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Results on Riemann Zeta functions using compositions

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Abstract

The main results of our paper are relations involving sums of Multiple Zeta Riemann functions (MZVs). Using the techniques described here, we found a simple relation for $\zeta(\{a\}^n) = \zeta(\overbrace{a, a, \ldots, a}^{n})$, for a positive integer n and a complex a where $Re\{a\} > 1$.

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1. Introduction

We recall some concepts and establish some notations. The well-known Riemann Zeta function,

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^{s}},$$

is defined for all complex number s where $Re\{s\}>0$ and its analytic continuation elsewhere. The extension of this function, the multiple Zeta function (MZV), is given by the series

$$\zeta(s)=\sum_{a_1>a_2>\cdots>a_k\geq 1}\frac{1}{a_1^{s_1}a_2^{s_2}\cdots n_k^{s_k}},$$

where $s=(s_1,\ldots,s_k)$, and the sum is over k-tuples of positive integers, $(a_1,a_2,\ldots,a_k)$, and converge when $Re\{s_1\}+Re\{s_2\}+\cdots+Re\{s_i\}>i$, $1\leq i\leq k$.

For simplicity, in this paper we use the notation $\zeta(\{a\}^n)$ for $\zeta(\overbrace{a,a,\ldots,a}^{n})$ and for convenience $\zeta(\{a\}^0)=1$. A multi-index $s=(s_1,s_2,\ldots,s_k)$ is said admissible if the series $\zeta(s)$ converges. There are many works that provided several interesting results involving the aforementioned functions, like Zagier [13, 14] and Burgos and Frésan [7].

In [11], Sitaramachandra Rao and Subbarao found a formula for $\zeta(a,a,a)$ for $a>1$ as given next:

$$\zeta(a,a,a)=\frac{1}{6}\zeta^3(a)+\frac{1}{3}\zeta(3a)-\frac{1}{2}\zeta(a)\zeta(2a).$$

We will obtain a similar result here for $\zeta(\{a\}^n)$, for all complex a, in which $Re\{a\}>1$. For example, the Corollary 3 of our paper asserts the following:

$$\begin{aligned}\zeta(\{a\}^n)=\frac{1}{n^n}&\sum_{w_1+\cdots+w_m\in C(n)}\frac{(-1)^{n+m}}{m!w_1w_2\cdots w_m}\left[\sum_{k_1+\cdots+k_r\in C^*(w_1)}c(w_1,r)\zeta((k_1+\cdots+k_r)a)\right]\\&\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_2)}c(w_2,r)\zeta((k_1+\cdots+k_r)a)\right]\times\\&\cdots\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_m)}c(w_m,r)\zeta((k_1+\cdots+k_r)a)\right],\end{aligned}$$

where $c(n,r)$ the multinomial coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_r^{k_r}$ in the expansion of $(x_1+x_2+\cdots+x_r)^n$, where $n=k_1+k_2+\cdots+k_r$. The number $c(n,r)$ is equal to

$$\frac{n!}{k_1!k_2!\cdots k_r!},$$

as seen in Charalambides [8], Balakrishnan [5]. In [10], M. Hoffman establish an identity involving sums of Multiple Zeta Riemann functions.

(MZVs), where the arguments of these are permutations of n elements. He managed to write this sum of MZVs as a combination of Riemann Zeta functions. Theorem 2.2 of the aforementioned paper asserts, for $i_1, i_2, \ldots, i_k > 1$, the following:

Theorem 1: *(Hoffman [13], Theorem 2.2)*

$$\sum_{\sigma\in\Sigma_k} A(i_{\sigma(1)} \ldots, i_{\sigma(k)}) = \sum_{partitions\ \Pi\ of\ \{1,2,\ldots,k\}} \tilde{c}(\Pi)\zeta(i,\Pi), \tag{1}$$

where

- $A(i_1 \ldots, i_k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \ldots n_k^{i_k}};$
- Σ_k *is the group of* k *- permutations;*
- $\Pi = \{P_1, \ldots, P_l\}$ *is a partition of the set* $1,2,\ldots,k$;
- $\zeta(i,\Pi) = \prod_{s=1}^{l} \zeta(\sum_{j\in P_s} i_j);$
- $\tilde{c}(\Pi) = (-1)^{k-l}(cardP_1 - 1)!(cardP_2 - 1)!\cdots(cardP_l - 1)!.$

Based in this theorem, in a sense, we can write the sum of the mentioned MZVs in a kind of a "basis", where the elements of this basis are products of simple Zeta functions. Similarly, in the paper [6], "Evaluation of triple Euler Sums", Borwein and Girgensohn found some results involving sums of permutation formulas for partial ζ-sums in three arguments.

In this work, our desire is to obtain similar results as Hoffman in [10] and Borwein and Girgensohn in [6], but using permutations with repetition. In the aforementioned results, these are valid for all real numbers greater than 1. However, ours is valid for all complex numbers in wherein the real part is greater than 1.

Definition 1: Given an positive integer n, an integer partition[1] of n (or simply a partition of n) is a way of expressing n as a sum of positive integers, where the order of addends does not matter.

[1] Several results on integer partitions is in Andrews et al.

If $j_1 + j_2 + \ldots + j_s$ is a partition of n, each j_i is called a part of the partition.

Example 1: There are seven partitions of $n = 5 : 5$, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1 and 1 + 1 + 1 + 1 + 1.

Definition 2: An integer composition is a way of writing a positive integer as an ordered sum of positive integers. We denote by $C(n)$ the set of all compositions of n.

Example 2: $C(5) = \{5, 4+1, 1+4, 3+2, 2+3, 3+1+1, 1+3+1, 1+1+3, 2+2+1,$ $2+1+2, 1+2+2, 2+1+1+1, 1+2+1+1, 1+1+2+1, 1+1+1+2, 1+1+1+1+1\}.$

A weak integer composition of n (or simply weak composition of n) is a composition of n in which the number zero is allowed. We denote by $C^*(n)$ the set of all weak compositions of n. More information about compositions and weak compositions can be found in Heubach and Mansour [9], Sills [12] and Alegri [1, 2, 3].

We denote by $pr_n(x_1, a_2, \ldots, x_r)$ the set of all n-permutations with repetition of the set $\{x_1, x_2, \ldots, x_r\}$, or equivalently the $n-tuples$ of elements of $\{x_1, x_2, \ldots, x_r\}$. For example,

$$pr_2(x_1, x_2) = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_2), (x_1, x_3), (x_3, x_1), (x_3, x_2), (x_2, x_3), (x_3, x_3)\}.$$

As in Charalambides [8], the number of n-permutations of r with unrestricted repetitions, denoted by $U(r, n)$, is $U(r, n) = r^n$.

Since in this work we deal with infinite products of a complex variable[2], it is important to establish a convergence criterion for these as given by the next theorem.

Theorem 2: *(Theorem A.1.5 of Andrews et al. [4]) If*

$$\sum_{n=1}^{\infty} |a_n(z)|$$

converges uniformly in some region, then the product $\prod_{n=1}^{\infty}(1 + a_n(z))$ *converges uniformly in that region.*

[2] In the reference Ponnusamy, one may found several properties of function of a complex variables

2. Results

Theorem 3: *For all complex numbers* $a_1, a_2, \ldots, a_r$ *with* $Re\{a_i\} > 1,\ 1 \le i \le r,$ *we have*

$$\sum_{(b_1,\ldots,b_r)\in pr_n(a_1,\ldots,a_r)} \zeta(b_1,\ldots,b_r)$$

$$= \sum_{w_1+\cdots+w_m\in C(n)} \frac{(-1)^{n+m}}{m! w_1 w_2 \cdots w_m}\left[\sum_{k_1+\cdots+k_r\in C^*(w_1)} C(w_1;k_1,\ldots,k_r)\zeta(k_1a_1+\cdots+k_ra_r)\right]$$

$$\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_2)} C(w_2;k_1,\ldots,k_r)\zeta(k_1a_1+\cdots+k_ra_r)\right]\times$$

$$\cdots\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_m)} C(w_m;k_1,\ldots,k_r)\zeta(k_1a_1+\cdots+k_ra_r)\right].$$

Proof: Let consider the infinite product

$$g(z) = \prod_{n=1}^{\infty}\left(1+z\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)\right), \tag{2}$$

wherein $Re\{a_i\} > 1$ and $|z| < 1$. For $a_n = z\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)$, we claim that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges uniformly for all complex $|z|$, in which $|z| < 1$. For $r = 1$, since $|z| < 1$ and $Re\{a_1\} > 1$, and the series $\sum_{n=1}^{\infty}\left|\frac{z}{n^{a_1}}\right|$ is uniformly convergent, then the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n^{a_1}}\right)$$

converges uniformly for $|z| < 1$ by the Theorem 2. Provided that

$$\left|z\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}\right)\right| \le \left|\frac{z}{n^{a_1}}\right| + \left|\frac{z}{n^{a_2}}\right|, \tag{3}$$

for all $n \ge 1$, the convergence of $\sum_{n=1}^{\infty}\left|\frac{z}{n^{a_1}}\right| + \sum_{n=1}^{\infty}\left|\frac{z}{n^{a_2}}\right|$ imply in the convergence of the series (3). Generally, as the series

$$\sum_{n=1}^{\infty}\left|z\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)\right|$$

is uniformly convergent in $\{z \in \mathbb{C}, |z| < 1\}$, then the product (2) is uniformly convergent in the same region of complex plane. Considering $|z| < 1$, we will find the coefficient of z^n in the expansion of $g(z)$. Firstly for $n = 1$, such coefficient is

$$\sum_{k=1}^{\infty}\left(\frac{1}{k^{a_1}}+\frac{1}{k^{a_2}}+\cdots+\frac{1}{k^{a_r}}\right)=\zeta(a_1)+\zeta(a_2)+\cdots+\zeta(a_r).$$

The coefficient of z^2 is

$$\sum_{1\le l_1<l_2}\left(\frac{1}{l_1^{a_1}}+\frac{1}{l_1^{a_2}}+\cdots+\frac{1}{l_1^{a_r}}\right)\left(\frac{1}{l_2^{a_1}}+\frac{1}{l_2^{a_2}}+\cdots+\frac{1}{l_2^{a_r}}\right)=\sum_{(b_1,b_2)\in pr_2(a_1,\ldots,a_r)}\zeta(b_1,b_2).$$

In general, the coefficient of z^n is given by the next sum:

$$\sum_{1\le l_1<l_2<\cdots<l_n}\left(\frac{1}{l_1^{a_1}}+\cdots+\frac{1}{l_1^{a_r}}\right)\cdots\left(\frac{1}{l_n^{a_1}}+\cdots+\frac{1}{l_n^{a_r}}\right)=\sum_{(b_1,\cdots,b_r)\in PR_n(a_1,\ldots,a_r)}\zeta(b_1,\ldots,b_r) \tag{4}$$

In other hand,

$$g(z)=exp\left(\sum_{n=1}^{\infty}ln\left(1+z\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)\right)\right), \tag{5}$$

wherein $exp(z)=e^z$. Whereas $|z|<1$ and $Re\{a_i\}>1,\ 1\le i\le r,$ we have $\left|z\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)\right|<1,$ for all positive integer n. With the Taylor expansion of the logarithm, we have

$$ln\left(1+z\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)^{k+1}z^{k+1}, \tag{6}$$

so that

$$g(z)=exp\left(\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)^{k+1}z^{k+1}\right).$$

Expanding on Taylor series of equation (5), we have

$$f(z)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\left(\frac{1}{n^{a_1}}+\frac{1}{n^{a_2}}+\cdots+\frac{1}{n^{a_r}}\right)^{k+1}z^{k+1}\right)^j. \tag{7}$$

Our eminent work is to find the coefficient of z^n in equation (7) to compare with the coefficient of the same z^n found in the equation (4). It may be useful to open some terms of $f(z)$ as given above in order to accomplish such an attempt.

$$f(z)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)z-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)^2z^2+\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)^3z^3+\cdots\right)^j$$

(8)

The coefficients of z^2 and z^3 in the expansion of $f(z)$ in the equation (8) are, respectively,

$$\left(-\frac{1}{2}\right)(\zeta(2a_1)+\zeta(2a_2)+\cdots+\zeta(2a_r)+2\zeta(a_1+a_2)+2\zeta(a_1+a_3)+\cdots+2\zeta(a_{r-1}+a_r))$$
$$+\frac{1}{2!}(\zeta(a_1)+\zeta(a_2)+\cdots+\zeta(a_r))^2$$

and

$$\frac{1}{3}(\zeta(3a_1)+\zeta(3a_2)+\cdots+\zeta(3a_r)+3\zeta(2a_1+a_2)+3\zeta(a_1+2a_2)+\cdots+3\zeta(a_{r-1}+2a_r))$$
$$-\frac{1}{2}(\zeta(a_1)+\zeta(a_2)+\cdots+\zeta(a_r))\zeta(2a_1)+\zeta(2a_2)+\cdots+\zeta(2a_r)+2\zeta(a_1+a_2)+\cdots+2\zeta(a_{r-1}+a_r)$$
$$+\frac{1}{3!}(\zeta(a_1)+\zeta(a_2)+\cdots+\zeta(a_r))^3.$$

For the coefficient of z^n, we shall consider the integer compositions $w_1+w_2+\cdots+w_m \in C(n)$. Firstly, we will find the coefficient of z^{w_i} in

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)z-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)^2 z^2+\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n^{a_1}}+\cdots+\frac{1}{n^{a_r}}\right)^3 z^3+\cdots. \quad (9)$$

If the coefficient of z^{w_i} is c_i, $1 \le i \le m$, as $z^n = z^{w_1+w_2+\cdots+w_m}$, then the coefficient of z^n is equal to

$$\sum_{w_1+\cdots+w_m \in C(n)} c_1 c_2 \cdots c_m.$$

The result is found easily if we find c_i, $1 \le i \le m$. For this purpose, we need to consider all weak compositions $k_1+k_2+\cdots+k_r \in C^*(w_1)$, to find the coefficient of z^{w_i} in the series (9). This coefficient is given by

$$\frac{(-1)^{\#(w_i)}}{w_i}\sum_{k_1+\cdots+k_r \in C^*(w_i)}\binom{n}{k_1, k_2, \cdots, k_{r-1}}\zeta(k_1a_1+k_2a_2+\cdots+k_ra_r),$$

where

$$\#(w_i)=\begin{cases} 2, \text{if } w_i \text{ is odd} \\ 1, \text{if } w_i \text{ is even}\end{cases}$$

Taking all parts of the integer composition $w_1+\cdots+w_m \in C(n)$, a parcel of the coefficient of z^n is

$$\frac{(-1)^{n+m}}{m! w_1 w_2 \cdots w_m}\left[\sum_{k_1+\cdots+k_r \in C^*(w_1)}\binom{w_1}{k_1, \cdots, k_{r-1}}\zeta(k_1a_1+\cdots+k_ra_r)\right]$$

$$\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_2)}\binom{w_2}{k_1,\cdots,k_{r-1}}\zeta(k_1a_1+\cdots+k_ra_r)\right]\times$$
$$\cdots\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_m)}\binom{w_m}{k_1,\cdots,k_{r-1}}\zeta(k_1a_1+\cdots+k_ra_r)\right].$$

Considering all integer compositions $w_1+\cdots+w_m\in C(n)$, we finally reach the coefficient of z^n, proving the theorem.

Example 3: For $a,b\in\mathbb{C}$, where $Re\{a\}>1$ and $Re\{b\}>1$, we have

$$\sum_{(c,d)\in PR_3(a,b)}\zeta(p_3(a,b))=\tfrac{1}{3}\left(\zeta(3a)+\zeta(3b)+3\zeta(2a+b)+3\zeta(2b+a)\right)$$
$$-\tfrac{1}{2}\left(\zeta(a)+\zeta(b)\right)\left(\zeta(2a)+\zeta(2b)+2\zeta(a)\zeta(b)\right)+\tfrac{1}{3}\left(\zeta(a)+\zeta(b)\right)^3.$$

For $a=a_1=a_2=\cdots=a_r$, with $Re\{a\}>1$, using the result of our main Theorem, we can state the next result.

Corollary 1:

$$\zeta(\{a\}^n)=\frac{1}{n^n}\sum_{w_1+\cdots+w_m\in C(n)}\frac{(-1)^{n+m}}{m!w_1w_2\cdots w_m}\left[\sum_{k_1+\cdots+k_r\in C^*(w_1)}c(w_1,r)\zeta((k_1+\cdots+k_r)a)\right]$$
$$\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_2)}c(w_2,r)\zeta((k_1+\cdots+k_r)a)\right]\times\cdots\times\left[\sum_{k_1+\cdots+k_r\in C^*(w_m)}c(w_m,r)\zeta((k_1+\cdots+k_r)a)\right].$$

We can reach this result in a different way by using $\prod_{n=1}^{\infty}\left(1+\frac{z}{n^a}\right)$, for $|z|<1$, as a generating function and performing the same steps in the main theorem to obtain the next result.

Theorem 4: *For all complex numbers a, wherein* $Re\{a\}>1$, *we have*

$$\zeta(\{a\}^n)=\frac{(-1)^{n+1}}{n}\zeta(na)+\frac{1}{n!}\zeta^n(a)+\sum_{x_1+\cdots+x_m\in C(n)}\frac{(-1)^{ne(x_1+\cdots+x_m)}}{m!x_1x_2\cdots x_m}\zeta(x_1a)\zeta(x_2a)\cdots\zeta(x_ma),$$

where $ne(x_1+\cdots+x_m)$ *is equals to the number of* x_i *even in the integer composition*

$$x_1+\cdots+x_m\in C(n).$$

Example 4:

$$\zeta(\{a\}^5)=\sum_{x_1+\cdots+x_m\in C(5)}\frac{(-1)^{ne(x_1+\cdots+x_m)}}{m!x_1x_2\cdots x_m}\zeta(x_1a)\zeta(x_2a)\cdots\zeta(x_ma)+\frac{1}{5}\zeta(5a)+\frac{1}{5!}\zeta^5(a)$$

$$= \frac{1}{5}\zeta(5a) + \frac{1}{5!}\zeta^5(a) - \frac{1}{12}\zeta^3(a)\zeta(2a) + \frac{1}{6}\zeta(3a)\zeta^2(a) - \frac{1}{6}\zeta(3a)\zeta(2a)$$
$$- \frac{1}{4}\zeta(a)\zeta(4a) + \frac{1}{8}\zeta^2(2a)\zeta(a).$$

Corollary 2: *For a complex number a, with* $Re\{a\} > 1$, *we have*

$$\frac{(-1)^{n+1}}{n}\zeta(na) + \frac{1}{n!}\zeta^n(a) + \sum_{x_1+\cdots+x_m \in C(n)} \frac{(-1)^{ne(x_1+\cdots+x_m)}}{m! x_1 x_2 \cdots x_m}\zeta(x_1 a)\zeta(x_2 a)\cdots\zeta(x_m a)$$
$$= \frac{1}{n^n}\sum_{w_1+\cdots+w_m \in C(n)} \frac{(-1)^{n+m}}{m! w_1 w_2 \cdots w_m}\left[\sum_{k_1+\cdots+k_r \in C^*(w_1)} c(w_1, r)\zeta((k_1+\cdots+k_r)a)\right]$$
$$\times\left[\sum_{k_1+\cdots+k_r \in C^*(w_2)} c(w_2, r)\zeta((k_1+\cdots+k_r)a)\right]\times$$
$$\cdots\times\left[\sum_{k_1+\cdots+k_r \in C^*(w_m)} c(w_m, r)\zeta((k_1+\cdots+k_r)a)\right].$$

For the next theorem we denote by Σ_r the set of all r-permutations.

Theorem 5: *For all complex numbers* a_i, $1 \le i \le r$, *where* $Re\{a_i\} > 1$, *we have*

$$\sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)}, \ldots, a_{\sigma(r)}) + \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)}, \ldots, a_{\sigma(r-2)}, a_{\sigma(r-1)} + a_{\sigma(r)})$$
$$+ \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)}, \ldots, a_{\sigma(r-3)}, a_{\sigma(r-2)} + a_{\sigma(r-1)}, a_{\sigma(r)}) + \cdots$$
$$+ \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)} + a_{\sigma(2)}, a_{\sigma(3)}, \ldots, a_{\sigma(r)}) + \cdots$$
$$+ \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)}, a_{\sigma(4)}, \ldots, a_{\sigma(r)}) + \cdots$$
$$+ \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)}, \ldots, a_{\sigma(1)}, a_{\sigma(r-2)} + a_{\sigma(r-1)} + a_{\sigma(r)}) + \cdots$$
$$+ \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)}, a_{\sigma(2)} + a_{\sigma(3)} + \cdots + a_{\sigma(r)}) + \cdots$$
$$+ \sum_{\sigma\in\Sigma_r} \zeta(a_{\sigma(1)} + a_{\sigma(2)} + \cdots + a_{\sigma(r-1)}, a_{\sigma(r)}) + \zeta(a_1 + a_2 + \cdots + a_r)$$
$$= \sum_{w_1+\cdots+w_m \in C(n)} \frac{(-1)^{n-m}}{m!}\prod_{j=1}^{m}(\zeta(w_j a_1) + \cdots + \zeta(w_j a_r)).$$

Proof: Let consider for $|z| < 1$ and $Re\{a_i\} > 1$, $1 \le i \le r$,

$$h(z) = \prod_{n=1}^{\infty}\left(1 + \frac{z}{n^{a_1}}\right)\left(1 + \frac{z}{n^{a_2}}\right)\cdots\left(1 + \frac{z}{n^{a_r}}\right).$$

function $h(z)$ is well-defined. Indeed, by Theorem 2, since $Re\{a_i\}>1$ for $1\le i\le r$, we have the convergence of each product $h_i(z)=\prod_{n=1}^{\infty}\left(1+\frac{z}{n^{a_1}}\right)$, and then the product $h(z)=\prod_{i=1}^{\infty}h_i(z)$ is uniformly convergent in $\{z\in\mathbb{C},|z|<1\}$.

As we did in Theorem 3, firstly we find the coefficient of z^n, $n\ge 1$ in $g(z)$. With some patience, it is not hard to understand that the coefficient of z^n in the expansion of $g(z)$ is equal to the left side of the equation in the Theorem. Since $|z|<1$, and $Re\{a_i\}>1$ for $1\le i\le r$, we can rewrite $h(z)$ as

$$h(z)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k z^{k+1}}{k+1}\left(\frac{1}{n^{(k+1)a_1}}+\frac{1}{n^{(k+1)a_2}}+\cdots+\frac{1}{n^{(k+1)a_r}}\right)\right)^j. \tag{10}$$

The coefficient of z^n in the equation (10) reveals the truth of the theorem.

Example 5: For $r=3$, and $Re\{a_i\}>1, i=1,2,3,$ we have:

$$\sum_{\sigma\in\Sigma_3}\zeta(a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)})+\sum_{\sigma\in\Sigma_3}\zeta(a_{\sigma(1)},a_{\sigma(2)}+a_{\sigma(3)})+\sum_{\sigma\in\Sigma_3}\zeta(a_{\sigma(1)}+a_{\sigma(2)},a_{\sigma(3)})+\zeta(a_1+a_2+a_3)$$
$$=\sum_{\sigma\in\Sigma_3}\zeta(3a_{\sigma(1)})-\left(\sum_{i=1}^{3}\zeta(a_i)\right)\left(\sum_{i=1}^{3}\zeta(2a_i)\right)+\frac{1}{6}\left(\sum_{i=1}^{3}\zeta(a_i)\right)^3.$$

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