Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 6, pp. 1339–1359
DOI : 10.47974/JIM-1783

New fixed point theorems in dynamic property under rational contractive condition with application

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Abstract

In this study, we enhance and generalize multiple findings in the field of fixed point theory within the context of metric spaces. We establish the existence of fixed points for self-maps denoted as $\mathcal{A}$, provided they meet certain rational contractive criteria. Furthermore, we present dynamic insights that establish connections between these fixed points. We provide illustrative examples that serve to validate the proposed hypotheses.

To support our results, we applied our theoretical results in solving a linear congruence problem.

Subject Classification: *(2010) 54H25, 47H10.*

Keywords: *Complete metric space, Fixed-point, Picard sequence, Rational-contraction self-maps.*

1. Introduction

One of the basal contributions in the area of fixed-point theory is Banach's contraction theorem[5]. It applies to a self-map $\mathcal{A}$ on a complete metric space Ω that fulfils the Banach contraction condition, which states that $\sigma(\mathcal{A}\eta, \mathcal{A}\vartheta) \leq k\sigma(\eta, \vartheta)$ for each $\eta, \vartheta \in \Omega,$ for some constant k in [0, 1). Then, the self-map $\mathcal{A}$ has a single fixed point in $\Omega,$ which corresponds to the limit of the iterative sequence $u_n = \mathcal{A}u_{n-1}$ for $n \in \mathbb{N},$ where u_0 is an arbitrary starting point in $\Omega.$ This principle assists in solving various nonlinear problems.

Fixed point theorems for maps are mainly related on the considered spaces that are defined using axioms. The classical one is the metric space introduced in 1906 by the French mathematician Maurice Rene Frechet. It is the most widespread and the richest in terms of the amount of fixed-point theorems published and the most applied in solving nonlinear problems, in which research is still ongoing until now, for example we mention some of these works: [25, 31, 14, 24, 3, 30, 2, 17, 15].

Thanks to this crucial role played by Banach's theorem, researchers raced to improve and generalized this classical result to become more widely applied in several branches of mathematics. This generalization in the usual metric space are made by imposing various contraction conditions on the mappings, we mention some of the initial generalizations for the contractive condition.

[Kannan [22]] It exists a fixed value a, where $0 < a < \frac{1}{2}$, ensuring that for every η and ϑ belonging to $\Omega,$

$$\sigma(\mathcal{A}(\eta), \mathcal{A}(\vartheta)) \leq a[\sigma(\eta, \mathcal{A}(\eta)) + \sigma(\vartheta, \mathcal{A}(\vartheta))]. \tag{1.1}$$

[Bianchini [8]] It exists a fixed value h, where $0 < h < 1$, ensuring that for every η and ϑ belonging to Ω,

$$\sigma(\mathcal{A}(\eta), \mathcal{A}(\vartheta)) \leq h \max\{\sigma(\eta, \mathcal{A}(\eta)), \sigma(\vartheta, \mathcal{A}(\vartheta))\}. \tag{1.2}$$

[Chatterjea [11]] It exists a fixed value a, where $0 < a < \frac{1}{2}$, ensuring that for every η and ϑ belonging to Ω,

$$\sigma(\mathcal{A}(\eta), \mathcal{A}(\vartheta)) \leq a[\sigma(\eta, \mathcal{A}(\vartheta)) + \sigma(\vartheta, \mathcal{A}(\eta))]. \tag{1.3}$$

It exists a fixed value h, where $0 < h < 1$, ensuring that for every η and ϑ belonging to Ω,

$$\sigma(\mathcal{A}(\eta), \mathcal{A}(\vartheta)) \leq h \max\{\sigma(\eta, \mathcal{A}(\vartheta)), \sigma(\vartheta, \mathcal{A}(\eta))\}. \tag{1.4}$$

[Ciric [13]] It exists a fixed value a, where $0 < a < 1$, ensuring that for every η and ϑ belonging to Ω,

$$\sigma(\mathcal{A}(\eta), \mathcal{A}(\vartheta)) < a \max\{\sigma(\eta, \vartheta), \sigma(\eta, \mathcal{A}(\eta)), \sigma(\vartheta, \mathcal{A}(\vartheta)), \sigma(\eta, \mathcal{A}(\vartheta)), \sigma(\vartheta, \mathcal{A}(\eta))\}. \tag{1.5}$$

Dass and Gupta [16] suggested a rational expression extension to the Banach contraction theorem in 1975. In the same direction, Jaggi [20] expanded the approach by proving unique fixed point theorems according to rational contractive conditions.

Another notable result in this field is due to M. S. Khan [23] who formulated a rational contractive condition and proved an existence result of fixed points under this condition. Their work provided a significant extension to previous results and demonstrated the versatility of rational contractive conditions in capturing a wide range of mappings.

In recent years, further refinements and generalizations of fixed-point theorems under rational contractive conditions have been proposed by numerous researchers, to name a few, Jebril et al [21] have proved the common fixed point of mappings satisfying rational inequality. Later on a number of works have appeared in which rational type have been used. Some authors have proved Common fixed point results of mappings satisfying different rational type contraction (see[6, 7]) and references there in.

Consequently, contractive conditions involving rational type expression gave interesting results in fixed point theory. Some well-known results in this direction are involved (see [28, 1, 12, 26, 19, 29, 32, 9, 4, 27, 10, 24, 3]).

Inspired by the generalizations of Banach contraction principle given by Kannan[22], Bianchini[8], Chatterjea[11] and Ciric [14, 13], we give a similar generalization to results of Khojasteh [24] and Aouine[3] in the frame work of a metric space satisfying generalized contractive conditions of rational type. In certain cases, we examine the uniqueness of a fixed point, in the other cases; the absence of fixed-point uniqueness was compensated by giving dynamic information that establishes connections between these fixed points. Appropriate examples are highlighted to support our results. Furthermore,,we apply our theorems to study congruence problems, verify the existence of solutions under specific conditions, and assess their uniqueness.

2. Main results

To begin, we will declare and establish our initial result.

Theorem 2.1: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.1) holds true for each* $\eta,\vartheta \in \Omega$,

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \leq \frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}\max\{\sigma(\eta,\mathcal{A}\vartheta),\sigma(\vartheta,\mathcal{A}\eta)\}, \tag{2.1}$$

Under these conditions, the subsequent statements are valid:

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the mapping $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.2):

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{\varepsilon}{\alpha+\beta}. \tag{2.2}$$

Proof: Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $u_{n+1} = \mathcal{A}u_n$, initiated from a random element u_0 in Ω. If there is an integer n_0 ensuring that $u_{n_0} = u_{n_0+1}$, then u_{n_0} represents a fixed point for the self-map $\mathcal{A}$, this concludes the proof.

However, if $u_n \neq u_{n+1}$ for each natural number n, we proceed as follows:

Claim 1: Let us demonstrate that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case one: Assuming that $\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$, by substituting $\eta = u_{n-1}$ and $\vartheta = u_n$ into inequality (2.1), we obtain

$$\begin{aligned}\sigma(u_n,u_{n+1}) &\le \frac{\alpha\sigma(u_{n-1},u_{n+1})}{\gamma\sigma(u_{n-1},u_n)+\delta\sigma(u_n,u_{n+1})+\varepsilon}\sigma(u_{n-1},u_{n+1});\\ &\le \frac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\gamma\sigma(u_{n-1},u_n)+\delta\sigma(u_n,u_{n+1})+\varepsilon}[\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})];\\ &\le \frac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon}[\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})].\end{aligned}$$

We define $\Lambda_n = \frac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon}$ for each $n \in \mathbb{N}$.

Given that $\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$ then $0 \le \Lambda_n < \frac{1}{2}$ for each $n \in \mathbb{N}$.

Furthermore, we can establish,

$$\sigma(u_n,u_{n+1}) \le \Lambda_n[\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1})],$$

which implies,

$$\sigma(u_n,u_{n+1}) \le \frac{\Lambda_n}{1-\Lambda_n}\sigma(u_{n-1},u_n).$$

We denote $\lambda_n = \frac{\Lambda_n}{1-\Lambda_n}$ for each $n \in \mathbb{N}$.

As $0 \le \Lambda_n < \frac{1}{2}$ for each n in $\mathbb{N}$, it follows that $\lambda_n \in [0,1)$.

This leads to $\sigma(u_n,u_{n+1}) < \sigma(u_{n-1},u_n)$ for each $n \in \mathbb{N}$, which further implies that $\sigma(u_{n+1},u_{n+2}) < \sigma(u_{n-1},u_n)$.

Moreover, the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is decreasing due to the following reasoning for each $n \in \mathbb{N}$,

$$\begin{aligned}\Lambda_{n+1}-\Lambda_n &= \frac{\alpha\varepsilon[\sigma(u_{n+1},u_{n+2})-\sigma(u_{n-1},u_n)]}{[\min\{\gamma;\delta\}(\sigma(u_n,u_{n+1})+\sigma(u_{n+1},u_{n+2}))+\varepsilon]}\\ &\quad\times\frac{1}{[\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon]},\\ &< 0.\end{aligned}$$

Therefore, the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ is also decreasing. Thus,

$$\begin{aligned}\sigma(u_n,u_{n+1}) &\le \lambda_n\sigma(u_{n-1},u_n);\\ &\le \lambda_n\lambda_{n-1}\sigma(u_{n-2},u_{n-1});\\ &\vdots\end{aligned}$$

$$\begin{aligned}&\leq \lambda_n \lambda_{n-1} \cdots \lambda_1 \sigma(u_0, u_1);\\ &\leq \lambda_1^n \sigma(u_0, u_1).\end{aligned}$$

Now, for any pair of natural numbers n and i where $m > n$, we observe,

$$\begin{aligned}\sigma(u_n, u_m) &\leq \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2}) + \cdots + \sigma(u_{m-1,m});\\ &\leq \sum_{k=n}^{m-1} \lambda_1^k \sigma(u_0, u_1);\\ &\leq \frac{\lambda_1^n - \lambda_1^m}{1-\lambda_1} \sigma(u_0, u_1).\end{aligned}$$

By executing the limit as n and m approaches infinity on both sides of the preceding inequality, we obtain,

$$\lim_{n,m\to+\infty} \sigma(u_n, u_m) = 0. \tag{2.3}$$

As a result, $(u_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence within Ω.

Case two: Assuming that $\beta \leq \frac{1}{2}\min\{\gamma, \delta\}$, by substituting $\eta = u_n$ and $\vartheta = u_{n-1}$ into inequality (2.1), we derive,

$$\sigma(u_n, u_{n+1}) \leq \frac{\beta\sigma(u_{n-1}, u_{n+1})}{\gamma\sigma(u_n, u_{n+1}) + \delta\sigma(u_{n-1}, u_n) + \varepsilon} \sigma(u_{n-1}, u_{n+1}).$$

Similar to Case one, we can conclude that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence within Ω.

Due to the completeness of the space (Ω, σ), there exists an element $\overline{\eta}$ in Ω for which,

$$\lim_{n\to+\infty} u_n = \overline{\eta}.$$

Claim 2: Let's confirm that $\overline{\eta}$ is indeed a fixed point for the self-map $\mathcal{A}$.

Case one: Assuming that $\alpha \leq \frac{1}{2}\min\{\gamma, \delta\}$, by substituting $\eta = u_n$, $\vartheta = \overline{\eta}$ into inequality (2.1), we obtain,

$$\sigma(\mathcal{A}\overline{\eta}, u_{n+1}) \leq \frac{\alpha\sigma(u_n, \mathcal{A}\overline{\eta}) + \beta\sigma(\overline{\eta}, u_{n+1})}{\gamma\sigma(u_n, u_{n+1}) + \delta\sigma(\overline{\eta}, \mathcal{A}\overline{\eta}) + \varepsilon} \max\{\sigma(\overline{\eta}, u_{n+1}), \sigma(u_n, \mathcal{A}\overline{\eta})\}. \tag{2.4}$$

By executing the limit as n and m approaches infinity on (2.4), we obtain,

$$\sigma(\mathcal{A}\overline{\eta}, \overline{\eta}) \leq \frac{\alpha\sigma(\overline{\eta}, \mathcal{A}\overline{\eta})}{\delta\sigma(\overline{\eta}, \mathcal{A}\overline{\eta}) + \varepsilon} \sigma(\overline{\eta}, \mathcal{A}\overline{\eta}). \tag{2.5}$$

Hence, $\sigma(\mathcal{A}\overline{\eta},\overline{\eta})=0$, that means $\mathcal{A}\overline{\eta}=\overline{\eta}$. Therefore, $\overline{\eta}$ is a fixed point of $\mathcal{A}$.

Case two: Assuming that $\beta\le\frac{1}{2}\min\{\gamma,\delta\}$, by substituting $\eta=\overline{\eta}$ and $\vartheta=u_n$ into inequality (2.1), we obtain,

$$\sigma(\mathcal{A}\overline{\eta},u_{n+1})\le\frac{\alpha\sigma(\overline{\eta},u_{n+1})+\beta\sigma(u_n,\mathcal{A}\overline{\eta})}{\gamma\sigma(\overline{\eta},\mathcal{A}\overline{\eta})+\delta\sigma(u_n,u_{n+1})+\varepsilon}\max\{\sigma(\overline{\eta},u_{n+1}),\sigma(u_n,\mathcal{A}\overline{\eta})\}.$$

In a manner similar to Case one, we conclude that $\mathcal{A}\overline{\eta}=\overline{\eta}$.

Claim 3: Let's assume that $\mathcal{A}$ has two different fixed points, $\overline{\eta}$ and $\overline{\vartheta}$, in the metric space Ω. Our objective is to determine the distance between them.

By substituting $\eta=\overline{\eta}$, $\vartheta=\overline{\vartheta}$ into inequality (2.1), we obtain,

$$\begin{aligned}\sigma(\overline{\eta},\overline{\vartheta})&\le\frac{\alpha\sigma(\overline{\eta},\overline{\vartheta})+\beta\sigma(\overline{\vartheta},\overline{\eta})}{\gamma\sigma(\overline{\eta},\overline{\eta})+\delta\sigma(\overline{\vartheta},\overline{\vartheta})+\varepsilon}\sigma(\overline{\eta},\overline{\vartheta});\\&\le\frac{(\alpha+\beta)\sigma(\overline{\eta},\overline{\vartheta})}{\varepsilon}\sigma(\overline{\eta},\overline{\vartheta}).\end{aligned}$$

Hence, $\sigma(\overline{\eta},\overline{\vartheta})\ge\frac{\varepsilon}{\alpha+\beta}$. The proof is finished.

Next, we will present and provide the proof for our second result.

Theorem 2.2: *Let (Ω,σ) represent a complete metric space, and consider the self-map $\mathcal{A}$ in Ω. If there are non-negative constants, denoted as $\alpha,\beta,\gamma,\delta$, ε satisfying either $\alpha\le\min\{\gamma,\delta\}$ or $\beta\le\min\{\gamma,\delta\}$ and the inequality shown in equation (2.6) holds true for each $\eta,\vartheta\in\Omega$*

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta)\le\frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}\max\{\sigma(\eta,\vartheta),\sigma(\eta,\mathcal{A}\eta),\sigma(\vartheta,\mathcal{A}\vartheta)\},\qquad(2.6)$$

Under these conditions, the subsequent statements are valid:

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the mapping $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.7)

$$\sigma(\overline{\eta},\overline{\vartheta})\ge\frac{\varepsilon}{\alpha+\beta}\qquad(2.7)$$

Proof: Consider a Picard sequence $\{u_n\}_{n\in\mathbb{N}}$ generated by the recurrence relation $(u_{n+1}=\mathcal{A}u_n)$ initiated from a random element $u_0\in\Omega.$ If there is an integer n_0 ensuring that $u_{n_0}=u_{n_0+1}$ then, u_{n_0} represents a fixed point for the self-map $\mathcal{A},$ this concludes the proof.

However, if $u_n\neq u_{n+1}$ for each natural number n, we proceed as follows:

Claim 1: Let us demonstrate that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case one: Assuming that $\alpha\leq\min\{\gamma,\delta\},$ by substituting $\eta=u_{n-1}$ and $\vartheta=u_n$ in inequality (2.6), we obtain,

$$\begin{aligned}
\sigma(u_n,u_{n+1}) &\leq \tfrac{\alpha\sigma(u_{n-1},u_{n+1})}{\gamma\sigma(u_{n-1},u_n)+\delta\sigma(u_n,u_{n+1})+\varepsilon}\\
&\quad \max\{\sigma(u_{n-1},u_n),\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1})\};\\
&\leq \tfrac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\gamma\sigma(u_{n-1},u_n)+\delta\sigma(u_n,u_{n+1})+\varepsilon}\max\{\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1})\};\\
&\leq \tfrac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon}\\
&\quad \max\{\sigma(u_{n-1},u_n),\sigma(u_n,u_{n+1})\};\\
&\leq \tfrac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon}\sigma(u_{n-1},u_n).
\end{aligned}$$

We define $\Lambda_n=\frac{\alpha\sigma(u_{n-1},u_n)+\alpha\sigma(u_n,u_{n+1})}{\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon}$ for each $n\in\mathbb{N}.$

As $\alpha\leq\min\{\gamma,\delta\},$ then $0\leq\Lambda_n<1$ for each $n\in\mathbb{N}.$

Moreover, the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ is non-increasing, as long as, for each $n\in\mathbb{N},$

$$\begin{aligned}
\Lambda_{n+1}-\Lambda_n &= \tfrac{\alpha\varepsilon[\sigma(u_{n+1},u_{n+2})-\sigma(u_{n-1},u_n)]}{[\min\{\gamma;\delta\}(\sigma(u_n,u_{n+1})+\sigma(u_{n+1},u_{n+2}))+\varepsilon]}\\
&\quad\times\tfrac{1}{[\min\{\gamma;\delta\}(\sigma(u_{n-1},u_n)+\sigma(u_n,u_{n+1}))+\varepsilon]},\\
&<0.
\end{aligned}$$

Besides, we observe,

$$\begin{aligned}
\sigma(u_n,u_{n+1}) &\leq \Lambda_n\sigma(u_{n-1},u_n);\\
&\leq \Lambda_n\Lambda_{n-1}\sigma(u_{n-2},u_{n-1});\\
&\ \ \vdots\\
&\leq \Lambda_n\Lambda_{n-1}\cdots\Lambda_1\sigma(u_0,u_1);\\
&\leq \Lambda_1^n\sigma(u_0,u_1).
\end{aligned}$$

Now, for any pair of natural numbers n and m where $m > n$, we observe,

$$\begin{aligned}\sigma(u_n, u_m) &\leq \sigma(u_n, u_{n+1}) + \sigma(u_{n+1}, u_{n+2}) + \cdots + \sigma(u_{m-1,m});\\ &\leq \sum_{k=n}^{m-1} \Lambda_1^k \sigma(u_0, u_1);\\ &\leq \frac{\Lambda_1^n - \Lambda_1^m}{1-\Lambda_1} \sigma(u_0, u_1).\end{aligned}$$

By executing the limit as n and m approaches infinity on the preceding inequality, we obtain,

$$\lim_{n,m\to+\infty} \sigma(u_n, u_m) = 0. \tag{2.8}$$

As a result, $(u_n)_{n\in\mathbb{N}}$ forms a Cauchy sequence within Ω.

Case two: Assuming that $\beta \leq \min\{\gamma, \delta\}$, by substituting $\eta = u_n$ and $\vartheta = u_{n-1}$ into inequality (2.6), we derive,

$$\begin{aligned}\sigma(u_n, u_{n+1}) \leq{}& \frac{\beta\sigma(u_{n-1}, u_{n+1})}{\gamma\sigma(u_n, u_{n+1}) + \delta\sigma(u_{n-1}, u_n) + \varepsilon}\\ &\max\{\sigma(u_{n-1}, u_n), \sigma(u_{n-1}, u_n), \sigma(u_n, u_{n+1})\};\\ \leq{}& \frac{\beta\sigma(u_{n-1}, u_n) + \beta\sigma(u_n, u_{n+1})}{\gamma\sigma(u_n, u_{n+1}) + \delta\sigma(u_{n-1}, u_n) + \varepsilon} \max\{\sigma(u_{n-1}, u_n), \sigma(u_n, u_{n+1})\};\\ \leq{}& \frac{\beta\sigma(u_{n-1}, u_n) + \beta\sigma(u_n, u_{n+1})}{\min\{\gamma, \delta\}[\sigma(u_n, u_{n+1}) + \sigma(u_{n-1}, u_n)] + \varepsilon}\\ &\max\{\sigma(u_{n-1}, u_n), \sigma(u_n, u_{n+1})\};\\ \leq{}& \frac{\beta\sigma(u_{n-1}, u_n) + \beta\sigma(u_n, u_{n+1})}{\min\{\gamma, \delta\}[\sigma(u_n, u_{n+1}) + \sigma(u_{n-1}, u_n)] + \varepsilon} \sigma(u_{n-1}, u_n).\end{aligned}$$

Similar to Case one, we can conclude that the sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence within Ω.

Due to the completeness of the space (Ω, σ), there exists an element $\overline{\eta}$ in Ω for which,

$$\lim_{n\to+\infty} u_n = \overline{\eta}.$$

Claim 2: Let's confirm that $\overline{\eta}$ is indeed a fixed point for the self-map $\mathcal{A}$.

Case one: Assuming that $a \leq \min\{\gamma, \delta\}$, by substituting $\eta = u_n$, $\vartheta = \overline{\eta}$ into inequality (2.6), we obtain,

$$\sigma(\mathcal{A}\overline{\eta}, u_{n+1}) \leq \frac{\alpha\sigma(u_n, \mathcal{A}\overline{\eta}) + \beta\sigma(u_{n+1}, \overline{\eta})}{\gamma\sigma(u_n, u_{n+1}) + \delta\sigma(\overline{\eta}, \mathcal{A}\overline{\eta}) + \varepsilon} \max\{\sigma(u_n, \overline{\eta}), \sigma(\overline{\eta}, \mathcal{A}\overline{\eta}), \sigma(u_n, u_{n+1})\}. \tag{2.9}$$

By executing the limit as n and m approaches infinity on (2.9), we obtain,

$$\sigma(\mathcal{A}\overline{\eta},\overline{\eta}) \leq \frac{\alpha\sigma(\overline{\eta},\mathcal{A}\overline{\eta})}{\delta\sigma(\overline{\eta},\mathcal{A}\overline{\eta})+\varepsilon}\sigma(\overline{\eta},\mathcal{A}\overline{\eta}). \tag{2.10}$$

Hence, $\sigma(\mathcal{A}\overline{\eta},\overline{\eta}) = 0$ that means $\mathcal{A}\overline{\eta} = \overline{\eta}$. Therefore, $\overline{\eta}$ is a fixed point of $\mathcal{A}$.

Case two: Assuming that $\beta \leq \min\{\gamma,\delta\}$, by substituting $\eta = \overline{\eta}$, $\vartheta = u_n$ into inequality (2.6), we obtain,

$$\sigma(\mathcal{A}\overline{\eta},u_{n+1}) \leq \frac{\alpha\sigma(\overline{\eta},u_{n+1})+\beta\sigma(u_n,\mathcal{A}\overline{\eta})}{\gamma\sigma(\overline{\eta},\mathcal{A}\overline{\eta})+\delta\sigma(u_n,u_{n+1})+\varepsilon}\max\{\sigma(u_n,\overline{\eta}),\sigma(\overline{\eta},\mathcal{A}\overline{\eta}),\sigma(u_n,u_{n+1})\}. \tag{2.11}$$

In a manner similar to Case one, we conclude that $\mathcal{A}\overline{\eta} = \overline{\eta}$.

Claim 3: Let's assume that $\mathcal{A}$ has two different fixed points, $\overline{\eta}$ and $\overline{\vartheta}$, in the metric space Ω. Our objective is to determine the distance between them.

By substituting $\eta = \overline{\eta}$, $\vartheta = \overline{\vartheta}$ into inequality (2.6), we obtain,

$$\sigma(\overline{\eta},\overline{\vartheta}) \leq \frac{\alpha\sigma(\overline{\eta},\overline{\vartheta})+\beta\sigma(\overline{\vartheta},\overline{\eta})}{\gamma\sigma(\overline{\eta},\overline{\eta})+\delta\sigma(\overline{\vartheta},\overline{\vartheta})+\varepsilon}\max\{\sigma(\overline{\eta},\overline{\vartheta}),\sigma(\overline{\eta},\overline{\eta}),\sigma(\overline{\vartheta},\overline{\vartheta})\}. \tag{2.12}$$

Hence, $\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{\varepsilon}{\alpha+\beta}$. The proof is finished.

If we consider all the constants in the above theorem to be equal to 1, i.e $\alpha = \beta = \gamma = \delta = \varepsilon = 1$, we could get the next corollary.

Corollary 2.3: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If the inequality shown in equation (2.13) holds true for each* $\eta,\vartheta \in \Omega$,

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \leq \frac{\sigma(\eta,\mathcal{A}\vartheta)+\sigma(\vartheta,\mathcal{A}\eta)}{\sigma(\eta,\mathcal{A}\eta)+\sigma(\vartheta,\mathcal{A}\vartheta)+1}\max\{\sigma(\eta,\vartheta),\sigma(\eta,\mathcal{A}\eta),\sigma(\vartheta,\mathcal{A}\vartheta)\}, \tag{2.13}$$

Hence,

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.14).

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{1}{2}. \tag{2.14}$$

We can generalize the first theorem as follow:

Theorem 2.4: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \le \frac{1}{2}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.15) holds true for each* $\eta,\vartheta \in \Omega$,

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \le \frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}\max\{\sigma(\eta,\mathcal{A}\vartheta),\sigma(\vartheta,\mathcal{A}\eta),\sigma(\eta,\vartheta)\}, \quad (2.15)$$

Under these conditions, the subsequent statements are valid:

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.16).

$$\sigma(\overline{\eta},\overline{\vartheta}) \ge \frac{\varepsilon}{\alpha+\beta}. \quad (2.16)$$

Proof: The procedure for proving the above theorem is the same as for proving Theorem 2.1.

Now, we give a generalization to the previous theorems.

Theorem 2.5: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \le \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \le \frac{1}{2}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.17) holds true for each* $\eta,\vartheta \in \Omega$,

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \le \frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}\max\{\sigma(\eta,\mathcal{A}\vartheta), \sigma(\vartheta,\mathcal{A}\eta),\sigma(\eta,\mathcal{A}\eta),\sigma(\vartheta,\mathcal{A}\vartheta),\sigma(\eta,\vartheta)\}, \quad (2.17)$$

Under these conditions, the subsequent statements are valid:

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.18).

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \tfrac{\varepsilon}{\alpha+\beta}. \tag{2.18}$$

Proof: The procedure for proving the above theorem is the same as for proving Theorem 2.1.

By ameliorating the previous results, we can give the following theorems.

Theorem 2.6: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.19) holds true for each* $\eta,\vartheta \in \Omega$,

$$\begin{aligned}\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \leq \tfrac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}\max\{&\sigma(\eta,\mathcal{A}\vartheta),\sigma(\vartheta,\mathcal{A}\eta),\\ &\sigma(\eta,\mathcal{A}\eta)+\sigma(\vartheta,\mathcal{A}\vartheta),\sigma(\eta,\vartheta)\},\end{aligned} \tag{2.19}$$

then,

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.20).

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \tfrac{\varepsilon}{\alpha+\beta}. \tag{2.20}$$

Proof: The procedure for proving the above theorem is the same as for proving Theorem 2.1.

Theorem 2.7: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.21) holds true for each* $\eta,\vartheta \in \Omega$,

$$\begin{aligned}\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \leq &\tfrac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}\\ &\max\{\sigma(\eta,\mathcal{A}\vartheta)+\sigma(\vartheta,\mathcal{A}\eta),\sigma(\eta,\mathcal{A}\eta),\sigma(\vartheta,\mathcal{A}\vartheta),\sigma(\eta,\vartheta)\},\end{aligned} \tag{2.21}$$

then,

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;

2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.22).

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{\varepsilon}{2(\alpha+\beta)}. \tag{2.22}$$

Proof: The procedure for proving the above theorem is the same as for proving Theorem 2.1.

Theorem 2.8: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \leq \frac{1}{2}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{2}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.23) holds true for each* $\eta,\vartheta\in\Omega$,

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \leq \frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon} \max\{\sigma(\eta,\mathcal{A}\vartheta)+\sigma(\vartheta,\mathcal{A}\eta),\sigma(\eta,\mathcal{A}\eta)+\sigma(\vartheta,\mathcal{A}\vartheta),\sigma(\eta,\vartheta)\}, \tag{2.23}$$

then,

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.24).

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{\varepsilon}{2(\alpha+\beta)}. \tag{2.24}$$

Proof: The procedure for proving the above theorem is the same as for proving Theorem 2.1.

Theorem 2.9: *Let* (Ω,σ) *represent a complete metric space, and consider the self-map* $\mathcal{A}$ *in* Ω. *If there are non-negative constants, denoted as* $\alpha,\beta,\gamma,\delta$ *and* ε *satisfying either* $\alpha \leq \frac{1}{5}\min\{\gamma,\delta\}$ *or* $\beta \leq \frac{1}{5}\min\{\gamma,\delta\}$ *and the inequality shown in equation (2.25) holds true for each* $\eta,\vartheta\in\Omega$,

$$\sigma(\mathcal{A}\eta,\mathcal{A}\vartheta) \leq \frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}[\sigma(\eta,\mathcal{A}\vartheta)+\sigma(\vartheta,\mathcal{A}\eta)+\sigma(\eta,\mathcal{A}\eta) +\sigma(\vartheta,\mathcal{A}\vartheta)+\sigma(\eta,\vartheta)], \tag{2.25}$$

Under these conditions, the subsequent statements are valid:

1. There is, at a minimum, one fixed point denoted as $\overline{\eta}$ within the space Ω for the self-map $\mathcal{A}$;
2. Each sequence of the form $(\mathcal{A}u_n)_{n\in\mathbb{N}}$ tends to one of the fixed points;
3. If $\mathcal{A}$ has several fixed points, then each pair of them must satisfy the inequality (2.26).

$$\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{\varepsilon}{3(\alpha+\beta)}. \tag{2.26}$$

Proof: The procedure for proving the above theorem is the same as for proving Theorem 2.1.

3. Illustrative examples

Our Theorem 2.1 is supported by the example below.

Example 3.1: Let $\Omega=\{0,1,2\}$ be a set equipped with a metric σ defined as follows: $\sigma(0,1)=0.75$, $\sigma(1,2)=0.25$ and $\sigma(0,2)=1$. Additionally, $\sigma(\eta,\eta)=0$ and $\sigma(\eta,\vartheta) = \sigma(\vartheta,\eta)$ for each $\eta,\vartheta\in\Omega$.

Consider a self-map $\mathcal{A}$ in Ω defined by $\mathcal{A}(0)=\mathcal{A}(2)=2$ and $\mathcal{A}(1)=1$.

With constants $\alpha=\beta=\frac{1}{2}$, $\gamma=\delta=1$ and $\varepsilon=\frac{1}{4}$. It may be readily inferred that the metric space (Ω,σ) is a complete. Over more, the inequality (2.1) has been hold for every $\eta,\vartheta\in\Omega$.

Theorem 2.1 leads us to the conclusion that $\mathcal{A}$ has a minimum of one fixed point. (precisely, $\mathcal{A}$ possesses two distinct fixed points 2 and 1). Furthermore, it can be observed that the distance separating each other, denoted as $\sigma(2,1)$, is greater than or equal to one-fourth.

The following example illustrate Theorem 2.2 and Corollary 2.3.

Example 3.2: Let $\Omega=\{m,p,q\}$ be a set of three different elements equipped with a metric σ in the following way $\sigma(p,q)=0.75$, $\sigma(q,m)=0.25$ and $\sigma(p,m)=1$. Additionally, the following standard equations hold for each $\eta,\vartheta\in\Omega$, $\sigma(\eta,\vartheta) = \sigma(\vartheta,\eta)$ and $\sigma(\eta,\eta)=0$.

Consider $\mathcal{A}$ represent a self-map in the set Ω. The self-map $\mathcal{A}$ is given by the following equations: $\mathcal{A}(m)=m$, $\mathcal{A}(p)=m$ and $\mathcal{A}(q)=q$.

It may be readily inferred that the metric space (Ω,σ) is a complete. Over more, the inequality (2.13) has been hold for every $\eta,\vartheta\in\Omega$.

Corollary 2.3 leads us to the conclusion that $\mathcal{A}$ has a minimum of one fixed point. (precisely, $\mathcal{A}$ possesses two distinct fixed points m and q).

Furthermore, it can be observed that the distance separating each other, is greater than or equal to one-half, i.e. $\sigma(m,q) \geq \frac{1}{2}$.

The following example confirm the validity of the Theorem 2.4.

Example 3.3: Consider the set $\Omega = \{m,p,q\}$ equipped with the metric σ by the following values: $\sigma(p,q) = 2$, $\sigma(m,p) = \sigma(q,m) = 1$. Furthermore, it holds that $\sigma(\eta,\vartheta) = \sigma(\vartheta,\eta)$ for every pair of elements η and ϑ belonging to the set Ω. Additionally, we have $\sigma(\eta,\eta) = 0$ for each elements η in the set Ω.

Consider a self-map $\mathcal{A}$ in the set Ω, where $\mathcal{A}(p) = m$, $\mathcal{A}(q) = q$, and $\mathcal{A}(m) = m$.

It can be immediately deduced that (Ω,σ) forms a complete metric space, and the inequality (2.15) has been established for every $\eta,\vartheta \in \Omega$ with the constants $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = 1$, and $\varepsilon = 1$.

From Theorem 2.4, we deduced that the self-map $\mathcal{A}$ possesses no less than one fixed point. Indeed, the given function possesses two distinct fixed points, q and m. Furthermore, it can be observed that the distance separating each other exceeds or equals 1, i.e., $\sigma(m,q) \geq 1$.

The following example confirm the validity of the Theorem 2.5.

Example 3.4: Consider a collection of three elements, $\Omega = \{m,p,q\}$, with a metric σ. The values of $\sigma(q,p)$ and $\sigma(q,m)$ are 2, 1, and 1 respectively. Furthermore, it holds that $\sigma(\eta,\vartheta) = \sigma(\vartheta,\eta)$ for every pair of elements η and ϑ belonging to the set Ω. Additionally, we have $\sigma(\eta,\eta) = 0$ for each elements η in the set Ω.

Consider a self-map $\mathcal{A}$ in the set Ω, where $\mathcal{A}(p) = p$, $\mathcal{A}(m) = p$ and $\mathcal{A}(q) = q$.

It can be immediately deduced that (Ω,σ) forms a complete metric space, and the inequality (2.17) has been established for every $\eta,\vartheta \in \Omega$ with the constants $\alpha = \beta = \varepsilon = \frac{3}{2}$ and $\gamma = \delta = 3$.

From Theorem 2.5, we deduced that the self-map $\mathcal{A}$ possesses no less than one fixed point. Indeed, the given function possesses two distinct fixed points q and p. Furthermore, it can be observed that the distance separating each other, is greater than or equal to $\frac{1}{2}$, i.e. $\sigma(p,q) \geq \frac{1}{2}$.

This example illustrate the validity of Theorem 2.9.

Example 3.5: Let $\Omega = \{m,p,q\}$ be a set of three different elements equipped with a metric σ in the following way $\sigma(p,q) = 2$, $\sigma(p,m) = \sigma(q,m) = 1$. Furthermore, it holds that $\sigma(\eta,\vartheta) = \sigma(\vartheta,\eta)$ for every pair of elements η

and ϑ belonging to the set Ω. Additionally, we have $\sigma(\eta,\eta)=0$ for each elements η in the set Ω.

Consider a self-map $\mathcal{A}$ in the set Ω, where $\mathcal{A}(p)=p$, $\mathcal{A}(q)=q$ and $\mathcal{A}(m)=p$.

It can be immediately deduced that (Ω,σ) forms a complete metric space, and the inequality (2.25) has been established for every $\eta,\vartheta\in\Omega$ with the constants $\alpha=\beta=\frac{1}{5}$, $\gamma=\delta=1$ and $\varepsilon=\frac{1}{2}$.

From Theorem 2.9, we deduced that the self-map $\mathcal{A}$ possesses no less than one fixed point. Indeed, the given function possesses two distinct fixed points q and p. Furthermore, it can be observed that the distance separating each other, is greater than or equal to $\frac{5}{12}$, i.e. $\sigma(p,q)\geq\frac{5}{12}$.

Notes

- The Corollary 2.3 enhances the outcomes regarding A. C. Aouine and A. Aliouche in [3] and F. Khojasteh et al[24]. Additionally, we notice the disappearance of the fixed point uniqueness after adding the term $\sigma(\eta,\vartheta)$ to the left-hand side of the inequality (2.13).
- It is important to acknowledge that selecting of constants associated to the inequalities (2.1), (2.6), (2.15), (2.17), (2.19), (2.21), (2.23) and (2.25) has a direct impact on the dynamic outcome of Theorems 2.1, 2.2, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9 respectively, example 4.2 illustrates this point effectively.
- It is important to note that, contrary to Banach's theorem, the ratio $\frac{\alpha\sigma(\eta,\mathcal{A}\vartheta)+\beta\sigma(\vartheta,\mathcal{A}\eta)}{\gamma\sigma(\eta,\mathcal{A}\eta)+\delta\sigma(\vartheta,\mathcal{A}\vartheta)+\varepsilon}$ in the inequalities (2.1), (2.6), (2.15), (2.17), (2.19), (2.21), (2.23) and (2.25) may exceed or be less than 1, Example 3.4 shows this point very clearly.
- If the range of the self-map $\mathcal{A}$ is a closed subset of Ω, then the inequalities (2.1), (2.6), (2.15), (2.17), (2.19), (2.21), (2.23), and (2.25) can be restrained to the range of $\mathcal{A}$. This restriction has no effect on the supporting proof or interesting outcomes. However, it allows the verification of its validity and enhances its applicability.
- The findings stated above may be extended to many generalized metric spaces, including partial and b-metric spaces.

4. Application to solve equations in $\mathbb{Z}/n\mathbb{Z}$

In this section, n is a natural number greater than or equal to 2.

In this part, we will discuss solving equations in $\mathbb{Z}/n\mathbb{Z}$. In our study, we will be interested in the existence of solutions and the distances between them without indicate their values.

The problem posed is that $\mathbb{Z}/n\mathbb{Z}$ is not a vector space but we can consider it as a metric space then we apply our theorem.

We begin by present our problem in the following form:

$$a\eta \equiv b[n] \tag{4.1}$$

such that $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, with the unknown $\eta \in \mathbb{Z}$.

Now, we reformulate the previous equation to make it applicable to our theorem.

$$\overline{a}\overline{\eta} = \overline{b} \tag{4.2}$$

such that $\overline{a}, \overline{b} \in \mathbb{Z}/n\mathbb{Z}$, with the unknown $\overline{\eta} \in \mathbb{Z}/n\mathbb{Z}$.

We know that the equations (4.1) and (4.2) are equivalent, so it is enough to study the second one.

We denote $\mathcal{A}$ the self-map

$$\begin{array}{rcl} \mathcal{A}:\mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Z}/n\mathbb{Z} \\ \eta & \mapsto & \mathcal{A}(\overline{\eta}) = (\overline{a+1})\overline{\eta} - \overline{b} \end{array}$$

Theorem 4.1: *Let $(\mathbb{Z}/n\mathbb{Z}, \sigma)$ be a metric space. If the following conditions are satisfies,*

1. $rang(\mathcal{A})$ is a closed set of $\mathbb{Z}/n\mathbb{Z}$,
2. If there are non-negative constants, denoted as $\alpha, \beta, \gamma, \delta$ and ε satisfying either

$\alpha \leq \frac{1}{2}\min\{\gamma, \delta\}$ or $\beta \leq \frac{1}{2}\min\{\gamma, \delta\}$ and the inequality shown in equation (4.3) holds true for each $\eta, \vartheta \in rang\mathcal{A}$

$$\sigma(\mathcal{A}\eta, \mathcal{A}\vartheta) \leq \frac{\alpha\sigma(\eta, \mathcal{A}\vartheta) + \beta\sigma(\vartheta, \mathcal{A}\eta)}{\gamma\sigma(\eta, \mathcal{A}\eta) + \delta\sigma(\vartheta, \mathcal{A}\vartheta) + \varepsilon}\max\{\sigma(\eta, \mathcal{A}\vartheta), \sigma(\vartheta, \mathcal{A}\eta)\}. \tag{4.3}$$

then, the equation (4.2) admit at least one solution. Over more, if the equation admit more than one solution we can give a dynamic information about solutions like equation (2.2).

Proof: Consider the metric space $\Omega = (\mathbb{Z}/n\mathbb{Z}, \sigma)$. It can be deduced that the space $(\mathbb{Z}/n\mathbb{Z}, \sigma)$ is complete, as the range of the self-map $\mathcal{A}$ is a closed subset of Ω.

Depending on the previous result and the equation (4.3) and by applying Theorem 2.1, we conclude that, No less than one fixed point $\overline{\eta} \in \Omega$ exists for $\mathcal{A}$. Since the fixed points of the function $\mathcal{A}$ are solutions of equation (4.2), we can get the results mentioned above.

Now, we present the following examples as confirmation of the applicability of our theorem to solve algebraic equations in $\mathbb{Z}$.

Example 4.2: Let $\mathbb{Z}/6\mathbb{Z}$ be the set of equivalence classes for congruence modulo 6 associated with the metric distance σ such as $\sigma(\overline{\eta},\overline{\vartheta}) = |\eta - \vartheta|$ were η,ϑ are refer to the minimum representation of the classes $\overline{\eta},\overline{\vartheta}$ respectively. It is immediately deduced that the metric space $(\mathbb{Z}/6\mathbb{Z},\sigma)$ is complete.

Let the following linear equation in $\mathbb{Z}/6\mathbb{Z}$ be

$$\overline{3}\overline{\eta} = \overline{3}. \tag{4.4}$$

Let $\mathcal{A}$ be the associated self-map to this equation,

$$\begin{array}{rcl} \mathcal{A} : \mathbb{Z}/n\mathbb{Z} & \to & \mathbb{Z}/n\mathbb{Z} \\ \eta & \mapsto & \mathcal{A}(\overline{\eta}) = \overline{4}\overline{\eta} + \overline{3}. \end{array}$$

$rang\mathcal{A} = \{\overline{1},\overline{3},\overline{5}\}$ is a closed subset of $\mathbb{Z}/6\mathbb{Z}$.

By choosing $\alpha = \beta = 1$, $\gamma = \delta = 2$ and $\varepsilon = 3$, the equation (4.3) is satisfied for each $\eta,\vartheta \in rang\mathcal{A}$, by applying Theorem 4.1, the equation (4.4) admit at least one solution.

We can observe that this equation has exactly 3 solutions $S = \{\overline{1},\overline{3},\overline{5}\}$, we can know some information about the solutions without knowing their number or even their values as follows, for each distinct pair solutions, we have $\sigma(\overline{\eta},\overline{\vartheta}) \geq \frac{3}{2}$.

In order to improve the previous result, we adjust the constants $\alpha,\beta,\gamma,\delta,$ and ε to the following values: $\alpha = \beta = \frac{1}{2}$, $\varepsilon = \delta = 2$, and $\gamma = 1$. This modification keeps the validity of equation (4.3) and the dynamic information has been upgraded to: for each distinct pair solutions, we have $\sigma(\overline{\eta},\overline{\vartheta}) \geq 2$.

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Received January, 2023