

Unit groups of semisimple group algebras of groups up to order 180

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Abstract

In this paper, we discuss the structure of the unit groups of the semisimple group algebras of 10 non-metabelian groups having orders 150, 160, 168 and 180, respectively. In particular, we consider the only one non-metabelian group of orders 150 and 180, two non-metabelian groups of order 160 and six non-metabelian groups of order 168 and study the unit groups of their corresponding semisimple group algebras. With this paper, the study of the unit groups of semisimple group algebras of all the groups up to order 180 is completed.

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1. Introduction

Let $K\mathcal{G}$ denote the group algebra of the finite group $\mathcal{G}$ and the finite field K of order $q = p^k$, where characteristics of K is p and $k > 0$. It is essential to remember that the unit group $U(K\mathcal{G})$ of $K\mathcal{G}$ is the collection of all invertible elements. Because the units are used in so many different fields like coding theory (see [10], [11]), number theory [8], cryptography [19] etc., the study of the structure of unit groups has naturally become an important research area, see [9, 14, 22, 23, 28, 29, 30, 32] for some of the recent works in this direction. Additionally, Manju et al. [15] discovered the expressions for the idempotent elements in the dihedral group algebras. These finding can also be used to construct unit elements in group ring.

We recall that the subgroup generated by the commutators of the group $\mathcal{G}$ is known as the derived subgroup $\mathcal{G}'$ of $\mathcal{G}$. Also, if $\mathcal{G}'$ is abelian, then $\mathcal{G}$ is known as a metabelian group. The units of the semisimple group algebras (SGA) generated by metabelian groups were identified by Bakshi et al. in [5]. Consequently, deducing the structure of the semisimple non-metabelian group algebras became a salient research area. Due to the seminal work of Pazderski [26], it is possible to identify the possible orders of non-metabelian groups (NMG). On that note, Khan et al. [12] and Maheshwari et al. [13] computed the unit groups for groups of order 24, whereas for the groups of order up to 120, Mittal et al. and Arvind et al. contributed in [17, 18, 20, 27, 24, 4]. In continuation, Abhilash et al. [1] considered all the NMG of order 128 and studied the unit groups of their corresponding SGA. Also, Abhilash et al. [2] and Mittal et al. [21] completed the work for groups of order 144. In addition to that, Abhilash et al. [3] discussed the unit groups for the groups of order 162.

In this paper, our main purpose is to completely deduce the unit groups for all the NMG up to order 180. Using [26], we can note that there are NMG of orders 150, 160, 168 and 180 for which the unit groups of their corresponding SGA are not yet studied in the literature. Up to isomorphism, we observe that there is only one non-metabelian group of order 150, two NMG of order 160, six NMG of order 168 and one NMG of order 180. We consider all the above-mentioned 10 groups and study the Wedderburn decompositions (WD) (see [16]) of their corresponding SGA. From the knowledge of WD, it is trivial to derive the unit group.

The flow of this paper is as follows. Section 2 discusses the important definitions and results prerequisite for our work. In Section 3, we introduce the 10 non-metablian groups to be studied in this paper. The main results

on the unit groups of the SGA are given in the subsequent sections. The final section concludes the paper.

2. Preliminaries

The following notations apply to this entire paper. K denotes the finite field consists of $q = p^k$ elements with characteristic p and $k \geq 1$. $\mathcal{G}$ denotes the finite group. The prime p is such that $p \nmid n$. Also, K_d , $d \in \mathbb{N}$ denotes the extension field of K with degree d . e denotes the exponent of $\mathcal{G}$ and ω denotes the primitive e -th root of unity over K . $\mathbb{G}$ denotes the $\text{Gal}(K(\omega)/K)$, where $K(\omega)$ is the splitting field of K . Let us denote

$$\mathcal{T}_{\mathcal{G},K} = \{s \mid \sigma(\omega) = \omega^s, \text{ where } \sigma \in \mathbb{G}\}.$$

Suppose the characteristic p does not divide the order of an element $x \in \mathcal{G}$, then x is said to be p' -element. Let C_x denote the conjugacy class of x and $\gamma_x = \sum_{h \in C_x} h$. The cyclotomic K -class of γ_x is the set

$$S_K(\gamma_x) = \{\gamma_{x^s} \mid s \in \mathcal{T}_{\mathcal{G},K}\}.$$

The next two results are given in [7]. The first result tells us about the number of cyclotomic K -classes and the second result is related to cardinality of a particular cyclotomic class.

Proposition 2.1: The number of components which are simple in $\frac{K\mathcal{G}}{J(K\mathcal{G})}$ and the total count of distinct cyclotomic K -classes in $\mathcal{G}$, where $J(K\mathcal{G})$ denotes the radical of $K\mathcal{G}$, is in 1-1 correspondence.

Lemma 2.1: The total count of distinct cyclotomic K -classes in $\mathcal{G}$ is denoted by l and let $\mathbb{G}$ is cyclic. Suppose the simple components of $Z(\frac{K\mathcal{G}}{J(K\mathcal{G})})$ are $K^{(1)}, K^{(2)}, \dots, K^{(l)}$ and the cyclotomic K -classes of $\mathcal{G}$ are $S_1, S_2, \dots, S_l$, then the cardinality of S_i denoted by $|S_i| = [K^{(i)} : K]$ with a appropriate ordering of the indices.

The next result is very crucial in our work and will help in the unique characterization of the WD of the semi-simple group algebras. For its proof, we refer to [16, Chapter 3].

Lemma 2.2:

- (i) Let $K\mathcal{G}$ be a semi-simple group algebra and let $\mathcal{N}$ be a normal subgroup of $\mathcal{G}$. Then

$$K\mathcal{G} \cong K\left(\frac{\mathcal{G}}{\mathcal{N}}\right) \oplus \Delta(\mathcal{G}, \mathcal{N}),$$

where $\Delta(\mathcal{G}, N)$ is an ideal of $K\mathcal{G}$ generated by the set $\{n-1 : n \in \mathcal{N}\}$.

(ii) Let $K\mathcal{G}$ be a semi-simple group algebra. Then

$$K\mathcal{G} \cong K\left(\frac{\mathcal{G}}{\mathcal{G}'}\right) \oplus \Delta(\mathcal{G}, \mathcal{G}'),$$

where $K\left(\frac{\mathcal{G}}{\mathcal{G}'}\right)$ is the sum of all commutative simple components of $K\mathcal{G}$ and $\Delta(\mathcal{G}, \mathcal{G}')$ is the sum of all others.

Further, we discuss a necessary condition for the dimension of the matrix algebra in the WD (see [6]).

Lemma 2.3: Suppose $\oplus_{i=1}^t M_{n_i}(K_i)$ is the summand of the semi-simple group algebra $K\mathcal{G}$, where p is the characteristics of K . Then, none of the n_i are divided by p .

Next, we discuss two very important results that plays a crucial role in our main results. The idea of the proof of Lemma 2.4 follows from [31].

Lemma 2.4: Let p_1 and p_2 be two primes. Let $K_1 = \mathbb{F}_{q_1}$ be a field with $q_1 = p_1^{k_1}$ elements and let $K_2 = \mathbb{F}_{q_2}$ be a field with $q_2 = p_2^{k_2}$ elements, where $k_1, k_2 \geq 1$. Let both the group algebras $K_1\mathcal{G}$, $K_2\mathcal{G}$ be semisimple. Suppose that

$$K_1\mathcal{G} \cong \oplus_{i=1}^t M(n_i, K_1), n_i \geq 1$$

and $M(n, K_2^{(r)})$ is one of the component in the WD of the group algebra $K_2\mathcal{G}$ for some $r \geq 2$ and any positive integer n , i.e.,

$$K_2\mathcal{G} \cong \oplus_{i=1}^{s-1} M(m_i, K_{2,i}) \oplus M(n, K_2^{(r)}), m_i \geq 1.$$

Here $K_{2,i}$ is a field extension of K_2 . Then $M(n, K_1)$ must be a component of the WD of the group algebra $K_1\mathcal{G}$ and it appears atleast r times.

Proof: Let $\overline{\mathbb{F}_{q_1}}$ denote the algebraic closure of $\mathbb{F}_{q_1}$. Then, using the concept of tensor products [16], we note that

$$\overline{\mathbb{F}_{q_1}\mathcal{G}} = \overline{\mathbb{F}_{q_1}}\mathcal{G} \otimes_{\mathbb{F}_{q_1}} \overline{\mathbb{F}_{q_1}} \cong \oplus_{i=1}^t M(n_i, \overline{\mathbb{F}_{q_1}}).$$

On the other hand, we observe that

$$\overline{\mathbb{F}_{q_1}\mathcal{G}} = \overline{\mathbb{F}_{q_2}\mathcal{G}} \otimes_{\mathbb{F}_{q_2}} \overline{\mathbb{F}_{q_1}} \cong \oplus_{i=1}^{s-1} M(m_i, \overline{\mathbb{F}_{q_2}} \otimes_{\mathbb{F}_{q_2}} \overline{\mathbb{F}_{q_1}}) \oplus M(n, \overline{\mathbb{F}_{q_2}} \otimes_{\mathbb{F}_{q_2}} \overline{\mathbb{F}_{q_1}}).$$

By utilizing the fact that $\overline{\mathbb{F}_{q_2}} \otimes_{\mathbb{F}_{q_2}} \overline{\mathbb{F}_{q_1}} \cong \overline{\mathbb{F}_{q_1}}^r$, we can get

$$M(n, \mathbb{F}_{q_2'} \otimes_{\mathbb{F}_{q_2}} \overline{\mathbb{F}_{q_1}}) \cong M(n, \overline{\mathbb{F}_q})^r.$$

From the uniqueness of the decomposition, we get the result.

Lemma 2.5: [4, corollary 3.8] *Let $K\mathcal{G}$ be a finite semisimple group algebra. Then if there exists an irreducible representations of degree n over K , then one of the Wedderburn components of $K\mathcal{G}$ is $M_n(K)$. Also, if there exists k irreducible representations of degree n over K , then $M_n(K)^k$ is a summand of the group algebra $K\mathcal{G}$.*

Throughout this paper, let $x^{-1}y^{-1}xy = [x, y]$ denote the commutator of $x, y \in \mathcal{G}$ and let 1 denote the identity element of $\mathcal{G}$. If there are j number of conjugacy classes (CC) in $\mathcal{G}_i$, then the representatives of the CC are denoted by $g_1 (= 1), g_2, g_3, \dots, g_j$.

3. NMG of orders 150, 160, 168 and 180

In this section, we discuss the structure of the non-metabelian groups of orders 150, 160, 168 and 180.

- (a) There are 13 non-isomorphic groups of order 150 and only one of them is non-metabelian, i.e., $\mathcal{G}_1 := (C_5 \times C_5) \rtimes S_3$.
- (b) There are 238 non-isomorphic groups of order 160 and only two of them are non-metabelian, i.e., $\mathcal{G}_2 := ((C_2 \times Q_8) \rtimes C_2) \rtimes C_5$ and $\mathcal{G}_3 := ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2$.
- (c) There are 57 non-isomorphic groups of order 168 and the following six of them are non-metabelian:
 - 1. $\mathcal{G}_4 := C_7 \times SL(2, 3)$
 - 2. $\mathcal{G}_5 := Q_8 \rtimes (C_7 \rtimes C_3)$
 - 3. $\mathcal{G}_6 := SL(3, 2)$
 - 4. $\mathcal{G}_7 := (C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3)$
 - 5. $\mathcal{G}_8 := C_7 \times S_4$
 - 6. $\mathcal{G}_9 := C_7 \rtimes S_4$
- (d) There are 37 non-isomorphic groups of order 180 and only one of them is non-metabelian, i.e., $\mathcal{G}_{10} := GL(2, 4) \simeq C_3 \times A_5$.

We note that groups $\mathcal{G}_4$, $\mathcal{G}_8$ and $\mathcal{G}_{10}$ are direct product of smaller groups. The WD of the SGA generated by these three groups can be computed using tensor products as in [21]. Also, the unit group of the

group algebra $K\mathcal{G}_6$ is given in [25]. Therefore, we compute the WD of the SGA generated by the remaining groups. More precisely, section 4 considers $\mathcal{G}_1$, section 5 considers $\mathcal{G}_2, \mathcal{G}_3$ and the remaining groups are considered in section 6.

Notation: In the remaining part of this paper, we denote $K = \mathbb{F}_q$ and $K_i = \mathbb{F}_{q^i}$ for $i \geq 2$.

4. Unit group of group algebras generated by $\mathcal{G}_1$

$\mathcal{G}_1 = (C_5 \times C_5) \rtimes S_3$. The group $\mathcal{G}_1$ has the following presentation:

$$\mathcal{G}_1 = \langle x_1, x_2, x_3, x_4 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1], [x_4, x_1]x_4^{-3}x_3^{-4}, \\ x_2^3, [x_3, x_2]x_4^{-3}, [x_4, x_2]x_4^{-2}x_3^{-4}, x_3^5, [x_4, x_3], x_4^5 \rangle.$$

The sizes (S), orders (O) and the representatives (R) of the 13 CC of $\mathcal{G}_1$ are given below:

R	1	x_1	x_2	x_3	x_4	x_1x_3	x_1x_4	x_3^2	x_4^2	$x_1x_3x_4$	$x_1x_4^2$	x_3^3	x_4^4
S	1	15	50	3	6	15	15	3	6	15	15	3	3
O	1	2	3	5	5	10	10	5	5	10	10	5	5

It is clear that the value of e in $\mathcal{G}_1$ is 30.

Theorem 4.1: The $U(K\mathcal{G}_1)$ of the group algebra $K\mathcal{G}_1$ is given below:

- (1) for $q \equiv \{1, 11\} \pmod{30}$, $U(K\mathcal{G}_1) \cong K^{*2} \oplus GL_2(K) \oplus GL_3(K)^8 \oplus GL_6(K)^2$.
- (2) for $q \equiv \{7, 13, 17, 23\} \pmod{30}$, $U(K\mathcal{G}_1) \cong K^{*2} \oplus GL_2(K) \oplus GL_3(K_4)^2 \oplus GL_6(K_2)$.
- (3) for $q \equiv \{19, 29\} \pmod{30}$, $U(K\mathcal{G}_1) \cong K^{*2} \oplus GL_2(K) \oplus GL_3(K_2)^4 \oplus GL_6(K)^2$.

Proof: The group algebra $K\mathcal{G}_1$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_1' \cong (C_5 \times C_5) \rtimes C_3$ and $\frac{\mathcal{G}_1}{\mathcal{G}_1'} \cong C_2$. Since $q = p^k$ and $p > 5$, we split the proof in the following 3 cases.

Case 1: $q \equiv \{1, 11\} \pmod{30}$. In this case, the cardinality of every cyclotomic K -class is 1. Therefore, the decomposition of the group algebra by using proposition 2.1 and lemma 2.1 is $K\mathcal{G}_1 \cong K \oplus \bigoplus_{i=1}^{n_i} M_{n_i}(K)$, $n_i \geq 1$. By applying (ii) of lemma 2.2, we further deduce that

$$K\mathcal{G}_1 \cong K^2 \bigoplus_{i=1}^{11} M_{n_i}(K), n_i \geq 2.$$

Since the dimensions of both the sides are same, we end up with $148 = \sum_{i=1}^{11} n_i^2$. This equation has 13 different solutions given as follows:

$$\begin{aligned} & (2^8, 4, 6, 8), (2^6, 3^3, 4, 9), (2^6, 3, 4, 5^2, 7), (2^5, 3^3, 4, 6, 7), (2^4, 3^4, 4^2, 8), \\ & (2^4, 3, 4^3, 5^3), (2^4, 4^6, 6)(2^3, 3^4, 5^4), (2^3, 3^3, 4^3, 5, 6), (2^2, 3^6, 5^2, 6), \\ & (2, 3^8, 6^2), (2, 3^7, 4^2, 7), (3^4, 4^7). \end{aligned}$$

By incorporating lemma 2.3, we conclude that p can not be 7. This means that we are remaining with 10 choices of n_i s. We define the group homomorphism $f_1: \mathcal{G}_1 \rightarrow GL_6(K(11))$, where $K(11)$ is the finite field having 11 elements, as follows:

$$\begin{aligned} x_1 &\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, x_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\ x_3 &\mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}, x_4 \mapsto \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}. \end{aligned}$$

Again, we define the group homomorphism $f_2: \mathcal{G}_1 \rightarrow GL_6(K(11))$ as follows:

$$\begin{aligned} x_1 &\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, x_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\ x_3 &\mapsto \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, x_4 \mapsto \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}. \end{aligned}$$

Both the group homomorphisms f_1 and f_2 are irreducible representations of $\mathcal{G}_1$ of degree 6. Therefore, lemma 2.5 implies that $M_6(K(11))^2$ must be a summand of $K(11)\mathcal{G}_1$. This confirms that $(2, 3^8, 6^2)$ is the only possible choice of the values of n_i s. Therefore, we have

$$K\mathcal{G}_1 \cong K^2 \oplus M_2(K) \oplus M_3(K)^8 \oplus M_6(K)^2.$$

Case 2: $q \equiv \{7, 13, 17, 23\} \pmod{30}$. In this case, the cyclotomic K -class corresponding to g_4 includes g_8, g_{13}, g_{12} . Similarly, g_6 includes g_7, g_{10}, g_{11} . Also, the cyclotomic K -class corresponding to g_5 includes g_9 , while rest of the g_i s forms individual classes. Therefore, as per proposition 2.1, lemma 2.1 and (ii) of lemma 2.2, the decomposition for this scenario is given by

$$\begin{aligned} K\mathcal{G}_1 &\cong K^2 \oplus M_{n_1}(K) \oplus M_{n_2}(K_2) \oplus_{i=3}^4 M_{n_i}(K_4), \quad n_i > 1 \Rightarrow 148 \\ &= n_1^2 + 2 \cdot n_2^2 + 4(n_3^2 + n_4^2). \end{aligned}$$

The possible choices of n_i s fulfilling above are $(2, 2, 3, 5)$, $(2, 6, 3, 3)$, $(4, 4, 3, 4)$, $(6, 4, 2, 4)$ and $(8, 4, 2, 3)$. We observe that the subgroup $N_1 := \langle x_3, x_4 \rangle$ is normal in $\mathcal{G}_1$ and $\mathcal{G}_1 / N_1 \cong S_3$. By [29], it follows that $M_2(K)$ must be a Wedderburn component of $K\mathcal{G}_1$. Therefore, the unique choice of n_i s is one of $(2, 2, 3, 5)$ or $(2, 6, 3, 3)$. By lemma 2.4, if $M_5(K_4)$ is a Wedderburn component of $K\mathcal{G}_1$, then $M_5(K)$ must be a Wedderburn component in case 1, which is not so. Therefore, $(2, 6, 3, 3)$ is the required choice, which means that

$$K\mathcal{G}_1 \cong K^2 \oplus M_2(K) \oplus M_3(K_4)^2 \oplus M_6(K_2).$$

Case 3: $q \equiv \{19, 29\} \pmod{30}$. In this case, the cyclotomic K -class corresponding to g_4 includes g_{13} , whereas g_6 includes g_{11} . Similarly, g_7 includes g_{10} and g_8 includes g_{12} , while rest of the g_i s forms individual classes. Therefore, similar to case 2, we note that

$$K\mathcal{G}_1 \cong K^2 \oplus_{i=1}^3 M_{n_i}(K) \oplus_{i=4}^7 M_{n_i}(K_2), \quad n_i > 1 \Rightarrow 148 = \sum_{i=1}^3 n_i^2 + \sum_{i=4}^7 2 \cdot n_i^2.$$

As in case 2, we can note that $M_2(K)$ must be a Wedderburn component. This means that we need to find the unique solution of $144 = \sum_{i=1}^2 n_i^2 + \sum_{i=3}^6 2 \cdot n_i^2$. Next, we define the group homomorphism $f: \mathcal{G}_1 \rightarrow GL_6(K(19))$, where $K(19)$ is the finite field having 19 elements, as follows:

$$\begin{aligned}
x_1 \mapsto & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \\
x_3 \mapsto & \begin{pmatrix} 12 & 2 & 0 & 0 & 0 & 0 \\ 18 & 11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & 2 & 0 & 0 \\ 0 & 0 & 18 & 11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 11 \\ 0 & 0 & 0 & 0 & 4 & 9 \end{pmatrix}, \quad x_4 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 11 & 17 & 0 & 0 \\ 0 & 0 & 1 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & 2 \\ 0 & 0 & 0 & 0 & 18 & 11 \end{pmatrix}.
\end{aligned}$$

Therefore, as in case 1, we see that $M_6(K(19))$ must be a Wedderburn component of $K(19)\mathcal{G}_1$. So, it remains to find the unique solution of $108 = n_1^2 + \sum_{i=2}^5 2 \cdot n_i^2$. The last equations has 3 solutions given by $(2^2, 4^3)$, $(2, 3^3, 5)$ and $(6, 3^4)$. Also, lemma 2.4 implies that $M_6(K_2)$ can not be the Wedderburn component of $K\mathcal{G}_1$. Furthermore, lemma 2.4 also implies that $(2^2, 4^3)$ can not be the choice of n_i s as if $M_2(K_2)$ is the Wedderburn component of $K\mathcal{G}_1$, then $M_2(K)^2$ must be a summand in case 1, which is not so. Therefore, the required choice of n_i s is $(6, 3^4)$, which means

$$K\mathcal{G}_1 \cong K^2 \oplus M_2(K) \oplus M_3(K_2)^4 \oplus M_6(K)^2.$$

This completes the proof.

5. Unit group of group algebras generated by $\mathcal{G}_2, \mathcal{G}_3$

5.1 $\mathcal{G}_2 = ((C_2 \times Q_8) \rtimes C_2) \rtimes C_5$.

The group $\mathcal{G}_2$ has the following presentation:

$$\begin{aligned}
\mathcal{G}_2 = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid & x_1^5, [x_2, x_1]x_5^{-1}x_2^{-1}, [x_3, x_1]x_5^{-1}x_3^{-1}x_2^{-1}, [x_4, x_1] \\
& x_6^{-1}x_5^{-1}x_4^{-1}x_3^{-1}x_2^{-1}, [x_5, x_1]x_6^{-1}x_4^{-1}x_3^{-1}x_2^{-1}, [x_6, x_1]x_2^2, [x_3, x_2]x_6^{-1}, [x_4, x_2]x_6^{-1}, \\
& [x_5, x_2]x_6^{-1}, [x_6, x_2], x_3^2x_6^{-1}, [x_4, x_3]x_6^{-1}, [x_5, x_3], [x_6, x_3], x_4^2x_6^{-1}, [x_5, x_4], \\
& [x_6, x_4], x_5^2, [x_6, x_5], x_6^2 \rangle.
\end{aligned}$$

The sizes, orders and the representatives of the 13 CC of $\mathcal{G}_2$ are given below:

R	1	x_1	x_2	x_3	x_4	x_6	x_1^2	x_1x_2	x_1^3	$x_1^2x_3$	x_1^4	$x_1^3x_4$	$x_1^4x_2$
S	1	16	10	10	10	1	16	16	16	16	16	16	16
O	1	5	2	4	4	2	5	10	5	10	5	10	10

the value of e in $\mathcal{G}_2$ is 20.

Theorem 5.1: *The $U(K\mathcal{G}_2)$ of the group algebra $K\mathcal{G}_2$ is given below:*

- (1) for $q \equiv \{1, 11\} \pmod{20}$, $U(K\mathcal{G}_2) \cong K^{*5} \oplus GL_4(K)^5 \oplus GL_5(K)^3$.
- (2) for $q \equiv \{3, 7, 13, 17\} \pmod{20}$, $U(K\mathcal{G}_2) \cong K^* \oplus K_4^* \oplus GL_4(K) \oplus GL_4(K_4) \oplus GL_5(K)^3$.
- (3) for $q \equiv \{9, 19\} \pmod{20}$, $U(K\mathcal{G}_2) \cong K^* \oplus K_2^{*2} \oplus GL_4(K) \oplus GL_4(K_2)^2 \oplus GL_5(K)^3$.

Proof: The group algebra $K\mathcal{G}_2$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_{2'} \cong (C_2 \times Q_8) \rtimes C_2$ and $\frac{\mathcal{G}_2}{\mathcal{G}_{2'}} \cong C_5$.

Case 1: $q \equiv \{1, 11\} \pmod{20}$. In this case, the cardinality of every cyclotomic K -class is 1. Therefore, as per proposition 2.1, lemma 2.1 and (ii) of lemma 2.2, the WD in this case is

$$K\mathcal{G}_2 \simeq K^5 \oplus_{i=1}^8 M_{n_i}(K), 155 = \sum_{i=1}^8 n_i^2, n_i > 1.$$

Let $N_1 := \langle x_5 \rangle$. One can note that N_1 is a normal subgroup of $\mathcal{G}_2$ and $\mathcal{G}_2 / N_1 \cong (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$. Using [5], we note that $K(\mathcal{G}_2 / N_1) \cong K^5 \oplus M_5(K)^3$ as the group $\mathcal{G}_2 / N_1$ is metabelian. By engaging (i) of lemma 2.2, we reach at

$$K\mathcal{G}_2 \simeq K^5 \oplus M_5(K)^3 \oplus_{i=1}^5 M_{n_i}(K), 80 = \sum_{i=1}^5 n_i^2, n_i > 1.$$

The possible choices of n_i 's fulfilling above are $(2^4, 8)$, $(2, 3^3, 7)$ and (4^5) . By lemma 2.3, $p=7$ is not possible. Next, to identify the unique choice of n_i 's, we define the group homomorphism $f: \mathcal{G}_2 \rightarrow GL_4(K(11))$, where $K(11)$ is the finite field having 11 elements as follows.

$$x_1 \mapsto \begin{pmatrix} 5 & 0 & 8 & 6 \\ 0 & 6 & 8 & 7 \\ 5 & 0 & 3 & 5 \\ 0 & 6 & 3 & 4 \end{pmatrix}, x_2 \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, x_3 \mapsto \begin{pmatrix} 8 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 10 \\ 0 & 0 & 3 & 8 \end{pmatrix}$$

$$x_4 \mapsto \begin{pmatrix} 8 & 3 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 8 \end{pmatrix}, x_5 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}, x_6 \mapsto \begin{pmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}.$$

The group homomorphism f is an irreducible representations of $\mathcal{G}_2$ of degree 4. Therefore, lemma 2.5 implies that $M_4(K(11))$ must be a Wedderburn component of $K(11)\mathcal{G}_1$. This confirms that (4^5) is the only possible choice of the values of n_i 's. Therefore, we have

$$K\mathcal{G}_2 \cong K^5 \oplus M_4(K)^5 \oplus M_5(K)^3.$$

Case 2: $q \equiv \{3, 7, 13, 17\} \pmod{20}$. In this case, the cyclotomic K -class of g_2 includes g_7, g_9, g_{11} . Similarly, g_8 includes g_{10}, g_{12}, g_{13} , while the rest of the g_i 's forms individual classes. Therefore, similar to case 1, the decomposition in this scenario is given by

$$K\mathcal{G}_2 \cong K \oplus K_4 \oplus_{i=1}^4 M_{n_i}(K) \oplus M_{n_5}(K_4) \Rightarrow 155 = \sum_{i=1}^4 n_i^2 + 4 \cdot n_5^2, n_i > 1.$$

This equation has 6 different solution sets for n_i 's given by $(2, 3, 5, 9, 3)$, $(3^3, 8, 4)$, $(3, 5, 6, 7, 3)$, $(4, 5^3, 4)$, $(4, 5, 7^2, 2)$, and $(5^3, 8, 2)$. Due to lemma 2.3, $M_3(K)$ and $M_7(K)$ can not be the Wedderburn components in this case. Consequently, the possible choice of n_i 's is either $(4, 5^3, 4)$ or $(5^3, 8, 2)$. For uniqueness, let $(5^3, 8, 2)$ be the final choice of n_i 's. This means $M_2(K_4)$ is a Wedderburn component in this case. Hence, lemma 2.4 implies that $M_2(K)^4$ must be a summand in case 1, which is not so. Therefore, the correct value of n_i 's is $(4, 5^3, 4)$. Thus, we have

$$K\mathcal{G}_2 \cong K \oplus K_4 \oplus M_4(K) \oplus M_4(K_4) \oplus M_5(K)^3.$$

Case 3: $q \equiv \{9, 19\} \pmod{20}$. In this case, the cyclotomic K -classes of g_2 and g_7 club, respectively, with g_{11} and g_9 . Similarly, g_8 and g_{10} club, respectively, with g_{13} and g_{12} . The remaining representatives form individual classes. Therefore, we note that

$$K\mathcal{G}_2 \cong K \oplus K_2^2 \oplus_{i=1}^4 M_{n_i}(K) \oplus_{i=5}^6 M_{n_i}(K_2) \Rightarrow 155 = \sum_{i=1}^4 n_i^2 + 2 \cdot n_5^2 + 2 \cdot n_6^2, n_i > 1.$$

By again considering the normal subgroup N_1 , we observe that $K(\mathcal{G}_2 / N_1) \cong K \oplus K_2^2 \oplus M_5(K)^3$ (see [5]). Hence, (i) of lemma 2.2 derives that

$$K\mathcal{G}_2 \cong K \oplus K_2^2 \oplus M_5(K)^3 \oplus M_{n_1}(K) \oplus_{i=2}^3 M_{n_i}(K_2) \Rightarrow 80 = n_1^2 + 2(n_2^2 + n_3^2), n_i > 1.$$

The last equation has only two choices given by (4^3) and $(8, 2^2)$. Following the same approach of case 2, we conclude that $(8, 2^2)$ cannot be a correct choice of the values of n_i s. Consequently, we have

$$K\mathcal{G}_2 \cong K \oplus K_2^2 \oplus M_4(K) \oplus M_4(K_2)^2 \oplus M_5(K)^3.$$

This completes the proof.

$$5.2 \quad \mathcal{G}_3 = ((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2.$$

The group $\mathcal{G}_3$ has the following presentation:

$$\begin{aligned} \mathcal{G}_3 = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid & x_1^2, [x_2, x_1]x_2^{-3}, [x_3, x_1]x_4^{-1}, [x_4, x_1], [x_5, x_1]x_6^{-1}x_5^{-1}x_4^{-1}x_3^{-1}, \\ & [x_6, x_1]x_6^{-1}x_5^{-1}x_3^{-1}, x_2^5, [x_3, x_2]x_4^{-1}, [x_4, x_2]x_5^{-1}, [x_5, x_2]x_6^{-1}, [x_6, x_2]x_6^{-1}x_3^{-1}, \\ & x_3^2, [x_4, x_3], [x_5, x_3], [x_6, x_3]x_4^2, [x_5, x_4], [x_6, x_4]x_5^2, [x_6, x_5]x_6^2 \rangle. \end{aligned}$$

The sizes, orders and the representatives of the 10 CC of $\mathcal{G}_3$ are given below:

R	1	x_1	x_2	x_3	x_4	x_5	x_1x_3	x_1x_5	x_1x_6	x_2^2
S	1	20	32	5	5	5	20	20	20	32
O	1	2	5	2	2	2	4	4	4	5

Clearly, the value of e in $\mathcal{G}_3$ is 20.

Theorem 5.2: The $U(K\mathcal{G}_3)$ of the group algebra $K\mathcal{G}_3$ is given below:

- (1) for $q \equiv \{1, 9, 11, 19\} \pmod{20}$, $U(K\mathcal{G}_3) \cong K^{*2} \oplus GL_2(K)^2 \oplus GL_5(K)^6$.
- (2) for $q \equiv \{3, 7, 13, 17\} \pmod{20}$, $U(K\mathcal{G}_3) \cong K^{*2} \oplus GL_2(K_2) \oplus GL_5(K)^6$.

Proof: The group algebra $K\mathcal{G}_3$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_3' \cong (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$ and $\frac{\mathcal{G}_3}{\mathcal{G}_3'} \cong C_2$.

Case 1: $q \equiv \{1, 9, 11, 19\} \pmod{20}$. In this case, the cardinality of every cyclotomic K -class is 1. Therefore, similar to previous theorem, we may write

$$K\mathcal{G}_3 \simeq K^2 \oplus_{i=1}^8 M_{n_i}(K), 158 = \sum_{i=1}^8 n_i^2, n_i > 1.$$

Let $N_1 := \langle x_3, x_4, x_5, x_6 \rangle$. One can note that N_1 is a normal subgroup of $\mathcal{G}_3$ and $\mathcal{G}_3 / N_1 \cong D_{10}$. Using [5], we note that $K(\mathcal{G}_3 / N_1) \cong K^2 \oplus M_2(K)^2$. By using (i) of lemma 2.2, we reach at

$$K\mathcal{G}_3 \simeq K^2 \oplus M_2(K)^2 \oplus_{i=1}^6 M_{n_i}(K), 150 = \sum_{i=1}^6 n_i^2, n_i > 1.$$

As per lemma 2.3, p can not be 3, 7. Moreover, $M_6(K)$ can not be a Wedderburn component, as otherwise $p=3$ divides 6 which is not possible due to lemma 2.3. Therefore, the possible choices of n_i s fulfilling the last equation are $(2, 4^2, 5^2, 8)$ and (5^6) . Next, similar to case 1 of theorem 5.1, we can define a group homomorphism from $\mathcal{G}_3$ to $GL_5(K(3))$ as follows:

$$\begin{aligned} x_1 \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, x_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, x_3 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\ x_4 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, x_5 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, x_6 \mapsto \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \end{aligned}$$

Similarly, we can define atleast two more distinct group homomorphisms from $\mathcal{G}_3$ to $GL_5(K(3))$. Therefore, as per lemma 2.5, the unique choice of n_i s is (5^6) , which means that

$$K\mathcal{G}_3 \cong K^2 \oplus M_2(K)^2 \oplus M_5(K)^6.$$

Case 2: $q \equiv \{3, 7, 13, 17\} \pmod{20}$. In this case, the cyclotomic K -class of g_3 includes g_{10} , while the rest of the g_i s form individual classes. The decomposition for this case is given by

$$K\mathcal{G}_3 \cong K^2 \oplus_{i=1}^6 M_{n_i}(K) \oplus M_{n_7}(K_2) \Rightarrow 158 = \sum_{i=1}^6 n_i^2 + 2 \cdot n_7^2, n_i > 1.$$

Due to [5], we know that $K(\mathcal{G}_3 / N_1) \cong K^2 \oplus M_2(K_2)$. By using (i) of lemma 2.2, we get

$$K\mathcal{G}_3 \simeq K^2 \oplus M_2(K_2) \oplus_{i=1}^6 M_{n_i}(K), 150 = \sum_{i=1}^6 n_i^2, n_i > 1.$$

Along the same lines as case 1, we conclude that we are remaining with two possible choices of n_i 's given by $(2, 4^2, 5^2, 8)$ and (5^6) . Further, as in case 1, we conclude that (5^6) is the required choice. Hence, we have

$$K\mathcal{G}_3 \cong K^2 \oplus M_2(K_2) \oplus M_5(K)^6.$$

The proof is now complete.

6. Unit group of group algebras generated by $\mathcal{G}_5, \mathcal{G}_7, \mathcal{G}_9$

$$6.1 \mathcal{G}_5 = Q_8 \rtimes (C_7 \rtimes C_3).$$

The group $\mathcal{G}_5$ has the following presentation:

$$\mathcal{G}_5 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^3, [x_2, x_1]x_5^{-1}x_3^{-1}x_2^{-1}, [x_3, x_1]x_5^{-1}x_2^{-1}, [x_4, x_1]x_4^{-1}, [x_5, x_1], \\ x_2^2x_5^{-1}, [x_3, x_2]x_5^{-1}, [x_4, x_2], [x_5, x_2], x_3^2x_5^{-1}, [x_4, x_3], [x_5, x_3], x_4^7, [x_5, x_4], x_5^2 \rangle.$$

The sizes, orders and the representatives of the 17 CC of $\mathcal{G}_5$ are given below:

R	1	x_1	x_2	x_4	x_5	x_1^2	x_1x_5	x_2x_4	x_3x_4	x_4x_5
S	1	28	6	3	1	28	28	6	6	3
O	1	3	4	7	2	3	6	28	28	14

$x_1^2x_2$	$x_2x_3x_4$	x_4^3	$x_2x_4^3$	$x_3x_4^3$	$x_4^3x_5$	$x_2x_3x_4^3$
28	6	3	6	6	3	6
6	28	7	28	28	14	28

Clearly, the value of e in $\mathcal{G}_5$ is 84.

Theorem 6.1 *The $U(K\mathcal{G}_5)$ of the group algebra $K\mathcal{G}_5$ is given below:*

- (1) for $q \equiv \{1, 43\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^{*3} \oplus GL_2(K)^3 \oplus GL_3(K)^9 \oplus GL_6(K)^2$.
- (2) for $q \equiv \{5, 17, 47, 59\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^* \oplus K_2^* \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K) \oplus GL_3(K_2) \oplus GL_3(K_6) \oplus GL_6(K_2)$.
- (3) for $q \equiv \{11, 23, 53, 65\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^* \oplus K_2^* \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K)^3 \oplus GL_3(K_3)^2 \oplus GL_6(K)^2$.
- (4) for $q \equiv \{13, 55\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^{*3} \oplus GL_2(K)^3 \oplus GL_3(K) \oplus GL_3(K_2)^4 \oplus GL_6(K_2)$.
- (5) for $q \equiv \{19, 31, 61, 73\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^{*3} \oplus GL_2(K)^3 \oplus GL_3(K) \oplus GL_3(K_2) \oplus GL_3(K_6) \oplus GL_6(K_2)$.

- (6) for $q \equiv \{25, 37, 67, 79\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^{*3} \oplus GL_2(K)^3 \oplus GL_3(K)^3 \oplus GL_3(K_3)^2 \oplus GL_6(K)^2$.
- (7) for $q \equiv \{29, 71\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^* \oplus K_2^* \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K)^9 \oplus GL_6(K)^2$.
- (8) for $q \equiv \{41, 83\} \pmod{84}$, $U(K\mathcal{G}_5) \cong K^* \oplus K_2^* \oplus GL_2(K) \oplus GL_2(K_2) \oplus GL_3(K) \oplus GL_3(K_2)^4 \oplus GL_6(K_2)$.

Proof: The group algebra $K\mathcal{G}_5$ is Artinian and semi simple. We observe that the commutator subgroup $\mathcal{G}_5' \cong C_7 \times Q_8$ and $\frac{\mathcal{G}_5}{\mathcal{G}_5'} \cong C_3$.

Case 1: $q \equiv \{1, 43\} \pmod{84}$. In this case, the cardinality of every cyclotomic K -class is 1. Therefore, the WD of the group algebra in this case is

$$K\mathcal{G}_5 \cong K^3 \oplus_{i=1}^{14} M_{n_i}(K), 165 = \sum_{i=1}^{14} n_i^2, n_i > 1.$$

We consider $N_1 = \langle x_5 \rangle$. It can be noted that N_1 is a normal subgroup of $\mathcal{G}_5$ and $\mathcal{G}_5 / N_1 \cong (C_{14} \times C_2) \rtimes C_3$. Using [5], we know that $K(\mathcal{G}_5 / N_1) \cong K^3 \oplus M_3(K)^9$. This and (i) of lemma 2.2 deduce that

$$K\mathcal{G}_5 \cong K^3 \oplus M_3(K)^9 \oplus_{i=1}^5 M_{n_i}(K), 84 = \sum_{i=1}^5 n_i^2, n_i > 1.$$

The last equation has only two solutions given by $(2^3, 6^2)$ and $(3^2, 4, 5^2)$. Further, lemma 2.3 guarantees that for $p = 5, (2^3, 6^2)$ is the only choice for n_i 's. This means that

$$K\mathcal{G}_5 \cong K^3 \oplus M_2(K)^3 \oplus M_3(K)^9 \oplus M_6(K)^2.$$

Case 2: $q \equiv \{5, 17, 47, 59\} \pmod{84}$. In this case, the cyclotomic K -class corresponding to g_2 includes g_6 , whereas g_4 includes g_{13} . Similarly, g_7 includes g_{11} and g_{10} includes g_{16} . Also, the cyclotomic K -class corresponding to g_8 includes $g_9, g_{12}, g_{14}, g_{15}, g_{17}$, while rest of the $g_{i'}$'s forms individual classes. The decomposition for this scenario is given by

$$\begin{aligned} K\mathcal{G}_5 &\cong K \oplus K_2 \oplus_{i=1}^2 M_{n_i}(K) \oplus_{i=3}^5 M_{n_i}(K_2) \oplus M_{n_6}(K_6), \\ &\Rightarrow 165 = \sum_{i=1}^2 n_i^2 + \sum_{i=3}^5 2 \cdot n_i^2 + 6 \cdot n_6^2, n_i > 1. \end{aligned}$$

Using the decomposition of $K(\mathcal{G}_5 / N_1)$ from [5] and (i) of lemma 2.2, we note that

$$K\mathcal{G}_5 \cong K \oplus K_2 \oplus M_3(K) \oplus M_3(K_2) \oplus M_3(K_6) \oplus M_{n_1}(K) \oplus_{i=2}^3 M_{n_i}(K_2),$$

with $84 = n_1^2 + 2(n_2^2 + n_3^2)$, $n_i > 1$. The possible choices of n_i 's are (2, 2, 6) and (4, 3, 5). Using lemma 2.3, we know that (4, 3, 5) can not be the Wedderburn component. Thus, we have

$$K\mathcal{G}_5 \cong K \oplus K_2 \oplus M_2(K) \oplus M_2(K_2) \oplus M_3(K) \oplus M_3(K_2) \oplus M_3(K_6) \oplus M_6(K_2).$$

We remark that for the rest of the possibilities, decompositions can be computed by following the same procedure as followed in cases 1 and 2. Therefore, we skip rest of the proof.

$$6.2. \mathcal{G}_7 = (C_2 \times C_2 \times C_2) \rtimes (C_7 \rtimes C_3).$$

The group $\mathcal{G}_7$ has the following presentation:

$$\mathcal{G}_7 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^3, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-1}, [x_4, x_1]x_4^{-1}, [x_5, x_1]x_5^{-1}x_3^{-1}, x_2^7, \\ [x_3, x_2]x_4^{-1}x_3^{-1}, [x_4, x_2]x_5^{-1}x_4^{-1}, [x_5, x_2]x_3^{-1}, x_3^2, [x_4, x_3], [x_5, x_3], x_4^2, [x_5, x_4], x_5^2 \rangle.$$

The sizes, orders and the representatives of the 8 CC of $\mathcal{G}_7$ are given below:

R	1	x_1	x_2	x_3	x_1^2	x_1x_4	$x_1^2x_4$	x_2^3
S	1	28	24	7	28	28	28	24
O	1	3	7	2	3	6	6	7

the value of e in $\mathcal{G}_7$ is 42.

Theorem 6.2: The $U(K\mathcal{G}_7)$ of the group algebra $K\mathcal{G}_7$ is given below:

- (1) for $q \equiv \{1, 5, 37\} \pmod{42}$, $U(K\mathcal{G}_7) \cong K^{*3} \oplus GL_3(K)^2 \oplus GL_7(K)^3$.
- (2) for $q \equiv \{5, 17, 41\} \pmod{42}$, $U(K\mathcal{G}_7) \cong K^* \oplus K_2^* \oplus GL_3(K_2) \oplus GL_7(K) \oplus GL_7(K_2)$.
- (3) for $q \equiv \{11, 23, 29\} \pmod{42}$, $U(K\mathcal{G}_7) \cong K^* \oplus K_2^* \oplus GL_3(K)^2 \oplus GL_7(K) \oplus GL_7(K_2)$.
- (4) for $q \equiv \{13, 19, 31\} \pmod{42}$, $U(K\mathcal{G}_7) \cong K^{*3} \oplus GL_3(K_2) \oplus GL_7(K)^3$.

Proof: The group algebra $K\mathcal{G}_7$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_7 \cong (C_2 \times C_2 \times C_2) \rtimes C_7$ and $\frac{\mathcal{G}_7}{\mathcal{G}_7'} \cong C_3$.

Case 1: $q \equiv \{1, 25, 37\} \pmod{42}$. In this case, the cardinality of every cyclotomic K -class is 1. Therefore, the WD of the group algebra is

$$K\mathcal{G}_7 \simeq K^3 \oplus_{i=1}^5 M_{n_i}(K_i), 165 = \sum_{i=1}^5 n_i^2, n_i > 1.$$

We consider the normal subgroup $N_1 = \langle x_3, x_4, x_5 \rangle$ of $\mathcal{G}_7$. The corresponding factor group $F = \mathcal{G}_7 / N_1 \cong C_7 \rtimes C_3$. Using [5], we note that $K(\mathcal{G}_7 / N_1) \cong K^3 \oplus M_3(K)^2$. This and (i) of lemma 2.2 imply that

$$K\mathcal{G}_7 \simeq K^3 \oplus M_3(K)^2 \oplus_{i=1}^3 M_{n_i}(K_i), 147 = \sum_{i=1}^3 n_i^2, n_i > 1.$$

The last equation has a unique solution given by (7^3) . Therefore, we have

$$K\mathcal{G}_7 \cong K^3 \oplus M_3(K)^2 \oplus M_7(K)^3.$$

Case 2: $q \equiv \{5, 17, 41\} \pmod{42}$. In this case, the cyclotomic K -class of g_2 includes g_5 , whereas g_3 includes g_8 . Similarly, g_6, g_7 comes under same class, while the rest of the g_i 's forms individual classes. The decomposition for this scenario is given by,

$$K\mathcal{G}_7 \cong K \oplus K_2 \oplus M_{n_1}(K) \oplus M_{n_2}(K_2) \oplus M_{n_3}(K_2) \Rightarrow 165 = n_1^2 + 2(n_2^2 + n_3^2), n_i > 1.$$

The last equation has a unique solution given by $(7, 3, 7)$. Therefore, we have

$$K\mathcal{G}_7 \cong K \oplus K_2 \oplus M_3(K_2) \oplus M_7(K) \oplus M_7(K_2).$$

The rest of the cases can be done on the similar lines of the previous cases.

$$6.3 \quad \mathcal{G}_9 = C_7 \rtimes S_4.$$

The presentation of the group $\mathcal{G}_9$ is given below:

$$\mathcal{G}_9 = \langle x_1, x_2, x_3, x_4, x_5 \mid x_1^2, [x_2, x_1]x_2^{-1}, [x_3, x_1]x_3^{-5}, [x_4, x_1]x_5^{-1}x_4^{-1}, [x_5, x_1]x_5^{-1}x_4^{-1}, x_2^3, [x_3, x_2], [x_4, x_2]x_5^{-1}x_4^{-1}, [x_5, x_2]x_4^{-1}, x_3^7, [x_4, x_3], [x_5, x_3], x_4^2, [x_5, x_4], x_5^2 \rangle.$$

The sizes, orders and the representatives of the 17 CC of $\mathcal{G}_9$ are given below:

R	1	x_1	x_2	x_3	x_4	x_1x_4	x_2x_3	x_3^2	x_3x_4	$x_2^2x_3$
S	1	42	8	2	3	42	8	2	6	8
O	1	2	3	7	2	4	21	7	14	21

$x_2x_3^2$	x_3^3	$x_3^2x_4$	$x_2^2x_3^2$	$x_2x_3^3$	$x_3^3x_4$	$x_2^2x_3^3$
8	2	6	8	8	6	8
21	7	14	21	21	14	21

the value of e in $\mathcal{G}_9$ is 84.

Theorem 6.3: *The $U(K\mathcal{G}_9)$ of the group algebra $K\mathcal{G}_9$ is given below:*

- (1) for $q \equiv \{1, 41, 43, 83\} \pmod{84}$, $U(K\mathcal{G}_9) \cong K^{*2} \oplus GL_2(K)^{10} \oplus GL_3(K)^2 \oplus GL_6(K)^3$.
- (2) for $q \equiv \{5, 25, 17, 37, 47, 59, 67, 79\} \pmod{84}$, $U(K\mathcal{G}_9) \cong K^{*2} \oplus GL_2(K) \oplus GL_2(K_2)^3 \oplus GL_3(K)^2 \oplus GL_6(K_2)$.
- (3) for $q \equiv \{11, 19, 23, 31, 53, 61, 65, 73\} \pmod{84}$, $U(K\mathcal{G}_9) \cong K^{*2} \oplus GL_2(K) \oplus GL_2(K_3) \oplus GL_2(K_6) \oplus GL_3(K)^2 \oplus GL_6(K_3)$
- (4) for $q \equiv \{13, 29, 55, 71\} \pmod{84}$, $U(K\mathcal{G}_9) \cong K^{*2} \oplus GL_2(K)^4 \oplus GL_2(K_2)^3 \oplus GL_3(K)^2 \oplus GL_6(K)^3$.

Proof: The group algebra $K\mathcal{G}_9$ is Artinian and semisimple. We observe that the commutator subgroup $\mathcal{G}_9' \cong C_4 \times A_4$ and $\frac{\mathcal{G}_9}{\mathcal{G}_9'} \cong C_2$.

Case 1: $q \equiv \{1, 41, 43, 83\} \pmod{84}$. In this case, the cardinality of every cyclotomic K -class is 1. Therefore, the WD of the group algebra is

$$K\mathcal{G}_9 \simeq K^2 \oplus_{i=1}^{15} M_{n_i}(K_i), 166 = \sum_{i=1}^{15} n_i^2, n_i > 1.$$

Consider the normal subgroup $N_1 = \langle x_4, x_5 \rangle$ of $\mathcal{G}_9$ with $\mathcal{G}_9 / N_1 \cong D_{42}$. Using [5], we see that $K(\mathcal{G}_9 / N_1) \cong K^2 \oplus M_2(K)^{10}$. This and (i) of lemma 2.2 imply that

$$K\mathcal{G}_9 \simeq K^2 \oplus M_2(K)^{10} \oplus_{i=1}^5 M_{n_i}(K_i), 126 = \sum_{i=1}^5 n_i^2, n_i > 1.$$

Again consider the normal subgroup $N_2 = \langle x_3 \rangle$ of $\mathcal{G}_9$. The corresponding factor group $\mathcal{G}_9 / N_2 \simeq S_4$. By [12], we know that $K(\mathcal{G}_9 / N_2) \cong K^2 \oplus M_2(K) \oplus M_3(K)^2$. This means that

$$K\mathcal{G}_9 \simeq K^2 \oplus M_2(K)^{10} \oplus M_3(K)^2 \oplus_{i=1}^3 M_{n_i}(K_i), 108 = \sum_{i=1}^3 n_i^2, n_i > 1.$$

The last equation has only two possibilities namely $(2, 2, 10)$ and (6^3) . Further, lemma 2.3 confirms that for $p = 5, (2, 2, 10)$ can not be the choice of n_i 's. Therefore, we must have

$$K\mathcal{G}_9 \cong K^2 \oplus M_2(K)^{10} \oplus M_3(K)^2 \oplus M_6(K)^3.$$

Case 2: $q \equiv \{5, 25, 17, 37, 47, 59, 67, 79\} \pmod{84}$. In this case, the cyclotomic K -class of g_4 includes g_8, g_{12} , whereas g_7 includes g_{11}, g_{17} . Similarly, g_9, g_{13}, g_{16} comes under same class and g_{10}, g_{14}, g_{15} comes under same class. The rest of the g_i 's forms individual classes. The decomposition for this scenario is given by

$$K\mathcal{G}_9 \cong K^2 \oplus_{i=1}^3 M_{n_i}(K) \oplus_{i=4}^7 M_{n_i}(K_3) \Rightarrow 166 = \sum_{i=1}^3 n_i^2 + \sum_{i=4}^7 3 \cdot n_i^2, n_i > 1.$$

By proceeding in the same manner as in case 1, we can note that

$$K\mathcal{G}_9 \cong K^2 \oplus M_2(K) \oplus M_2(K_2)^3 \oplus M_3(K)^2 \oplus M_6(K_2).$$

Case 3: $q \equiv \{11, 19, 23, 31, 53, 61, 65, 73\} \pmod{84}$. In this case, the cyclotomic K -class of g_4 includes g_8, g_{12} , whereas g_9 includes g_{16}, g_{13} . Also, g_7 clubs with $g_{10}, g_{11}, g_{14}, g_{15}, g_{17}$. Other representatives forms individual classes. The group algebra $K\mathcal{G}_9$ decomposes as follows:

$$K\mathcal{G}_9 \cong K^2 \oplus_{i=1}^3 M_{n_i}(K) \oplus_{i=4}^5 M_{n_i}(K_3) \oplus M_{n_6}(K_6) \Rightarrow 166 = \sum_{i=1}^3 n_i^2 + 3 \sum_{i=4}^5 n_i^2 + 6 \cdot n_6^2, n_i > 1.$$

Using [5], we know that $K(\mathcal{G}_9 / N_1) \cong K^2 \oplus M_2(K) \oplus M_2(K_3) \oplus M_2(K_6)$. This and (i) of lemma 2.2 give

$$\begin{aligned} K\mathcal{G}_9 &\cong K^2 \oplus M_2(K) \oplus M_2(K_3) \oplus M_2(K_6) \oplus_{i=1}^2 M_{n_i}(K) \oplus M_{n_3}(K_3) \\ &\Rightarrow 126 = \sum_{i=1}^2 n_i^2 + 3 \cdot n_3^2, n_i > 1. \end{aligned}$$

This equation has a unique solution given by $(3, 3, 6)$. Hence, we get

$$K\mathcal{G}_9 \cong K^2 \oplus M_2(K) \oplus M_2(K_3) \oplus M_2(K_6) \oplus M_3(K)^2 \oplus M_6(K_3).$$

Case 4: $q \equiv \{13, 29, 55, 71\} \pmod{84}$. In this case, the cyclotomic K -classes of g_7 clubs with g_{10} . Similarly, g_{11} and g_{15} clubs, respectively with g_{14} and g_{17} . The other representatives forms individual classes. The decomposition for this scenario is given by

$$K\mathcal{G}_9 \simeq K^2 \oplus_{i=1}^9 M_{n_i}(K) \oplus_{i=10}^{12} M_{n_i}(K_2), n_i > 1 \Rightarrow 166 = \sum_{i=1}^9 n_i^2 + \sum_{i=10}^{12} 2 \cdot n_i^2, n_i > 1.$$

By following the procedure of previous cases, we get that

$$K\mathcal{G}_9 \cong K^2 \oplus M_2(K)^4 \oplus M_2(K_2)^3 \oplus M_3(K)^2 \oplus M_6(K)^3.$$

This completes the proof.

7. Conclusion

The study of the unit group of SGA of NMG up to order 180 is completed with this paper. It is clear that with the increase in the size of the group, new techniques are required to uniquely characterize the unit groups. This paper motivates researchers to find an algorithm that finds the WD of a semisimple group algebra of any finite group.

References

- [1] N. Abhilash, E. Nandakumar, R. K. Sharma, G. Mittal, Structure of the unit group of the group algebras of NMG of order 128, Preprint (2023).
- [2] N. Abhilash, E. Nandakumar, R. K. Sharma, G. Mittal, On the unit group of the SGA of groups up to order 144, Preprint (2023).
- [3] N. Abhilash, E. Nandakumar, G. Mittal, R. K. Sharma, Units of the SGA of groups of order 162, Preprint (2023).
- [4] N. Arvind, S. Panja, Unit group of some finite semi-simple group algebras, *J. Egypt. Math. Soc.*, 30, Article No. 17 (2022).
- [5] G. K. Bakshi, S. Gupta, I. B. S. Passi, The algebraic structure of finite metabelian group algebras, *Comm. Algebra*, 43(1), 2240-2257 (2015).
- [6] C. Dietzel, G. Mittal, Summands of finite group algebras, *Czech. Math. J.*, 71, 1011-1014 (2021).
- [7] R. Ferraz, Simple components of center of $\mathbb{F}G / J(\mathbb{F}G)$, *Comm. Algebra*, 36(9), 3191- 3199 (2008).
- [8] W. Gao, A. Geroldinger, F. Halter- Koch, Group algebras of finite abelian groups and their applications to combinatorial problems, *Rocky Mountain J. Math.*, 39, 805-823 (2008).
- [9] J. Gildea, The structure of the unit group of the group algebra $\mathbb{F}_{2^k}A_4$, *Czechoslovak Math. J.*, 61, 531-539 (2011).

- [10] T. Hurley, Group rings and rings of matrices, *Int. J. Pure Appl. Math.*, 31(3), 319-335 (2006).
- [11] T. Hurley, Convolutional codes from units in matrix and group rings, *Int. J. Pure Appl. Math.*, 50(3), 431-463 (2009).
- [12] M. Khan, R. K. Sharma, J. Srivastava, The unit group of FS_4 , *Acta Math. Hungar.*, 118, 105-113 (2008).
- [13] S. Maheshwari, R. K. Sharma, The unit group of the group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_3)$, *J. Algebra Comb. Discrete Appl.*, 3(1), 1-6 (2015).
- [14] N. Makhijani, R. K. Sharma, J. Srivastava, A note on the structure of $F_{p^k} A_5 / J(F_{p^k} A_5)$, *Acta Sci. Math. (Szeged)*, 82, 29-43 (2016).
- [15] Manju Pruthi, Sudesh, Idempotents in the group algebra of dihedral groups, *Journal of Information and Optimization Sciences*, 39:2, 461-465 (2018).
- [16] C. P. Miles, S. K. Sehgal, An introduction to group rings, Kluwer Academic Publishers, Dordrecht (2002).
- [17] G. Mittal, R. K. Sharma, On unit group of finite group algebras of NMG up to order 72, *Math. Bohemica*, 146(4), 1-27 (2021).
- [18] G. Mittal, R. K. Sharma, On unit group of finite group algebras of NMG of order 108, *J. Algebra Comb. Discrete Appl.*, 8(2), 59-71 (2021).
- [19] G. Mittal, S. Kumar, S. Narain, S. Kumar, Group ring based public key cryptosystems, *J. Disc. Math. Sci. Crypt.*, 25(6), 1683-1704 (2022).
- [20] G. Mittal, R. K. Sharma, Unit group of SGA of some NMG of order 120, *Asian-European J. Math.*, 15(3), 2250059 (2022).
- [21] G. Mittal, R. K. Sharma, The unit groups of SGA of some NMG of order 144, *Math. Bohemica*, Online first (2022).
- [22] G. Mittal, R. K. Sharma, WD of a semisimple group algebra $\mathbb{F}_q G$ from a subalgebra of factor group of G , *Int. Elect. J. Algeb.*, 32, 91-100 (2022).
- [23] G. Mittal, R. K. Sharma, Computation of WD of groups algebras from their subalgebra, *Bull. Korean Math. Soc.*, 59(3), 781-787 (2022).
- [24] G. Mittal, R. K. Sharma, Computation of the WD of SGA of groups up to order 120, Preprint (2022).
- [25] N. Arvind, S. Panja, The unit group of $K_q SL(3, 2)$, $p > 11$, Arxiv (2021).
- [26] G. Pazderski, The orders to which only belong metabelian groups, *Math. Nachr.*, 95, 7-16 (1980).
- [27] R. K. Sharma, G. Mittal, On the unit group of a semisimple group algebra $\mathbb{F}_q SL(2, \mathbb{Z}_5)$, *Math. Bohemica*, 147(1), 1-10 (2022).

- [28] R. K. Sharma, J. B. Srivastava, M. Khan, The unit group of $\mathbb{F}A_4$, Publ. Math. Debrecen, 71(1-2), 21-26 (2007).
- [29] R. K. Sharma, J. B. Srivastava, M. Khan, The unit group of $\mathbb{F}S_3$, Acta Math. Acad. Paedagog. Nyhzi., 23(2), 129-142 (2007).
- [30] R. K. Sharma, P. Yadav, The unit group of Z_pQ_8 , Algebras Groups Geom., 25(4), 425-429 (2008).
- [31] Re'em Waxman, On the Wedderburn component of a group algebra, <https://math.stackexchange.com/q/4682495>.
- [32] G. Tang, Y. Wei, Y. Li, Units group of group algebras of some small groups, *Czechoslovak Math. J.*, 64(1), 149-157 (2014).

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