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Two numerical methods for solving conformable fractional KGE equation

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Abstract

The conformable fractional derivatives in the sense of Khalil is considered and the Homotopy analysis method and the (G'/G)-Expansion method are applied to solve the Klein-Gordon equation; new approximate solutions and new travelling wave solutions are obtained. Some numerical simulations are graphically illustrated. At the end, a conclusion is given.

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1. Introduction

There are many different methods used for solving the Klein-Gordon equation (KGE). We can cite, for instance, the variational iteration method [23], the tanh method, the sine-cosine method [21], the decomposition method [6], the differential transform method [17], the Sobolev gradient method [16], and the homotopy-perturbation method [5]. Other methods for finding the solutions of the fractional KGE are also presented in the works [19, 15, 22, 8].

In this paper, we consider the Homotopy Analysis Method (HAM) [2, 10, 11, 12, 13] and the $\left(\frac{G'}{G}\right)$-expansion method [20, 14, 24, 18, 3] to solve the fractional differential equations.

In addition to the above two applied methods, we will also use the Khalil conformable fractional derivatives to derive new approximate solutions and some "other" traveling wave solutions to the FDEs. These derivatives have been shown to have some advantages over other types of fractional derivatives, see [1, 4, 7, 9].

So, let us consider the following fractional KGE differential problem:

$$T_t^{2\alpha} u(x,t) + a u_{xx}(x,t) + b u(x,t) + d(u(x,t))^3 = 0, 0 < \alpha \leq 1, \tag{1.1}$$

where T_t^{α} is the conformable fractional derivative, $u(x,t)$ shows the wave displacement at position x at time t and a, b, d are constants.

It is to note that when $\alpha = 1$, the above problem is reduced to the following standard KGE equation:

$$u_{tt}(x,t) + a u_{xx}(x,t) + b u(x,t) + d(u(x,t))^3 = 0. \tag{1.2}$$

The aim of this paper is to study the conformable KGE equation (1.1) using the HAM mthod and the $\left(\frac{G'}{G}\right)$-expansion method.

The paper is organized as follows. In Sec. II, some necessary details on the conformable fractional derivatives are provided. In Sec. III, the KGE with conformable fractional derivative is studied with the HAM and the $\left(\frac{G'}{G}\right)$-Expansion method. Some graphs are plotted to achieve this study. Finally, a conclusion follows.

2. Conformable Fractional Derivatives

In this section, we recall some definitions and properties for the conformable fractional approach in the sense of Khalil. Fore more details on this important approach, see [1, 9].

Definition 2.1: Let $f:(0,\infty)\to\mathbb{R}.$ The conformable fractional derivative of order $0<\alpha\le 1$ is defined by

$$(T^{\alpha}f)(t)=\frac{\partial^{\alpha}f(t,x)}{\partial t^{\alpha}}=\lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}\right),\ t>0. \tag{2.1}$$

Remark that when $\alpha=1,$ the above formula can be reduced to the derivative or order one.

Definition 2.2: The conformable fractional integral of a function $f:(0,\infty)\to\mathbb{R}$ of order $0<\alpha\le 1$ is defined as

$$(I^{\alpha}f)(t)=\int_0^t \tau^{\alpha-1}f(\tau)d\tau. \tag{2.2}$$

We need also the following properties:

$$T^{\alpha}(af+bg)=aT^{\alpha}(f)+bT^{\alpha}(g),\quad \text{for all } a,b\in\mathbb{R}. \tag{2.3}$$

$$T^{\alpha}(\lambda)=0,\quad \text{for all constant } \lambda. \tag{2.4}$$

$$T^{\alpha}(t^{p})=pt^{p-\alpha},\quad \text{for all } p\in\mathbb{R}. \tag{2.5}$$

$$I^{\alpha}T^{\alpha}f(t)=f(t)-f(0) \tag{2.6}$$

and

$$(T^{\alpha}f)(t)=t^{1-\alpha}\frac{df(t)}{dt}. \tag{2.7}$$

3. The HAM and the $\left(\frac{G'}{G}\right)$-Expansion Methods

In this section, we present the HAM method and the $\left(\frac{G'}{G}\right)$-expansion methods in the case of Khalil fractional theory.

3.1 *The HAM Method*

Consider the following form of the equations

$$T_t^{\alpha}u(x,t)+au_{xx}(x,t)+bu(x,t)+d(u(x,t))^3=0,1<\alpha\le 2, \tag{3.1}$$

with the conditions

$$u(x,0)=f(x)=x. \tag{3.2}$$

For application of the above proposed method, in view of (3.1) and the conditions (3.2), it in convenient to choose

$$u_0(x,t) = f(x). \tag{3.3}$$

We put also

$$L[\phi(x,t,q)] = T_t^{\alpha} \tag{3.4}$$

with the property $L[c]$, where c is a constant of integration. Furthermore, we define a nonlinear operator as

$$N[\phi(x,t,q)] = T_t^{\alpha} u(x,t) + a u_{xx}(x,t) + b u(x,t) + d(u(x,t))^3 \tag{3.5}$$

We construct the following equations

$$R_m(\overleftarrow{u_{m-1}}) = T_t^{\alpha} u_{m-1} + a\frac{\partial^2 u_{m-1}}{\partial x^2} + b u_{m-1} + d(u_{m-1})^3 \tag{3.6}$$

Now, the solution of the m-th-order deformation equations (3.6) for $m \geq 1$ is given by:

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hL^{-1}(R_m(\overleftarrow{u_{m-1}})). \tag{3.7}$$

Consequently, the first few terms of the HAM series solution are given by:

$$u_0(x,t) = f(x) = x \tag{3.8}$$

$$\begin{aligned}
u_1(x,t) &= \chi_1 u_0(x,t) + hL^{-1}(R_1(\overleftarrow{u_0}))\\
&= hL^{-1}\left(T_t^{\alpha} u_0 + a\frac{\partial^2 u_0}{\partial x^2} + b u_0 + d(u_0)^3 \right)\\
&= h u_0(x,t) I^{\alpha}(1) + h + a\frac{\partial^2 (f(x))}{\partial x^2} + hbf(x) I^{\alpha}(1) + df^3(x) I^{\alpha}(1)\\
&= h(f(x) + af''(x) + bf(x) + df^3(x))\frac{t^{\alpha}}{\alpha}\\
&= h(x + bx + dx^3)\frac{t^{\alpha}}{\alpha}
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
u_2(x,t) &= \chi_2 u_1(x,t) + hL^{-1}(R_2(\overleftarrow{u_1}))\\
&= u_1(x,t) + hL^{-1}\left(T_t^{\alpha} u_1 + a\frac{\partial^2 u_1}{\partial x^2} + b u_1 + d(u_1)^3 \right)\\
&= h(x + bx + dx^3)\frac{t^{\alpha}}{\alpha} + h^2(x + bx + dx^3)\frac{t^{\alpha}}{\alpha} + ah^2(6dx) I^{\alpha}\left(\frac{t^{\alpha}}{\alpha}\right)\\
&\quad + bh^2(x + bx + dx^3) I^{\alpha}\left(\frac{t^{\alpha}}{\alpha}\right) + dh^4(x + bx + dx^3)^3 I^{\alpha}\left(\frac{t^{3\alpha}}{\alpha^3}\right)\\
&= h(1+h)(x + bx + dx^3)\frac{t^{\alpha}}{\alpha} + 3adh^2 x\frac{t^{2\alpha}}{\alpha^2} + bh^2(x + bx + dx^3)\frac{t^{2\alpha}}{2\alpha^2}\\
&\quad + dh^4(x + bx + dx^3)^3\frac{t^{5\alpha}}{5\alpha^5}
\end{aligned} \tag{3.10}$$

Hence, the HAM series solution for $h = -1$ is:

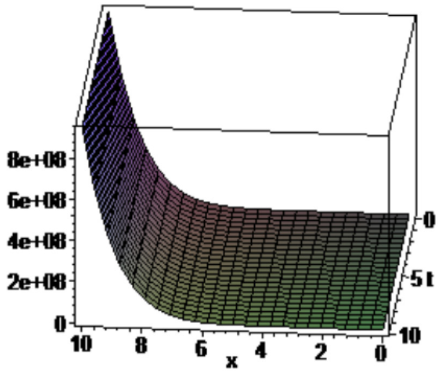

Figure 4.1

3D plot of the solution (3.12) of (3.1)

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+\ldots \\ &= x-(x+bx+dx^3)\frac{t^\alpha}{\alpha}+((6ad+b)x+b^2x+bdx^3)\frac{t^{2\alpha}}{2\alpha^2}+d(x+bx+dx^3)^3\frac{t^{5\alpha}}{5\alpha^5}+\ldots \end{aligned} \tag{3.11}$$

For the special case, $\alpha=2$ we obtain:

$$\begin{aligned} u(x,t) &= u_0(x,t)+u_1(x,t)+\ldots \\ &= x-(x+bx+dx^3)\frac{t^2}{2} \\ &+((6ad+b)x+b^2x+bdx^3)\frac{t^4}{8}+d(x+bx+dx^3)^3\frac{t^{10}}{20}+\ldots \end{aligned} \tag{3.12}$$

Figure 4.1 presents the graph of solution (3.12) for Eq. (3.1) with $a=1,\ b=1,\ d=-1,\ k=\frac{1}{2},\ c=1,\ \alpha=\frac{1}{2}$ and $\beta=\frac{3}{4}$.

3.2 *The* $\left(\frac{G'}{G}\right)$*-Expansion Method*

We consider the following problem:

$$T_t^{2\alpha}u(x,t)+au_{xx}(x,t)+bu(x,t)+d(u(x,t))^3=0, \tag{3.13}$$

where T_x^α, T_t^α are the conformable fractional derivatives, with $0<\alpha\le 1$.

For our purpose, we introduce the transformations

$$u(t,x)=U(\zeta), \tag{3.14}$$

$$\zeta=\frac{k}{\alpha}t^\alpha+\omega x, \tag{3.15}$$

where k and ω are constants.

Substituting (3.15) into (3.13), we can reduce (3.13) into an ODE as follows:

$$(\omega^2 + ak^2)U_{\zeta\zeta} + bu(x,t) + d(u(x,t))^3 = 0. \tag{3.16}$$

By considering the homogeneous balance between the highest order derivatives and nonlinear term presented in the above Equation, we have the following form

$$U(\zeta) = a_1(G'/G) + a_0 + a_{-1}(G'/G)^{-1}, \tag{3.17}$$

where a_1, a_0 and a_{-1} are arbitrary constants and $G(\xi)$ satisfies the following second order linear ordinary differential equation.

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{3.18}$$

with λ and μ as constants.

To determine the desired constants, we substitute Eq(3.17) into Eq(3.16), and we collecting all the terms with the same power of $\left(\frac{G'}{G}\right)$ together, equating each coefficient equal to zero, yields a set of algebraic equations. It is given by:

$$\begin{aligned}
(\omega^2 + ak^2)(\lambda\mu a_{-1} + \lambda a_{-1}) + ba_0 + da_0^3 + 6a_0a_1a_{-1} &= 0\\
(\omega^2 + ak^2)(2\mu a_1 + \lambda^2 a_1) + ba_1 + d(3a_0^2a_1 + 3a_1^2a_{-1}) &= 0\\
3\lambda a_1(\omega^2 + ak^2) + 3a_0a_1^2 d &= 0\\
2a_1(\omega^2 + ak^2) + a_1^3 d &= 0\\
(\omega^2 + ak^2)(\lambda^2 a_{-1} + 2\mu a_{-1}) + ba_{-1} + d(3a_0^2a_{-1} + 3a_{-1}^2a_1) &= 0\\
3\lambda\mu a_{-1}(\omega^2 + ak^2) + 3a_0a_{-1}^2 d &= 0\\
2\mu^2 a_{-1}(\omega^2 + ak^2) + da_{-1}^3 &= 0
\end{aligned} \tag{3.19}$$

We need now to solve the algebraic equations with the aid of Maple to obtain the following traveling wave solutions:

Family 1

$$\begin{gathered}
a_1 = 0\,, a_0 = \sqrt{-\tfrac{b}{d}}\,, a_{-1} = 0\\
\lambda = A\,, \mu = B\\
U(\zeta) = \sqrt{-\tfrac{b}{d}}
\end{gathered} \tag{3.20}$$

Family 2

$$a_1 = \sqrt{\frac{-2D}{d}}\ , a_0 = 0, a_{-1} = 0$$

$$\lambda = 0\ , \mu = -\frac{b}{2D}$$

$$U(\zeta) = \sqrt{\frac{-2D}{d}}(G'/G) = \sqrt{\frac{-2D}{d}}\sqrt{\frac{b}{2D}}\,\frac{c_1 e^{\sqrt{\frac{b}{2D}}\xi} - c_2 e^{-\sqrt{\frac{b}{2D}}\xi}}{c_1 e^{\sqrt{\frac{b}{2D}}\xi} + c_2 e^{-\sqrt{\frac{b}{2D}}\xi}} \tag{3.21}$$

Family 3

$$a_1 = \sqrt{\frac{-2D}{d}}\ , a_0 = \sqrt{-\frac{b}{d}}, a_{-1} = 0$$

$$\lambda = -\frac{d}{D}\sqrt{-\frac{b}{d}}\sqrt{-\frac{2D}{d}}, \mu = 0$$

$$U(\zeta) = \sqrt{-\frac{b}{d}} + \sqrt{\frac{-2D}{d}}\,(G'/G)$$

$$= \sqrt{-\frac{b}{d}} - 2c_2\sqrt{-\frac{b}{d}}\,\frac{e^{\frac{d}{D}\sqrt{-\frac{b}{d}}\sqrt{-\frac{2D}{d}}\xi}}{c_1 + c_2 e^{\frac{d}{D}\sqrt{-\frac{b}{d}}\sqrt{-\frac{2D}{d}}\xi}} \tag{3.22}$$

Family 4

$$a_1 = 0\ , a_0 = 0, a_{-1} = \sqrt{\frac{-1}{2dD}}$$

$$\lambda = 0\ , \mu = \frac{-}{d2D}$$

$$U(\zeta) = \sqrt{\frac{-1}{2dD}}\,(G'/G)^{-1}$$

$$= \frac{2dD}{b}\sqrt{\frac{-1}{2dD}}\,\frac{c_1 e^{\frac{b}{d2D}\xi} + c_2 e^{-\frac{b}{d2D}\xi}}{c_1 e^{\frac{b}{d2D}\xi} - c_2 e^{-\frac{b}{d2D}\xi}} \tag{3.23}$$

Family 5

$$a_1 = \sqrt{-\frac{2D}{d}}\ , a_0 = 0, a_{-1} = \frac{b}{16dD}\left(3d\sqrt{\frac{-2D}{d}} + \sqrt{-2dD}\right)$$

$$\lambda = 0\ , \mu = -\frac{b}{4D^2}\left(2D + \frac{3}{8}\sqrt{\frac{-2D}{d}}\left(3d\sqrt{\frac{-2D}{d}} + \sqrt{-2dD}\right)\right)$$

$$U(\zeta) = \sqrt{-\frac{2D}{d}}\,(G'/G) + \frac{b}{16dD}\left(3d\sqrt{\frac{-2D}{d}} + \sqrt{-2dD}\right)(G'/G)^{-1}$$

$$= \mu\sqrt{-\frac{2D}{d}}\,\frac{c_1 e^{\mu\xi} - c_2 e^{-\mu\xi}}{c_1 e^{\mu\xi} + c_2 e^{-\mu\xi}} + \frac{b}{16\mu dD}\left(3d\sqrt{\frac{-2D}{d}} + \sqrt{-2dD}\right)\frac{c_1 e^{\mu\xi} + c_2 e^{-\mu\xi}}{c_1 e^{\mu\xi} - c_2 e^{-\mu\xi}} \tag{3.24}$$

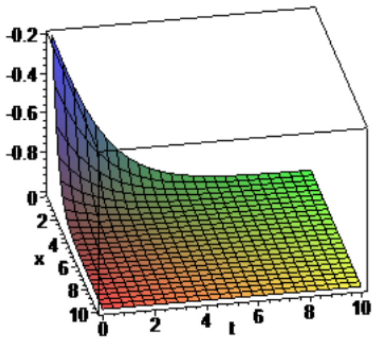

Figure 4.2

3D plot of the solution (3.22) of (3.13)

Family 6

$$a_1 = \sqrt{\frac{-2D}{d}}\,, a_0 = \frac{\sqrt{-12dD-db+24D}}{d}\,, a_{-1} = -\frac{3}{d}(d-2)\sqrt{\frac{-2D}{d}}$$
$$\lambda = -\frac{1}{D}\sqrt{-12dD-db+24D}\,\sqrt{\frac{-2D}{d}}, \mu = \frac{6}{d}-3$$
$$U(\zeta) = \frac{\sqrt{-12dD-db+24D}}{d} + \sqrt{\frac{-2D}{d}}(G'/G) + -\frac{3}{d}(d-2)\sqrt{\frac{-2D}{d}}(G'/G)^{-1} \tag{3.25}$$

where $D = \omega^2 + ak^2$ and A, B are constants.

Figure 4.2 presents the graph of solution (3.22) for Eq. (3.13) with $a = 1,\ b = 1,\ d = -1,\ k = -3,\ \omega = -5,\ c_1 = 2,\ c_2 = 3$ and $\alpha = 1$.

Conclusion

In this paper, the HAM method has been successfully applied to derive numerical solutions for the fractional KGE equation in the sense of Khalil (the solutions are not necessary of traveling wave type). Then, the $\left(\frac{G'}{G}\right)$-expansion method has also been used to derive new traveling wave solutions for the same equation. Based on the above results, we can state that these two numerical methods are very promising to study a large class of fractional differential models in the Khalil sense.

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