

On explicit formulas of hyperbolic tangent matrix function

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Abstract

The hyperbolic tangent matrix function is a mathematical tool with diverse applications in various fields of mathematics, applied science, and engineering. The hyperbolic tangent matrix $th(M)$, ($M \in M, (\mathbb{C})$) can be determined through different techniques. In this paper, we establish several explicit formulas for $th(tM)$. The first approach is based on Hermite matrix polynomial expansions, the second utilizes the decomposition of Fibonacci-Hörner, and the third involves polynomial decomposition utilizing the expression for the n -th power of the matrix and the polynomial decomposition of the exponential matrix. An example is provided to illustrate these methods.

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1. Introduction

Matrix functions serve a significant position in numerous fields of science and engineering, encompassing applications in solving differential equations, control theory, quantum mechanics, and signal processing. Among these functions, matrix trigonometric functions, and specifically the hyperbolic tangent matrix function, are of significant interest due to their widespread applicability and the challenges associated with their computation [14].

The hyperbolic tangent matrix function finds applications across numerous domains in mathematics, applied sciences, and engineering. It is particularly valuable for deriving analytical solutions to the radiative transfer equation [3], analyzing heat transfer phenomena [10], investigating symplectic systems [16], exploring graph theory [1, 5], and developing specialized exponential integrators [2].

Using the hyperbolic tangent function's Taylor series expansion:

$$th(z) = \sum_{k \geq 1} \frac{4^k (4^k - 1) \mathcal{B}_{2k}}{(2k)!} z^{2k-1}, |z| < \frac{\pi}{2}$$

We obtain another option for calculating the hyperbolic tangent matrix function

$$th(M) = \sum_{k \geq 1} \frac{4^k (4^k - 1) \mathcal{B}_{2k}}{(2k)!} M^{2k-1}, \rho(M) < \frac{\pi}{2} \quad (1)$$

In this article the set of eigenvalues of matrix $M \in M_r(\mathbb{C})$ will be referred to as $Sp(M)$ and $\rho(M)$ denote the spectral radius associated with the matrix M , identified by

$$\rho(M) = \max\{|\alpha|; \alpha \in Sp(M)\}.$$

We can express the hyperbolic tangent of a matrix in terms of the matrix the hyperbolic matrix function:

$$th(M) = sh(M)ch^{-1}(M), \rho(M) < \frac{\pi}{2} \quad (2)$$

Furthermore, for an invertible matrix $M \in \mathbb{C}^{n \times n}$ with $\rho(M) < \frac{\pi}{2}$, its hyperbolic tangent can be calculated using the matrix exponential function as follows:

$$th(M) = (e^M - e^{-M})(e^M + e^{-M})^{-1}, \rho(M) < \frac{\pi}{2} \quad (3)$$

This form is particularly useful as it relates $\tanh(M)$ to the more well-understood exponential matrix function, enabling the use of existing algorithms for matrix exponentiation.

The calculation of hyperbolic matrix functions remains an intriguing topic, and the main objective of this article is to present several methods for computing the matrix function $th(tM)$, where M is a given matrix and t is a scalar parameter. The first technique we employ relies on expansions using Hermite matrix polynomials, which involves expressing the matrix function as a sum of matrix polynomials. Specifically, we use a series of Hermite matrix polynomials to approximate the function, which allows us to derive an explicit expression for $th(tM)$. The next technique we employ involves the Fibonacci-Hörner decomposition, which involves expressing the matrix exponential in terms of a sum of Fibonacci-Hörner polynomials. This technique is particularly useful for matrices that have a recursive structure, as it allows us to decompose the exponential matrix into a sum of simpler polynomials [8, 9]. The third technique we employ is polynomial decomposition via the expression for the matrix raised to the power of n and the polynomial decomposition of the exponential matrix to obtain an explicit formula for $th(tM)$. [13]

Overall, these techniques provide several different approaches for computing the matrix function $th(tM)$, each of which has its own advantages and limitations. By presenting these techniques in detail, we hope to provide a comprehensive understanding of how to compute this function for a given matrix. By providing explicit and efficient formulas for the hyperbolic tangent matrix function, we hope to facilitate further advancements in the computational methods for matrix functions and broaden their applicability in scientific and engineering problems.

2. Hermite matrix polynomial series expansions

The Hermite matrix polynomial series expansions refer to methods where matrices are expressed as polynomials in terms of Hermite polynomials, allowing for intricate expansions that facilitate various computational and theoretical analyses in matrix algebra and related fields.

2.1 Introduction

In this section, we discuss specific characteristics that define Hermite matrix polynomials found in [6] that is fundamental to our approach.

The n -th Hermite matrix polynomial is defined as

$$\mathcal{H}_n(x, M) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (\sqrt{2M})^{n-2k}}{k!(n-2k)!} x^{n-2k},$$

For $M \in M_r(\mathbb{C})$ for a positively stable matrix, the n -th Hermite matrix polynomial satisfy the recurrence relationship

$$\mathcal{H}_{-1}(x, M) = 0, \quad \mathcal{H}_0(x, M) = I,$$

$$\mathcal{H}_n(x, M) = Ix\sqrt{(2M)}\mathcal{H}_{n-1}(x, M) - 2(n-1)\mathcal{H}_{n-2}(x, M), \quad n \geq 1$$

with I represent the identity matrix of $M_r(\mathbb{C})$.

In [6] it was established that

$$e^{xt\sqrt{2M}-t^2t} = \sum_{n \geq 0} \frac{1}{n!} \mathcal{H}_n(x, M) t^n, \quad |t| < +\infty, \quad (4)$$

and for $\mathcal{H}_n(x)$ the n th scalar Hermite polynomial, We obtain:

$$\begin{aligned} \|\mathcal{H}_n(x, M)\|_2 &\leq \frac{\mathcal{H}_n\left(\frac{\|a\|\sqrt{2M}\|_2}{2}\right)}{i^n}, \quad |x| < a, \quad n \in \mathbb{N}, \\ \sum_{n \geq 0} \frac{\mathcal{H}_n\left(\frac{\|\sqrt{2M}\|_2 ia}{2}\right)}{n! i^n} |t|^n &= \exp(\|t\| a \|\sqrt{2M}\|_2 + |t|^2), \quad |t| < \infty. \end{aligned}$$

Let $C(l, m)$ and $D(l, m)$ be matrices on $M_r(\mathbb{C})$ for $m \geq 0, l \geq 0$, In [13, Lemma 11 p.57] Elementary series manipulations it was established that:

$$\sum_{m \geq 0} \sum_{l \geq 0} C(l, m) = \sum_{m \geq 0} \sum_{l=0}^{\lfloor m/2 \rfloor} C(l, m-2l), \quad \sum_{m \geq 0} \sum_{l \geq 0} D(l, m) = \sum_{m \geq 0} \sum_{l=0}^m D(l, m-l). \quad (5)$$

For M a positive stable matrix. Using (4) we obtain:

$$e^{xt\sqrt{2M}+t^2t} = \sum_{n \geq 0} \frac{\mathcal{H}_n(x, M)}{n!} t^n$$

Thus

$$\sum_{n \geq 0} \frac{(\sqrt{2M}x)^n}{n!} t^n = \left(\sum_{n \geq 0} \frac{t^{2n}}{n!} \right) \left(\sum_{n \geq 0} \frac{\mathcal{H}_n(x, M)}{n!} t^n \right) \quad (6)$$

and by (5) one gets

$$\left(\sum_{n \geq 0} \frac{t^{2n}}{n!} \right) \left(\sum_{n \geq 0} \frac{\mathcal{H}_n(x, M)}{n!} t^n \right) = \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\mathcal{H}_{n-2k}(x, M)}{k!(n-2k)!} t^n. \quad (7)$$

Determined by identifying the coefficient of t^n in (6) and (7) we get:

$$x^n I = (\sqrt{2M})^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{k!(n-2k)!} \mathcal{H}_{n-2k}(x, M), \quad -\infty < x < +\infty \quad (8)$$

Hermite matrix polynomials possess several properties that make them suitable for various applications in mathematical and engineering fields. One of the notable properties is their ability to be expressed in terms of series expansions, which is particularly useful for solving matrix differential equations.

2.2 Series expansions of hyperbolic tangent matrices using Hermite matrix polynomials

Let $B \in M_n(\mathbb{C})$

In this section we present the Hermite matrix polynomial series expansion of $th(tB)$ with $\rho(tB) < \frac{\pi}{2}$

Theorem 2.1: Let B be a matrix in $M_r(\mathbb{C})$ satisfying the spectral condition

$$|\operatorname{Re}(z)| > |\operatorname{Im}(z)| \quad \forall z \in \rho(B).$$

We have:

$$th(tB) = \sum_{n \geq 0} \psi(n) \mathcal{H}_{2n+1} \left(x, \frac{1}{2} B^2 \right), \quad -\infty < x < +\infty \quad (9)$$

$$\text{With } \psi(n) = \sum_{k \geq 0} \frac{4^{n+k+1} (4^{n+k+1} - 1)}{(2(n+k)+2)!} B_{2(n+k)+2} \frac{(2(n+k)+1)!}{k!(2n+1)!}$$

Proof: Using the the series expansion of $th(Mx)$ and (5) we obtain

$$\begin{aligned} th(Mx) &= \sum_{n \geq 0} \frac{4^{n+1} (4^{n+1} - 1) \mathcal{B}_{2n+2}}{(2n+2)!} M^{2n+1} x^{2n+1} \\ &= \sum_{n \geq 0} \frac{4^{n+1} (4^{n+1} - 1) \mathcal{B}_{2n+2}}{(2n+2)!} \sum_{k=0}^n \frac{(2n+1)!}{k!(2n-2k+1)!} \mathcal{H}_{2n-2k+1} \left(x, \frac{1}{2} M^2 \right), \\ &= \sum_{n \geq 0} \sum_{k \geq 0} \frac{4^{n+k+1} (4^{n+k+1} - 1)}{(2(n+k)+2)!} B_{2(n+k)+2} \frac{(2(n+k)+1)!}{k!(2n+1)!} \mathcal{H}_{2n+1} \left(x, \frac{1}{2} M^2 \right). \end{aligned}$$

Another approach is to consider the formula:

$$th(Bx) = I - 2\rho(Bx)$$

$$\text{with } \rho(Bx) = (e^{2Bx} - I)^{-1}$$

$$\text{In [4] it was shown that } e^{2Bx} = e \sum_{n \geq 0} \frac{\mathcal{H}_n(x+2B^2)}{n!}$$

$$\text{a simple calculation yields: } \rho(Bx) = [e \sum_{n \geq 1} \frac{\mathcal{H}_n(x+2B^2)}{n!} + (1+e)I]^{-1}$$

In [4], Hermite matrix polynomials have been employed to solve second-order matrix differential equations. The solutions can be represented in terms of Hermite matrix polynomials, providing a structured and systematic approach to these types of problems.

Furthermore, Hermite matrix polynomials are instrumental in deriving explicit formulas for matrix functions. For instance, in [6], an explicit representation of the exponential matrix function in terms of Hermite polynomials involving matrices are given, which facilitates the calculation of the matrix exponential.

The Hermite matrix polynomial expansions technique has the advantage of being relatively straightforward to apply and can be used for a wide range of matrices. It involves expressing the matrix function as a sum of matrix polynomials and using a series of Hermite matrix polynomials to approximate the function. However, this technique can become computationally expensive for large matrices, as it requires the computation of many matrix products and powers.

3. Hyperbolic Tangent Matrix Function Decomposition Through Recursive Relationships

3.1 Recursive Relations for Linear Combinations of Square Matrices

Let $M \in M_r(\mathbb{C})$ and consider its characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \dots - a_{r-1}$. According to the theorem of Cayley-Hamilton, $P(M) = O_r$ (the null matrix of order $r \times r$). Consequently, the sequence $\{M^n\}_{n \geq 0}$ in $M_r(\mathbb{C})$ behaves like an r -generalized Fibonacci sequence, with initial conditions $M^0 = I_r$ (the identity matrix), $M, \dots, M^{r-1}$, and coefficients $a_0, \dots, a_{r-1}$. For nonsingular matrices M , the focus is particularly on the r -generalized Fibonacci sequence:

$$\{M^n\}_{n \geq -r+1},$$

with initial conditions $M^{-r+1}, \dots, M^{-1}, M^0 = I_r$, and coefficients $a_0, \dots, a_{r-1}$, as discussed in [13].

As detailed in [13, 11], the matrix powers M^n ($n \geq r$) decompose as:

$$M^n = u_M(n)M_0 + u_M(n-1)M_1 + \dots + u_M(n-r+1)M_{r-1}, \text{ for all } n \geq r, \quad (10)$$

where

$$M_0 = I_r; M_i = M^i - a_0 M^{i-1} - \dots - a_{i-1} I_r, \text{ for each } i = 1, \dots, r-1. \quad (11)$$

The sequence $\{u_M(n)\}_{n \geq -r+1}$ is defined by:

$$u_M(n) = \sum_{k_0+2k_1+\dots+rk_{r-1}=n} \frac{(k_0+k_1+\dots+k_{r-1})!}{k_0!k_1!\dots k_{r-1}!} a_0^{k_0} a_1^{k_1} \dots a_{r-1}^{k_{r-1}}, \quad (12)$$

for all $n \geq 1$, where $u_M(0) = 1$ and $u_M(n) = 0$ for $-r+1 \leq n \leq -1$.

Furthermore, research in [13, 11] establishes that $\{u_M(n)\}_{n \geq -r+1}$ satisfies the linear recurrence relation:

$$u_M(n+1) = a_0 u_M(n) + a_1 u_M(n-1) + \dots + a_{r-1} u_M(n-r+1), \text{ for all } n \geq 0. \quad (13)$$

In summary, the sequence $\{u_M(n)\}_{n \geq 1-r}$ constitutes an r -generalized Fibonacci sequence.

Let $M \in M_r(\mathbb{C})$. By induction, we can demonstrate that the decomposition (10) of M^n ($n \geq r$) holds universally for M , whether reversible or irreversible, where M_k ($0 \leq k \leq r-1$) is defined by (11), and $\{u_M(n)\}_{n \geq 1-r}$ is defined by (12).

The collection of matrices $\{M_0, \dots, M_{r-1}\}$ is termed the Fibonacci-Horner basis of the decomposition into powers (10) for the matrix M .

The coefficients of $u_M(n)$ ($n \geq 1$) can alternatively be computed using the recursive formula (13). Additionally, they can be obtained by employing the $n \times n$ upper Hessenberg matrix determinant, as discussed in [7, 15].

$$u_M(n) = \begin{vmatrix} a_0 & a_1 & \dots & a_{r-1} & 0 & \dots & 0 \\ -1 & a_0 & \dots & a_{r-2} & a_{r-1} & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \dots & \dots & \vdots \\ \vdots & \vdots & \dots & \ddots & \dots & \dots & \vdots \\ 0 & 0 & \dots & \dots & \dots & a_0 & a_1 \\ 0 & 0 & \dots & \dots & \dots & -1 & a_0 \end{vmatrix}. \quad (14)$$

For $r+1 \leq n$, the expression (14) arises from the determinant of a Toeplitz banded matrix of size $r+1$.

$$M^m = \sum_{l=0}^{r-1} M^{r-1-l} \times u_M^{(l)}(m-r+1) = \sum_{l=0}^{r-1} M^l \times u_M^{(r-1-l)}(m-r+1), \quad (15)$$

Here, $u_M^{(l)}(m-r+1)$, the determinant for $(m-r+1) \times (m-r+1)$ matrix, corresponds to $u_M(m-r+1)$ provided in the expression (14), however the first row shifted forward by l positions (see proposition 2.2 [12]).

3.2 Fibonacci-Hörner decomposition of hyperbolic tangent matrix

The next theorem introduces the decomposition based on Fibonacci-Horner of the hyperbolic tangent matrix functions.

Theorem 3.1: Let $M \in M_r(\mathbb{C})$, with $P_M(z) = z^r - a_0 z^{r-1} - \dots - a_{r-1}$ as the characteristic polynomial of M , where $a_{r-1} \neq 0$ and $t \in \mathbb{R}$ so that $\rho(tM) < \frac{\pi}{2}$. the Fibonacci-Horner basis for M is denoted by $\{M_i\}_{0 \leq i \leq r-1}$. Consequently, we get:

$$\tanh(tM) = \sum_{i=0}^{r-1} \Theta_i(t) M_i, \quad (16)$$

where

$$\Theta_i(t) = \sum_{n \geq 1} \frac{2^{2n} (2^{2n-1} \mathcal{B}_{2n})}{(2n)!} t^{2n-1} u_M(2n-i-1).$$

Here, $u_M(n)$ is given by (12).

Proof: Substituting (10) into the series expansion of the hyperbolic tangent function yields:

$$\begin{aligned} th(tM) &= \sum_{n \geq 1} \frac{2^{2n} (2^{2n-1} \mathcal{B}_{2n})}{(2n)!} t^{2n-1} M^{2n-1}, \\ &= \sum_{n \geq 1} \frac{2^{2n} (2^{2n-1} \mathcal{B}_{2n})}{(2k)!} t^{2n-1} \sum_{i=0}^{r-1} u_M(2n-i-1) M_i \\ &= \sum_{i=0}^{r-1} \left[\sum_{n \geq 1} \frac{2^{2n} (2^{2n-1} \mathcal{B}_{2n})}{(2n)!} t^{2n-1} u_M(2n-i-1) \right] M_i \\ &= \sum_{i=0}^{r-1} \Theta_i(t) M_i \end{aligned}$$

Example 3.1: Let $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$

$R(z) = z^2 + 1$ is a polynomial that annihilates M .

Therefore, we get $a_0 = 0$, $a_1 = -1$ after calculation, we obtain the Fibonacci-Horner basis. $M_0 = I_2$ and $M_1 = M$

The sequence $(u_M(n))_{n \geq -1}$ is represented by
$$\begin{cases} u_M(l) = (-1)^{\frac{l}{2}} & \text{if } l \text{ is even} \\ u_M(l) = 0 & \text{if } l \text{ is odd} \end{cases}$$

Calculation using the expression (16) we obtain:

$$th(tM) = \Theta_0(t) M_0 + \Theta_1(t) M_1 = \tan(t) M$$

The decomposition of the hyperbolic tangent matrix using Fibonacci-Horner provides a valuable tool in the study and application of hyperbolic tangent matrices. Its efficiency and accuracy make it a suitable choice for a wide range of applications that require the evaluation of hyperbolic tangent matrices. This technique is particularly useful for matrices that have a recursive structure, as it allows us to decompose the exponential

matrix into a sum of simpler polynomials. This technique has the advantage of being computationally efficient and can handle large matrices. However, it may not be suitable for matrices that do not have a recursive structure, as the computation of the Fibonacci-Horner polynomials may become difficult or impossible.

3.3 Decomposing the hyperbolic tangent matrix using polynomials

This section introduces an approach to compute the polynomial decomposition of the hyperbolic tangent matrix functions that relies on the representation of the matrix raised to the power of n (15).

Theorem 3.2: Let $M \in M_r(\mathbb{C})$ with $P_M(z) = z^r - a_0 z^{r-1} - \dots - a_{r-1}$ as the characteristic polynomial of M , where $a_{r-1} \neq 0$ and $t \in \mathbb{R}$ such that $\rho(tM) < \frac{\pi}{2}$, we get:

$$\tanh(tM) = \sum_{k=0}^{r-1} \Omega_k(t) M^k \quad (17)$$

where

$$\Omega_k(t) = \begin{cases} \sum_{n=\lfloor \frac{r}{2} \rfloor}^{+\infty} \frac{4^{k+1}(4^{k+1}-1)\mathcal{B}_{2k+2}}{(2(k+1))!} t^{2k+1} u_M^{(r-1-k)}(2n-r+2), & \text{if } k \text{ is even} \\ \frac{2^{k+1}(2^{k+1}-1)}{(k+1)!} \mathcal{B}_{k+1} t^k + \sum_{n=\lfloor \frac{r}{2} \rfloor}^{+\infty} \frac{4^{k+1}(4^{k+1}-1)\mathcal{B}_{2k+2}}{(2(k+1))!} t^{2k+1} u_M^{(r-1-k)}(2n-r+2), & \text{if } k \text{ is odd} \end{cases}$$

and the coefficients $u_M^{(r-1-s)}(n-r+1)$ are those specified in (15).

Proof: From (1) we obtain,

$$th(tM) = \sum_{k=0}^{\lfloor \frac{r}{2} \rfloor - 1} \frac{4^{k+1}(4^{k+1}-1)\mathcal{B}_{2k+2}}{(2(k+1))!} t^{2k+1} M^{2k+1} + \sum_{k=\lfloor \frac{r}{2} \rfloor}^{+\infty} \frac{4^{k+1}(4^{k+1}-1)\mathcal{B}_{2k+2}}{(2(k+1))!} t^{2k+1} M^{2k+1}$$

Therefore, by using (15) we deduce that

$$th(tM) = \sum_{k=0}^{\lfloor \frac{r-2}{2} \rfloor} \frac{4^{k+1}(4^{k+1}-1)\mathcal{B}_{2k+2}}{(2(k+1))!} t^{2k+1} M^{2k+1} + \sum_{k=\lfloor \frac{r}{2} \rfloor}^{+\infty} \frac{4^{k+1}(4^{k+1}-1)\mathcal{B}_{2k+2}}{(2(k+1))!} t^{2k+1} \sum_{j=0}^{r-1} u_{2k-r+2}^{(r-1-j)} M^j$$

Considering the uniform convergence, we get

$$th(tM) = \sum_{k=0}^{r-1} \Phi_k(t) M^k$$

The polynomial decomposition via the expression for the matrix raised to the n -th power and the polynomial decomposition of the exponential matrix technique is particularly effective for matrices that have complex eigenvalues or non-diagonalizable matrices. This technique

has the advantage of being able to handle a wide range of matrices and can provide accurate approximations of the matrix function. However, it may not be the most computationally efficient technique, particularly for large matrices.

4. Conclusion

Extensive research has been conducted in the literature on defining matrix functions using interpolating polynomials. In the realm of functions involving matrices, the study of constituent matrices holds great importance and is an ongoing subject of investigation.

In this paper, we have explored three distinct techniques for computing the hyperbolic trigonometric matrix. The first technique involves Hermite matrix polynomial expansions, while the second technique employs the Fibonacci-Hörner decomposition. Additionally, we have presented a third method that uses the polynomial decomposition of the hyperbolic tangent functions of matrices.

After reviewing the expressions presented in this paper and comparing them to the existing literature in this field, we have concluded that they are original. To the best of our knowledge, our approach is distinct from those currently being explored. We believe that our research provides a basis for creating an algorithm and investigating the feasibility of applying these techniques to other matrix functions.

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