

$$L[\kappa(\gamma,\sigma)]=S^{-1}\gamma^2+S^{-\alpha}L\left[\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}\right] \tag{4.8}$$

Applying Laplace inverse on both side, we get

$$\kappa(\gamma,\sigma)=\gamma^2+L^{-1}\left[S^{-\alpha}\frac{1}{2}\gamma^2\kappa_{\gamma\gamma}\right] \tag{4.9}$$

using Adomian Decompostion Method we have , $\kappa_0(\gamma,\sigma)=\gamma^2$

$$\kappa_{n+1}(\gamma,\sigma)=L^{-1}\left[S^{-\alpha}\frac{1}{2}\gamma^2(\kappa_n^*)_{\gamma\gamma}\right] \tag{4.10}$$

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[\kappa(0,\sigma)-\kappa_n(0,\sigma)]+\gamma[\kappa(1,\sigma)-\kappa_n(1,\sigma)] \tag{4.11}$$

Put n = 0 we get ,

$$\kappa_0^*(\gamma,\sigma)=\gamma^2+(1-\gamma)[0-0]+\gamma(\mathrm{e}^{\sigma}-1) \tag{4.12}$$

$$\kappa_0^*(\gamma,\sigma)=\gamma^2+\gamma(\mathrm{e}^{\sigma}-1) \tag{4.13}$$

Similarly we find

$$\kappa_1(\gamma,\sigma)=L^{-1}[S^{-\alpha}L(\frac{1}{2}\gamma^2(\kappa_0^*)_{\gamma\gamma})] \tag{4.14}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2L^{-1}(S^{-\alpha}L[1]) \tag{4.15}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2L^{-1}[S^{-(\alpha+1)}] \tag{4.16}$$

$$\gamma_1(\gamma,\sigma)=\gamma^2\frac{\sigma^\alpha}{\Gamma(1+\alpha)}$$

in similar manner we obatin $\kappa_1^*(\gamma,\sigma)$ by putting n =1 therefore

$$\kappa_1^*(\gamma,\sigma)=\kappa_1(\gamma,\sigma)+(1-\gamma)[\kappa(0,\sigma)-\kappa_1(0,\sigma)]+\gamma[\kappa(1,\sigma)-\kappa_1(1,\sigma)] \tag{4.17}$$

$$\kappa_1^*(\gamma,\sigma)=\gamma^2\frac{\sigma^\alpha}{\Gamma(1+\alpha)}+(1-\gamma)[0-0]+\gamma\left[\mathrm{e}^\sigma-\frac{\sigma^\alpha}{\Gamma(1+\alpha)}\right] \tag{4.18}$$

$$\kappa_2(\gamma,\sigma)=L^{-1}[S^{-\alpha}L(\tfrac{1}{2}\gamma^2(\kappa_1^*)_{\gamma\gamma})] \tag{4.19}$$

$$\kappa_2(\gamma,\sigma)=\frac{\gamma^2}{\Gamma(1+\alpha)}L^{-1}[S^{-\alpha}L(\sigma^\alpha)]=\gamma^2\frac{\sigma^{2\alpha}}{\Gamma(1+2\alpha)} \tag{4.20}$$

In similar fashion we get

$$\kappa_3(\gamma,\sigma)=\gamma^2\frac{\sigma^{3\alpha}}{\Gamma(1+3\alpha)} \tag{4.21}$$

and so on ...

$$\kappa=\kappa_0+\kappa_1+\kappa_2+\kappa_3+\ldots \tag{4.22}$$

$$\kappa=\gamma^2+\gamma^2\frac{\sigma^\alpha}{\Gamma(1+\alpha)}+\gamma^2\frac{\sigma^{2\alpha}}{\Gamma(1+2\alpha)}+\gamma^2\frac{\sigma^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots \tag{4.23}$$

$$\kappa=\gamma^2\left[1+\frac{\sigma^\alpha}{\Gamma(1+\alpha)}+\frac{\sigma^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{\sigma^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots\right] \tag{4.24}$$

$$\kappa=\gamma^2\mathrm{e}^{\sigma^\alpha}$$

where L is the Laplace operator , so that L^{-1} is Laplace inverse operator

$$\kappa(\gamma,\sigma)=\gamma^2e^{\sigma^\alpha}$$

Which is required solution.

Example 2: Consider wave-like equation

$$\frac{\partial^{\alpha+1}\kappa}{\partial\sigma^{\alpha+1}}-\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}=0,\quad 0<\gamma<1,\sigma>0 \tag{4.25}$$

Condition is given as

$$\kappa(\gamma,0)=\gamma,\quad \frac{\partial\kappa(\gamma,0)}{\partial\sigma}=\gamma^2,\quad 0<\gamma<1, \tag{4.26}$$

also

$$\kappa(0,\sigma)=0, \quad \kappa(1,\sigma)=1+\sinh\sigma, \sigma>0. \tag{4.27}$$

Now by employing a novel approximation κ_n^* in equation (3.5) we get

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[\kappa(0,\sigma)-\kappa_n(0,\sigma)]+\gamma[\kappa(1,\sigma)-\kappa_n(1,\sigma)] \tag{4.28}$$

where $n=0,1,2,\ldots$ the initial approximation is

$$\kappa_0(\gamma,\sigma)=\kappa(\gamma,0)+\frac{\partial\kappa(\gamma,0)}{\partial\sigma}\sigma=\gamma+\gamma^2\sigma \tag{4.29}$$

Now, we Start with a new initial approximation κ_0^* (when $n = 0$)

$$\kappa_0^*(\gamma,\sigma)=\gamma+\gamma^2\sigma+\gamma(\sinh\sigma-\sigma) \tag{4.30}$$

$$\kappa_{n+1}(\gamma,\sigma)=L^{-1}[S^{-(\alpha+1)}\tfrac{1}{2}\gamma^2(\kappa_n^*)_{\gamma\gamma}] \tag{4.31}$$

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[\kappa(0,\sigma)-\kappa_n(0,\sigma)]+\gamma[\kappa(1,\sigma)-\kappa_n(1,\sigma)] \tag{4.32}$$

$$\kappa_n^*(\gamma,\sigma)=\kappa_n(\gamma,\sigma)+(1-\gamma)[0-\kappa_n(0,\sigma)]+\gamma[1+\sinh\sigma-\kappa_n(1,\gamma)], \tag{4.33}$$

Put n = 0 we get ,

$$\kappa_0^*(\gamma,\sigma)=\kappa_0(\gamma,\sigma)+(1-\gamma)[0-\kappa_0(0,\sigma)]+\gamma[1+\sinh\sigma-\kappa_0(1,\sigma)], \tag{4.34}$$

$$\kappa_0^*(\gamma,\sigma)=\gamma+\gamma^2\sigma+(1-\gamma)[0-0]+\gamma(\sinh\sigma-\sigma) \tag{4.35}$$

$$\sigma_0^*(\gamma,\sigma)=\gamma+\gamma^2\sigma+\gamma(\sinh\sigma-\sigma) \tag{4.36}$$

Similarly we find

$$\kappa_1(\gamma,\sigma)=L^{-1}[S^{-(\alpha+1)}L(\tfrac{1}{2}\gamma^2(\kappa_0^*)_{\gamma\gamma})] \tag{4.37}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2L^{-1}(S^{-(\alpha+1)}L[\sigma]) \tag{4.38}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2L^{-1}[S^{-(\alpha+1)}\frac{\Gamma 2}{S^2}] \tag{4.39}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2L^{-1}[\frac{\Gamma 2}{S^{(\alpha+2)+1}}] \tag{4.40}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2L^{-1}[\frac{1}{S^{(\alpha+2)+1}}] \tag{4.41}$$

$$\kappa_1(\gamma,\sigma)=\gamma^2\frac{\sigma^{(\alpha+2)}}{\Gamma(\alpha+2+1)} \tag{4.42}$$

in similar manner we obatin $\kappa_1^*(\gamma,\sigma)$ by putting n =1 therefore

$$\kappa_1^*(\gamma,\sigma)=\kappa_1(\gamma,\sigma)+(1-\sigma)[\kappa(0,\sigma)-\kappa_1(0,\sigma)]+\gamma[\kappa(1,\sigma)-\kappa_1(1,\sigma)] \tag{4.43}$$

$$\kappa_1^*(\gamma,\sigma)=\gamma^2\frac{\sigma^{(\alpha+2)}}{\Gamma(\alpha+2+1)}+(1-\gamma)[0-0]+\gamma[1+\sinh\sigma-\frac{\sigma^{(\alpha+2)}}{\Gamma(\alpha+2+1)}] \tag{4.44}$$

$$\kappa_2(\gamma,\sigma)=L^{-1}[S^{-(\alpha+1)}L(\frac{1}{2}\gamma^2(\kappa_1^*)_{\gamma\gamma})] \tag{4.45}$$

$$\kappa_2(\gamma,\sigma)=\frac{\gamma^2}{\Gamma(\alpha+2+1)}L^{-1}[S^{-(\alpha+1)}L(\sigma^{(\alpha+2)})]=\gamma^2\frac{\sigma^{(2\alpha+3)}}{\Gamma(2\alpha+3+1)} \tag{4.46}$$

In similar fashion we get

$$\kappa_3(\gamma,\sigma)=\gamma^2\frac{\sigma^{(3\alpha+4)}}{\Gamma(3\alpha+4+1)} \tag{4.47}$$

and so on .....

$$\kappa=\kappa_0(\gamma,\sigma)+\kappa_1(\gamma,\sigma)+\kappa_2(\gamma,\sigma)+\kappa_3(\gamma,\sigma)+......$$

$$\kappa=\gamma+\gamma^2\sigma+\gamma^2\frac{\sigma^{(\alpha+2)}}{\Gamma(\alpha+2+1)}+\sigma^2\frac{\sigma^{(2\alpha+3)}}{\Gamma(2\alpha+3+1)}+\gamma^2\frac{\sigma^{(3\alpha+4)}}{\Gamma(3\alpha+4+1)}+\dots \tag{4.48}$$

$$\kappa=\gamma+\gamma^2[\sigma+\frac{\sigma^{(\alpha+2)}}{\Gamma(\alpha+2+1)}+\frac{\sigma^{(2\alpha+3)}}{\Gamma(2\alpha+3+1)}+\frac{\sigma^{(3\alpha+4)}}{\Gamma(3\alpha+4+1)}+\dots] \tag{4.49}$$

Which is required solution.

If we consider $\alpha=1$ then wave equation become like

$$\frac{\partial^2\kappa}{\partial\sigma^2}-\frac{1}{2}\sigma^2\frac{\partial^2\kappa}{\partial\gamma^2}=0,\quad 0<\gamma<1,\sigma>0 \tag{4.50}$$

subject to conditions as above

Therefore we get

$$\kappa_1(\gamma,\sigma)=\frac{\gamma^2\sigma^3}{3!} \tag{4.51}$$

$$\kappa_2(\gamma,\sigma)=\frac{\gamma^2\sigma^5}{5!} \tag{4.52}$$

$$\kappa_3(\gamma,\sigma)=\frac{\gamma^2\sigma^7}{7!} \tag{4.53}$$

and so on, the series solution is provided by

$$\kappa(\gamma,\sigma)=\gamma+\gamma^2\left(\sigma+\frac{\sigma^3}{3!}+\frac{\sigma^5}{5!}+\frac{\sigma^7}{7!}+\cdots\right) \tag{4.54}$$

hence

$$\kappa(\gamma,\sigma)=\gamma+\gamma^2\sinh\sigma \tag{4.55}$$

Which is required solution.

Example 3: Consider

$$\frac{\partial^{\alpha+1}\kappa}{\partial\sigma^{\alpha+1}}=\frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\gamma^2}+\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\delta^2}, \quad 0<\gamma,\delta<1,\sigma>0 \tag{4.56}$$

and initial conditions

$$\kappa(\gamma,\delta,0)=\gamma^2+\delta^2,\frac{\partial\kappa(\gamma,\delta,0)}{\partial\sigma}=-(\gamma^2+\delta^2) \tag{4.57}$$

with boundary conditions

$$\begin{aligned}&\kappa(0,\delta,\sigma)=\delta^2e^{-\sigma}, \quad \kappa(1,\delta,\sigma)=(1+\delta^2)e^{-\sigma},\\ &\kappa(\gamma,0,\sigma)=\gamma^2e^{-\sigma}, \quad \kappa(\gamma,1,\sigma)=(1+\gamma^2)e^{-\sigma}\end{aligned} \tag{4.58}$$

Now by employing a novel approximation κ_n^* in Eq. (3.8)

$$\begin{aligned}\kappa_n^*(\gamma,\delta,\sigma)=&\kappa_n(\gamma,\delta,\sigma)+(1-\gamma)[\delta^2e^{-\sigma}-\kappa_n(0,\delta,\sigma)]+\gamma[(1+\delta^2)e^{-\sigma}-\kappa_n(1,\delta,\sigma)]\\ &+(1-\delta)[\gamma^2e^{-\sigma}-\kappa_n(\gamma,0,\sigma)]+\delta[(1+\gamma^2)e^{-\sigma}-\kappa_n(\gamma,1,\sigma)]\end{aligned}$$

where $n=0,1,2,\dots$ the initial approximation is

$$\kappa_0(\gamma,\delta,\sigma)=(\gamma^2+\delta^2)-(\gamma^2+\delta^2)\sigma$$

We Start with a new initial approximation κ_0^* (when $n=0$)

$$\begin{aligned}\kappa_0^*(\gamma,\delta,\sigma)=&\gamma^2+\delta^2-(\gamma^2+\delta^2)\sigma+(1-\gamma)[\delta^2e^{-\sigma}-\delta^2+\delta^2\sigma]\\ &+\gamma[(1+\delta^2)e^{-\sigma}-(1+\delta^2)+(1+\delta^2)\sigma]+(1-\delta)[\gamma^2e^{-\sigma}-\gamma^2+\gamma^2\sigma]\\ &+\delta[(1+\gamma^2)e^{-\zeta}-(\gamma^2+1)+(\gamma^2+1)\sigma]\\ =&-\gamma-\delta+\gamma e^{-\sigma}+\delta e^{-\sigma}+\gamma^2e^{-\sigma}+\delta^2e^{-\sigma}+(\gamma+\delta)\sigma\end{aligned} \tag{4.59}$$

we have

$$L[\frac{\partial^{\alpha+1}\kappa}{\partial\sigma^{\alpha+1}}]=L[\frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\gamma^2}+\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\delta^2}], \quad 0<\gamma<1,\sigma>0, \tag{4.60}$$

where L is the Laplace operator, and L^{-1} is Laplace inverse operator

Using a novel approach to initial solutions κ_n^* The formula of iteration becomes,

$$\kappa_{n+1}^*(\gamma,\delta,\sigma) = L^{-1}\left[S^{-(\alpha+1)}L\left(\frac{1}{2}\delta^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2} + \frac{1}{2}\gamma^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right] \quad (4.61)$$

By Using above Equation we substitute n = 0 to obtain a first iteration as given below

$$\kappa_1(\gamma,\delta,\sigma) = L^{-1}\left[S^{-(\alpha+1)}L\left(\frac{1}{2}\delta^2\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\gamma^2} + \frac{1}{2}\gamma^2\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right]$$

$$\kappa_1(\gamma,\delta,\sigma) = (\gamma^2+\delta^2))L^{-1}\left[S^{-(\alpha+1)}\left(\frac{1}{S+1}\right)\right]$$

$$\kappa(\gamma,\delta,\sigma) = \kappa_0(\gamma,\delta,\sigma) + \kappa_1(\gamma,\delta,\sigma)$$

$$\kappa(\gamma,\delta,\sigma) = (\gamma^2+\delta^2) - (\gamma^2+\delta^2)\sigma + L^{-1}\left[S^{-(\alpha+1)}\left(\frac{1}{s+1}\right)\right] \quad (4.62)$$

if α =1 then above example becomes

$$\frac{\partial^2\kappa}{\partial\sigma^2} = \frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\gamma^2} + \frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\delta^2}, \quad 0<\gamma,\delta<1, \sigma>0 \quad (4.62)$$

with above boundary condition We can easily check

$$\kappa(\gamma,\delta,\sigma) = \kappa_0(\gamma,\delta,\sigma) + \kappa_1(\gamma,\delta,\sigma)$$

$$\kappa(\gamma,\delta,\sigma) = (\sigma^2+\delta^2) - (\sigma^2+\delta^2)\sigma + L^{-1}\left[S^{-(2)}\left(\frac{1}{s+1}\right)\right]$$

$$\kappa(\gamma,\delta,\sigma) = (\gamma^2+\delta^2)e^{-\sigma}.$$

Which is required solution.

Example 4: Consider the equation

$$\frac{\partial^{\alpha+1}\kappa}{\partial\sigma^{\alpha+1}} = \frac{15}{2}(\gamma\kappa_{\gamma\gamma}^2 + \delta\kappa_{\delta\delta}^2) + 2\gamma^2 + 2\delta^2, \quad 0<\gamma,\delta<1, \sigma>0 \quad (4.64)$$

and initial conditions

$$\kappa(\gamma,\delta,0) = 0, \quad \frac{\partial\kappa(\gamma,\delta,0)}{\partial\sigma} = 0 \quad (4.65)$$

with boundary conditions

$$\kappa(0,\delta,\sigma) = \delta^2\sigma^2 + \delta\sigma^6, \quad \kappa(1,\delta,\sigma) = (1+\delta^2)\sigma^2 + (1+\delta)\sigma^6,$$

$$\kappa(\gamma,0,\sigma) = \gamma^2\sigma^2 + \gamma\sigma^6, \quad \kappa(\gamma,1,\sigma) = (1+\gamma^2)\sigma^2 + (1+\gamma)\sigma^6. \quad (4.66)$$

Now by employing a novel approximation κ_n^* in equation (3.8) we get

$$\begin{aligned}\kappa_n^*(\gamma,\delta,\sigma) = \kappa_n(\gamma,\delta,\sigma) &+ (1-\gamma)[\delta^2\sigma^2 + \delta\sigma^6 - \kappa_n(0,\delta,\sigma)]\\ &+\gamma[(1+\delta^2)\sigma^2 + (1+\delta)\sigma^6 - \kappa_n(1,\delta,\sigma)]\\ &+(1-\delta)[\gamma^2\sigma^2 + \gamma\sigma^6 - \kappa_n(\gamma,0,\sigma)]\\ &+\delta[(1+\gamma^2)\sigma^2 + (1+\gamma)\sigma^6 - \kappa_n(\gamma,1,\sigma)], \quad n = 0,1,2,\ldots\end{aligned}$$

The initial approximation is

$$\kappa_0(\gamma,\delta,\sigma) = \kappa(\gamma,\delta,0) + \frac{\partial\kappa(\gamma,\delta,0)}{\partial\sigma}\sigma = 0.$$

We begin with a novel approximation κ_0^* (when $n=0$)

$$\kappa_0^*(\gamma,\delta,\sigma) = (\gamma^2+\delta^2)\sigma^2 + (1-\gamma)\delta\sigma^6 + \gamma(1+\delta)\sigma^6 + (1-\delta)\gamma\sigma^6 + \delta(1+\gamma)\sigma^6 \tag{4.67}$$

$$\kappa_{n+1}(\gamma,\delta,\sigma) = \frac{15}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2} + \delta\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right] \tag{4.68}$$

$$\kappa_{n+1}(\gamma,\delta,\sigma) = \frac{15}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right] + \frac{15}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\delta\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right]$$

if n=0 then equation becomes

$$\kappa_1(\gamma,\delta,\sigma) = \frac{15}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right] + \frac{15}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\delta\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right]$$

$$\kappa_1(\gamma,\delta,\sigma) = 15(\gamma+\sigma)\left[L^{-1}\left(\frac{2}{S^{(\alpha+3+1)}}\right)\right]$$

$$\kappa_1(\gamma,\delta,\sigma) = 30(\gamma+\delta)\left[\left(\frac{\sigma^{\alpha+3}}{\Gamma(\alpha+3+1)}\right)\right] \tag{4.69}$$

if we consider $\alpha = 1$ then equation rewrite as below

$$\frac{\partial^2\kappa}{\partial\sigma^2} = \frac{15}{2}(\gamma\kappa_{\gamma\gamma}^2 + \delta\kappa_{\delta\delta}^2) + 2\gamma^2 + 2\delta^2, \quad 0<\gamma,\delta<1, \zeta>0 \tag{4.70}$$

subject to given condition as above $\kappa_1(\gamma,\delta,\sigma) = \frac{5}{4}(\gamma+\delta)\sigma^4$

We can readily check

$$\kappa(\gamma,\delta,\sigma) = \kappa_0(\gamma,\delta,\sigma) + \kappa_1(\gamma,\delta,\sigma) = (\gamma^2+\delta^2)\sigma^2 + \frac{5}{4}(\gamma+\delta)\sigma^4$$

Which is required solution .

Example 5: Consider

$$\frac{\partial^{\alpha+1}\kappa}{\partial\sigma^{\alpha+1}} = \frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2} + \frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\delta^2}, \quad 0<\gamma,\delta<1, \sigma>0$$

Subject to initial conditions

$$\kappa(\gamma,\delta,0) = \gamma^2+\delta^2, \quad \frac{\partial\kappa(\gamma,\delta,0)}{\partial\sigma} = \delta^2-\gamma^2, \tag{4.71}$$

and boundary conditions

$$\kappa(0,\delta,\sigma) = \delta^2 e^{\sigma}, \quad \kappa(1,\delta,\sigma) = e^{-\sigma}+\delta^2 e^{\sigma}, \tag{4.72}$$

$$\kappa(\gamma,0,\sigma) = \gamma^2 e^{-\sigma}, \quad \kappa(\gamma,1,\sigma) = e^{\sigma}+\gamma^2 e^{-\sigma}. \tag{4.73}$$

Now by employing a novel approximation κ_n^* in equation (3.8) we get

$$\kappa_n^*(\gamma,\delta,\sigma) = \kappa_n(\gamma,\delta,\sigma) + (1-\gamma)[\delta^2 e^{\sigma} - \kappa_n(0,\delta,\sigma)] + \gamma[e^{-\sigma}+\delta^2 e^{\sigma} - \kappa_n(1,\delta,\sigma)]$$
$$+(1-\delta)[\gamma^2 e^{-\sigma} - \kappa_n(\gamma,0,\sigma)] + \delta[e^{\sigma}+\gamma^2 e^{-\sigma} - \kappa_n(\gamma,1,\sigma)]$$

consider $\kappa_0 = (\gamma^2+\delta^2)+(\delta^2-\gamma^2)\sigma$

$$\kappa_{n+1}(\gamma,\delta,\sigma) = \frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2} + \delta^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right] \tag{4.74}$$

$$\kappa_{n+1}(\gamma,\delta,\sigma) = \frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right] + \frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\delta^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right]$$

if n=0 then equation becomes

$$\kappa_1(\gamma,\delta,\sigma) = \frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma^2\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right] + \frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\delta^2\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right]$$

$$\kappa_1(\gamma,\delta,\sigma) = \frac{1}{2}L^{-1}[S^{-(\alpha+1)}L(2\gamma^2 e^{-\sigma})] + \frac{1}{2}L^{-1}[S^{-(\alpha+1)}L(2\delta^2 e^{\sigma})]$$

$$\kappa_1(\gamma,\delta,\sigma) = \gamma^2 L^{-1}\left[S^{-(\alpha+1)}\left(\frac{1}{s+1}\right)\right] + \delta^2 L^{-1}\left[S^{-(\alpha+1)}\left(\frac{1}{s-1}\right)\right]$$

from Eq. (4.52) we get first iteration

$$\kappa_1(\gamma,\delta,\sigma) = \gamma^2[(e^{-\sigma}-1)+\sigma] + \delta^2[(e^{\sigma}-1)-\sigma] \tag{4.75}$$

if we consider $\alpha=1$ then equation rewrite as below

$$\frac{\partial^2\kappa}{\partial\sigma^2} = \frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\sigma^2} + \frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\delta^2}, \quad 0<\gamma,\delta<1, \sigma>0$$

subject to given conditions as above We can readily check

$$\kappa(\gamma,\delta,\sigma)=\kappa_0(\gamma,\delta,\sigma)+\kappa_1(\gamma,\delta,\sigma)=\gamma^2 e^{-\sigma}+\delta^2 e^{\sigma}.$$

Which is required solution.

Example 6: Consider the problem

$$\frac{\partial^{\alpha+1}\kappa}{\partial\sigma^{\alpha+1}}=\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}+\frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\delta^2},\quad 0<\gamma,\delta<1,\sigma>0 \tag{4.76}$$

subject to the conditions

$$\kappa(\gamma,\delta,0)=\delta^2,\frac{\partial\kappa(\gamma,\delta,0)}{\partial\delta}=\gamma^2 \tag{4.77}$$

and boundary conditions

$$\kappa(0,\delta,\sigma)=\delta^2\cosh\sigma,\quad \kappa(1,\delta,\sigma)=\sinh\sigma+\delta^2\cosh\sigma, \tag{4.78}$$

$$\kappa(\gamma,0,\sigma)=\gamma^2\sinh\sigma,\quad \kappa(\gamma,1,\sigma)=\gamma^2\sinh\sigma+\cosh\sigma. \tag{4.79}$$

by employing a novel approximation κ_n^* in Eq. (3.8) we get

$$\kappa_n^*(\gamma,\delta,\sigma)=\kappa_n(\gamma,\delta,\sigma)+(1-\gamma)[\delta^2\cosh\sigma-\kappa_n(0,\delta,\sigma)]+\gamma[\sinh\sigma+$$
$$\delta^2\cosh\sigma-\kappa_n(1,\delta,\sigma)]+(1-\delta)[\gamma^2\sinh\sigma-\kappa_n(\gamma,0,\sigma)]+\delta[\gamma^2\sinh\sigma+$$
$$\cosh\sigma-\kappa_n(\gamma,1,\sigma)],$$

where $n=0,1,2,\ldots$. The initial approximation is

$$\kappa_0(\gamma,\delta,\sigma)=\delta^2+\gamma^2\sigma$$

Now,Consider new approximation κ_0^* (when $n=0$)

$$\kappa_0^*(\gamma,\delta,\sigma)=\delta^2+\gamma^2\sigma+(1-\gamma)[\delta^2\cosh\sigma-\delta^2]+\gamma[\sinh\sigma+\delta^2\cosh\sigma-\delta^2-\sigma]$$
$$+(1-\delta)[\gamma^2\sinh\sigma-\gamma^2\sigma]+\delta[\gamma^2\sinh\sigma+\cosh\sigma-1-\gamma^2\sigma]$$

$$\kappa_{n+1}(\gamma,\delta,\sigma)=\frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2}+\delta^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right] \tag{4.80}$$

$$\kappa_{n+1}(\gamma,\delta,\sigma)=\frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right]+\frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\delta^2\frac{\partial^2\kappa_n^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right] \tag{4.81}$$

if n=0 then equation becomes

$$\kappa_1(\gamma,\delta,\sigma)=\frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\gamma^2\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\gamma^2}\right)\right]+\frac{1}{2}L^{-1}\left[S^{-(\alpha+1)}L\left(\delta^2\frac{\partial^2\kappa_0^*(\gamma,\delta,\sigma)}{\partial\delta^2}\right)\right]$$

$$\kappa_1(\gamma,\delta,\sigma)=\frac{1}{2}L^{-1}[S^{-(\alpha+1)}L(2\gamma^2 sinh\sigma)]+\frac{1}{2}L^{-1}[S^{-(\alpha+1)}L(2\delta^2 cosh\sigma)]$$

$$\kappa_1(\gamma,\delta,\sigma)=\gamma^2L^{-1}[S^{-(\alpha+1)}L(sinh\sigma)]+\delta^2L^{-1}[S^{-(\alpha+1)}L(cosh\sigma)]$$

$$\kappa_1(\gamma,\delta,\sigma)=\gamma^2L^{-1}\left[S^{-(\alpha+1)}\left(\frac{1}{s^2-1}\right)\right]+\delta^2L^{-1}\left[S^{-(\alpha+1)}\left(\frac{s}{s^2-1}\right)\right]$$

for $\alpha=1$ then equation rewrite as

$$\frac{\partial^2\kappa}{\partial\sigma^2}=\frac{1}{2}\gamma^2\frac{\partial^2\kappa}{\partial\gamma^2}+\frac{1}{2}\delta^2\frac{\partial^2\kappa}{\partial\delta^2},\quad 0<\gamma,\delta<1,\sigma>0 \tag{4.82}$$

and it's solution is given as below

$$\kappa_1(\gamma,\delta,\sigma)=\gamma^2L^{-1}\left[S^{-(2)}\left(\frac{1}{S^2-1}\right)\right]+\delta^2L^{-1}\left[S^{-(2)}\left(\frac{S}{S^2-1}\right)\right]$$

we obtain a first iteration

$$\kappa_1(\gamma,\delta,\gamma)=\gamma^2[\sinh\sigma-\sigma]+\delta^2[\cosh\sigma-1]$$

We can readily check

$$\kappa(\gamma,\delta,\sigma)=\kappa_0(\gamma,\delta,\sigma)+\kappa_1(\gamma,\delta,\sigma)=\gamma^2\sinh\sigma+\delta^2\cosh\sigma$$

Which is required solution.

5. Conclusions

In this study, a very powerful technique for creating new initial consecutive solutions θ_n^* by combining boundary and initial conditions is described in formula form (3.5) and (3.8) when $k=2$ This is used to determine approximate values repeatedly θ_n of the solution θ by ADM approach is used to address initial boundary value problems.. This strategy for building fresh sequential initial solutions can provide a more precise result.This paper provides Various examples to demonstrate the efficacy and ease of a novel approach. It is crucial to note that the precise answers for the final four cases were determined right from the first iteration, but if we simply utilise beginning conditions, we would obtain an accurate result by computing infinite successive solutions θ_n.

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Received January, 2023