Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 6, pp. 1257–1271
DOI : 10.47974/JIM-1748

Dynamics analysis of the modified Beverton-Holt model

Chabane Bedjguelel *
Hacene Gharout †
Bakir Farhi §
Université de Bejaia
Faculté des Sciences Exactes
Laboratoire des Mathématiques Appliquées
06000 Bejaia
Algérie

Abstract

This work focuses on a novel Beverton-Holt population model generalized by four parameters. The essential conditions for the existence and stability of equilibrium points, also known as fixed points, have been determined. Certain biological phenomena are observed in this model through the existence of the Allee effect and the regions of extinction. To demonstrate the theoretical results, numerical simulations are performed.

Subject Classification: *(2010) Primary 39A30, 37C25, 39A28.*

Keywords: *Extinction, Stability , Fixed point , Allee effect, Beverton-Holt model.*

1. Introduction

In this article, we will be interested in a new Beverton-Holt model with four real parameters (1), obtained by generalizing the Beverton-Holt model, a model used in population dynamics [3]. The model that will be the subject of our work is:

$$x_{n+1} = f(x_n), \text{ with } f(x) = \frac{rx^{\gamma}}{1+\delta x^{m}}, \tag{1.1}$$

where r, γ, δ, m are positive real parameters, and f is called the production function or the stock-recruitment relationship. The model defined by

* *E-mail:* **chabane.bedjguelel@univ-bejaia.dz (Corresponding Author)**
† *E-mail:* **hacene.gharout@univ-bejaia.dz**
§ *E-mail:* **bakir.farhi@univ-bejaia.dz**

© Taru Publications

Equation (1) is also a generalization of the Beverton-Holt equation studied by E. Liz [8], and the model proposed by Maynard Smith and Slatki [10, 15]. Moreover, given the importance of this model in the evolution of populations, many researchers are interested in this type of recurrence or unimodal functions, citing, for example, the works of J.L. Rocha [11, 12] in the γ -Ricker model and the works of S. Schreiber [13, 14].

Recently, many authors in the field of dynamical systems have been interested in population models incorporating the Allee effect, using various approaches and applications. For example, see J. A. Walter[16] and S. Elaydi et al.[6] which use higher dimensional triangular maps to explore multi-species hierarchical models and three-species Ricker competition moels. F, Bozkurt et al. [4] established a fractional logistic model of the cancer growth. The Allee effect is so named by Warder-C-Allee [1], popularized by F. Courchamp's work [5], and refers to the reduced capacity or decline in the growth of a population due to low population size or density. In other words, when population densities are low, it is difficult to find mates and birth rates decline, which can lead to population extinction. This phenomenon occurs in the models when a specific threshold is reached, below which the population becomes extinct. This threshold is called the "Allee fixed point". In the dynamical systems literature, a function representing the Allee effect must have three equilibrium or fixed points: a trivial fixed point that is asymptotically stable, an unstable positive fixed point called a threshold point, and a larger positive point that is asymptotically stable [2].

The principal objective of this study is to investigate the dynamic properties of this specific model with four real parameters. We observe that the model incorporates the Allee effect for certain parameter values. In this work, we will restrict our study to the parameter set:

$$R_0 = \{(r,\gamma,\delta,m) \in \mathbf{R}^4 : r,\alpha,\delta,m > 0 \text{ and } m > \gamma\},$$

with $x \in [0,+\infty[$, and the condition $m > \gamma$ being a unimodality condition for the function *f*. The new model's production function *f* satisfies the properties $(A_1)-(A_5)$, which are specified in the following properties:

Property: Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be the function given by the equation (1). The following is verified:

- (A_1) f is continuous $[0,+\infty[$ and $f(0) = \lim_{x\to+\infty} f(x) = 0.$
- (A_2) f has one critical point $c = \sqrt[m]{\frac{\gamma}{\delta(m-\gamma)}}$ which represents a global maximum of *f*.

(A_3) $f'(x) \neq 0, \forall x \in [0,+\infty[\setminus\{c\}, \quad f'(c) = 0,$ and $f''(c) < 0.$

(A_4) f is of class $C^3(]0,+\infty[\setminus\{c\})$.

(A_5) The Schwarz derivative of f is negative for all $x \in [0,+\infty[\setminus\{c\}$.

2. Bifurcation of the function *f*

In this section, we determine the equations of the fold and flip bifurcation curves, which help us study the stability of the proposed model in the parameter set R_0. We consider r to be the parameter of bifurcation in the plan of parameters (γ, δ, m), which means that we write r as a function of (γ, δ, m).

Recall that fold bifurcation occurs when two fixed points or cycles of order k appear, with one stable and the second unstable, and verify:

$$\frac{\partial f(x_i)}{\partial x} = 1 \text{ for all } i = 1, 2...k.$$

The flip bifurcation is a change in the stability of the fixed point or cycle of order k and the appearance of a fixed point or cycle of order $2k$, verifying

$$\frac{\partial f(x_i)}{\partial x} = -1 \text{ for all } i = 1, 2...k.$$

A cycle of order one is a fixed point.

2.1 *Fold bifurcation*

The equation fold bifurcation, associated with the function f is obtained using this system:

$$\begin{cases} f(x) = x, \\ \frac{\partial f(x)}{\partial x} = 1. \end{cases} \tag{2.1}$$

This is equivalent to

$$\begin{cases} \frac{rx^{\gamma}}{1+\delta x^m} = x, \\ \frac{rx^{\gamma-1}(\gamma+\delta x^m(\gamma-m))}{(1+\delta x^m)^2} = 1. \end{cases} \tag{2.2}$$

It is therefore possible to find a Φ solution where the equation of the curve obtained at the bifurcation of the fold is:

$$r = \Phi(x) = \frac{m}{m-\gamma+1}\left(\frac{\gamma-1}{\delta(m+1-\gamma)}\right)^{\frac{1-\gamma}{m}}, \text{ with } x = \sqrt[m]{\frac{\gamma-1}{\delta(m+1-\gamma)}}. \tag{2.3}$$

2.2 *Flip bifurcation*

The flip bifurcation is obtained using:

$$\begin{cases} f(x) = x, \\ \frac{\partial f(x)}{\partial x} = -1. \end{cases} \tag{2.4}$$

This is identically equivalent to

$$\begin{cases} \frac{rx^{\gamma}}{1+\delta x^{m}} = x, \\ \frac{rx^{\gamma-1}(\gamma+\delta x^{m}(\gamma-m))}{(1+\delta x^{m})^{2}} = -1. \end{cases} \tag{2.5}$$

So there exists a solution Ψ such that the obtained curve equation of the flip bifurcation is:

$$r = \Psi(x) = \frac{m}{m-\gamma-1}\left(\frac{1+\gamma}{\delta(m-\gamma-1)}\right)^{\frac{1-\gamma}{m}}, \tag{2.6}$$

with $x = \sqrt[m]{\frac{\gamma+1}{\delta(m-\gamma-1)}}$ and $m > \gamma + 1$.

3. Fixed point and stability

We recall that if a fixed point x_p satisfies: $|\frac{\partial f}{\partial x}(x_p)| < 1$, then it is stable and if it satisfies

$|\frac{\partial f}{\partial x}(x_p)| > 1$, then it is unstable. The fixed point x_p, is said to be expanding if *f* is continuous at x_p and there exist an *U* neighborhood of x_p, a natural number n_0 such that, $f^{n_0}(U) \not\subset U$ and a local inverse f^{-1} satisfies

$$\bigcap_{n\geq 0} f^{-n}(U) = \{x_p\}.$$

Proposition 3.1: Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be the function given by the equation (1) and $\Psi(x)$ is the flip bifurcation. If $0 < \gamma < 1$ and $m > \gamma + 1$, then *f* has two fixed points, the trivial fixed point and x_1, such that $x = 0$ is unstable and x_1 is stable if only if $r < \Psi(x)$.

Proof: Let $x \geq 0$, the fixed point equation $f(x) = x$, corresponds to:

$$\frac{rx^{\gamma}}{1+\delta x^{m}} = x.$$

Obviously, in all cases, $x = 0$ is a fixed point of f for all $(r, \gamma, \delta, m) \in R_0$. For $x > 0$, the fixed point equation becomes $\frac{rx^{\gamma-1}}{1+\delta x^{m}} = 1$. Let $h(x) = rx^{\gamma-1}$ and $g(x) = 1 + \delta x^{m}$, let's look at the solutions to the equation

$$h(x) = g(x). \tag{3.1}$$

We have h is decreasing, $\lim_{x\to 0} h(x) = +\infty$ and $\lim_{x\to+\infty} h(x) = 0$. Similarly, g is increasing,

$\lim_{x\to+\infty} g(x) = +\infty$ and admits 1 as a minimal value at the point $x = 0$. Applying the Theorem of intermediate value, we can deduce that equation (3.1) has only one solution noted by x_1, which represents the fixed point of f.

The derivative of f verifies:

$$\begin{cases} \lim_{x\to 0} f'(x) = \infty, \text{ if } 0 < \gamma < 1, \\ \lim_{x\to 0} f'(x) = r, \text{ if } \gamma = 1, \\ \lim_{x\to 0} f'(x) = 0, \text{ if } \gamma > 1. \end{cases} \tag{3.2}$$

Hence $x = 0$ is unstable. Let us now study the stability of x_1. For $0 < \gamma < 1$, we have f is strictly decreasing in $]c, +\infty[$. In addition, we have $x_1 \in]c, +\infty[$, then it follows that $f'(x_1) < 0$. Thus we have $f'(x_1) < 1$. To show that x_1 is stable, it is sufficient to solve the following system

$$\begin{cases} f(x_1) = x_1, \\ f'(x_1) > -1. \end{cases} \tag{3.3}$$

This is identically equivalent to

$$\begin{cases} r = x_1^{1-\gamma}(1 + \delta x_1^{m}), \\ x_1 < \sqrt[m]{\frac{\gamma+1}{\delta(m-\gamma-1)}}. \end{cases} \tag{3.4}$$

As the function $\Psi(x) = x^{1-\gamma}(1 + \delta x^{m})$ is increasing for $\gamma < 1$, then $\Psi(x_1) < \Psi\left(\sqrt[m]{\frac{\gamma+1}{\delta(m-\gamma-1)}}\right)$, i.e., $r < \Psi\left(\sqrt[m]{\frac{\gamma+1}{\delta(m-\gamma-1)}}\right) = \Psi(x, \gamma, \delta, m)$.

Proposition 3.2: Lets $\gamma > 1$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ the function defined by Equation (1) and $\Phi(x)$ the fold bifurcation curve then:

(i) If $r < \Phi(x)$, f has one single fixed point $x = 0$.

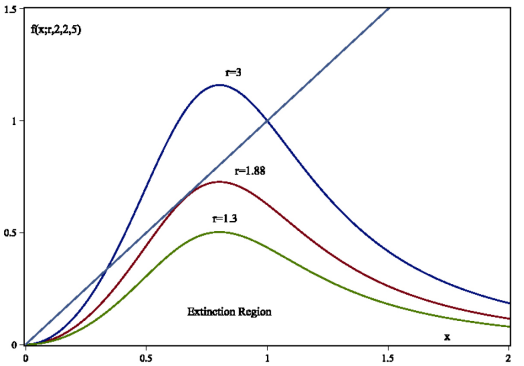


Figure 1

The graph of f when $\gamma = 2 > 1$ and $\delta = 2, m = 5$ the color green corresponding to f with one fixed point (Extinction region), the color red corresponding to f with two fixed points, and the color blue corresponding to f with three fixed points.

(ii) If $r = \Phi(x)$, f must have two fixed points $x = 0$ and $x = x_1$.

(iii) If $r > \Phi(x)$, f has precisely three fixed points $x = 0$, x_1 and x_2, where $0 < x_1 < x_2$.

Proof: Let $f_1(x) = \frac{rx^{\gamma-1}}{1+\delta x^m}$, f_1 is differentiable and has only one critical point. $c = \sqrt[m]{\frac{\gamma-1}{\delta(m-\gamma+1)}}$. Consider the monotonicity of f_1, as it is strictly increasing on $[0, c[$ and strictly decreasing on $]c, +\infty[$, $\lim_{x\to 0} f_1(x) = 0$, $\lim_{x\to+\infty} f_1(x) = 0$, then the critical point c represents a maximum of f_1. Therefore, if $f_1(c) < 1$ this condition is equivalent to $r < \phi(x)$, the function f_1 cannot take the value 1 which is equivalent to saying that the Equation (3.1) cannot have a positive solution. Consequently, the only fixed point of f is the origin.

For $r = \Phi(x)$ this condition is equivalent to $f_1(c) = 1$, in this case, f_1 passes through the value 1 only once. Consequently, the function f admits two fixed points: $x = 0$ and $x = x_1$. A similar proof applies to point *iii*.

Proposition 3.3: Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be the function given by the equation (1), $\Phi(x)$, $\Psi(x)$ the fold and the flip bifurcations, respectively, and x_1, x_2 the fixed point defined in Proposition 3.2 then:

(I) For $\gamma > 1$ and $r < \Phi(x)$, $x = 0$ is stable, and this condition is satisfied:

$$\lim_{n\to+\infty} f^n(x) = 0 \text{ for all } x \in \mathbb{R}_+.$$

(II) If $\gamma > 1$ and $r = \Phi(x)$, $x = 0$ is stable and it is satisfied that

$$\lim_{n\to+\infty} f^n(x) = 0 \text{ for all } x \in \mathbb{R}_+ \setminus [x_1, x_1^*],$$

and

$$\lim_{n\to+\infty} f^n(x) = x_1 \text{ for all } x \in [x_1, x_1^*], \text{ where } x_1^* = \max\{f^{-1}(x_1)\}.$$

(III)If $\gamma > 1$ and $m > \gamma + 1$, $x = 0$ is stable, x_1 is unstable, and x_2 is stable if and only if:

$$\Phi(x) < r < \Psi(x).$$

Proof: We suppose that $\gamma > 1$ and $r < \Phi(x)$, according to Proposition 3.2, the function f admits a single fixed point (the trivial fixed point $x = 0$) and from Equation (3.2), then $x = 0$ is stable. In this case, we have $f(x) < x$ for all $x \in \mathbb{R}_+$. The sequence $f^n(x)$ is monotone decreasing and delimited below by 0. Through the continuity of f, we get that the sequence $f^n(x)$ converges at the single fixed point $x = 0$ which will present an attractor. Therefore, we conclude that

$$\lim_{n\to+\infty} f^n(x) = 0 \text{ for all } x \in \mathbb{R}_+ \text{(see Figure 1)}$$

If $\gamma > 1$ and $r = \Phi(x)$, according to Proposition 3.2, f admits two fixed points $x = 0$ and x_1. From Equation (3.2), $x = 0$ is stable and taking into account the properties of the function f exactly (A_3) and (A_4), we conclude the following inequality:

$$0 < x_1 < x_2 < x_1^*, f'(x) > 0 \text{ for all } x \in [0, x_1[\text{ and } f'(x) < 0 \text{ for all } x \in [x_1^*, +\infty[.$$

i.e, $f(x) > x_1$ for all $x \in [x_1, x_1^*]$. Hence, $f^n(x)$ $f^n(x)$ represents a decreasing sequence which converges towards the attractor $x = 0$ for any $x \in [0, +\infty[\setminus[x_1, x_1^*]$. Consequently,

$$\lim_{n\to+\infty} f^n(x) = 0 \text{ for all } x \in \mathbb{R}_+ \setminus [x_1, x_1^*].$$

Let $x \in [x_1, x_1^*]$, from the monotonicity and the unimodality of f, we have :

$$x_1 < c < x_1^* \text{ and } f'(x) > 0 \text{ for all } x \in [x_1, c[\text{ and } f'(x) < 0 \text{ for all } x \in [c, x_1^*[$$

In addition,

$$x_1 < f(x) < f(c) \text{ for all } x \in [x_1, x_1^*].$$

Consequently, the range $[x_1, x_1^*]$ has just one fixed point, which is x_1. As $f(x) < x$ for all $x \in [x_1, x_1^*]$, then we deduce that $f^n(x)$, is a monotone decreasing sequence and bounded below by x_1. Continuity of the function f implies that the sequence $f^n(x)$ converge to the only fixed point $x = x_1$. Hence,

$$\lim_{n \to +\infty} f^n(x) = x_1 \text{ for all } x \in [x_1, x_1^*].$$

We suppose that $\gamma > 1$ and $\Phi(x) < r < \Psi(x)$, from the Proposition 3.2, we have f admits three fixed points $x = 0$, x_1 and x_2 such as $0 < x_1 < x_2$. Let us now consider the result obtained in (2), i.e., after the fold bifurcation curve and before the bifurcation of flip. The fixed point x_1 bifurcates and gives rise to two fixed points x_1 and x_2, so that $x_1 < c < x_2 < x_1^*$, with $x_1^* = \max f^{-1}(x_1)$. Otherwise, for $r < \Phi(x)$ and from Proposition 3.2, the function f admits one fixed point which contradicts the existence of three fixed points. However, if $r > \Psi(x)$, taking into account the equation (2.6), we notice the appearance of a new bifurcation, the flip bifurcation, in which the fixed point x_2 goes unstable and gives rise to a new attractive cycle of period 2. This new attractive cycle contradicts the hypothesis.

Remark 3.1:

(i) For $\gamma > 1$, we have established that for $r < \Phi(x)$, there is a single fixed point $x = 0$ for the function f, which represents a global attractor. In this case, it represents extinction region (see Figure 1). If $r = \Phi(x)$, the function f admits two fixed points, the trivial fixed point and x_1. According to these particular conditions, the trivial fixed point is stable then $\lim_{n \to +\infty} f^n(x) = x_1$ for everyone $x \in [x_1, x_1^*]$, the fixed point x_1 is attractive on the right, but repulsive on the left (see Figure 1) and is not expanding. If $r > \Phi(x)$, the function f admits three fixed points $x = 0$, the Allee fixed point x_1 and x_2 such that $x = 0$ is stable, x_1 is unstable and the stability of the fixed point x_2 will depend from the existence of the flip bifurcation. Note in this case that x_1 becomes expanding.

(ii) The case where $m = 1$ has been studied in [8].

(iii) When $\gamma = m$, the function f is strictly increasing but is not unimodal, and it does not verify the properties $(A_1) - (A_5)$, it can have at most three fixed points depending on the values of the parameter γ. Indeed, a simple study of the variations of f shows that the function f admits exactly two fixed points for $\gamma < 1$ and admits at least one

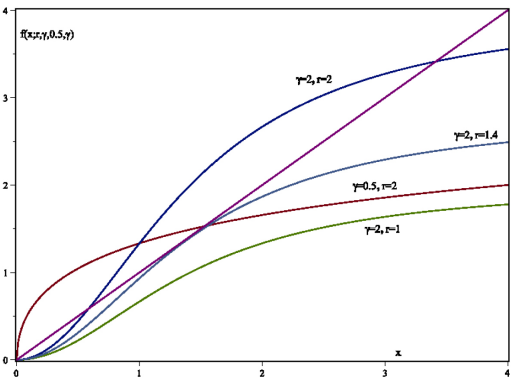


Figure 2

The graph of f when $\gamma = m$, $\delta = 0.5$ with various values of the parameters r and γ.

fixed point the trivial fixed point and no more than three fixed points $\gamma > 1$ (see Figure 2).

4. Allee Effect

This section concentrates on the Allee effect, which occurs when the parameter γ exceeds the critical value 1 and $r > \Phi(x)$. The graph of the function f in this case resembles a camel's hump (see Figure 3).

Definition 4.1: [12] The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the equation (1), is an Allee function if the parameters $(r, \gamma, \delta, m) \in R_0$ with $\gamma > 1$ and $r > \Phi(r)$, and real numbers x_1 and x_2, given in Proposition ??, such that:

(a) $0 < x_1 < c < x_2 < +\infty$, where $c = \sqrt[m]{\frac{\gamma}{\delta(m-\gamma)}}$ is the only critical point of f.

(b) For all $x \in \mathbb{R}_+ \setminus [x_1, x_2]$ we have $f(x) < x$.

(c) For all $x \in]x_1, x_2[$ we have $f(x) > x$.

Definition 4.2: [13] Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be the function given by the equation (1), in the parameter set R_0, with Allee effect. We define the chaotic semi-stability surface S_{ch}, by:

$$S_{ch} = \{(r, \gamma, \delta, m) \in R_0 : f^2(c) = x_1, \gamma > 1, \Phi(x) < r < \Psi(x)\}. \quad (4.1)$$

This is equivalent to

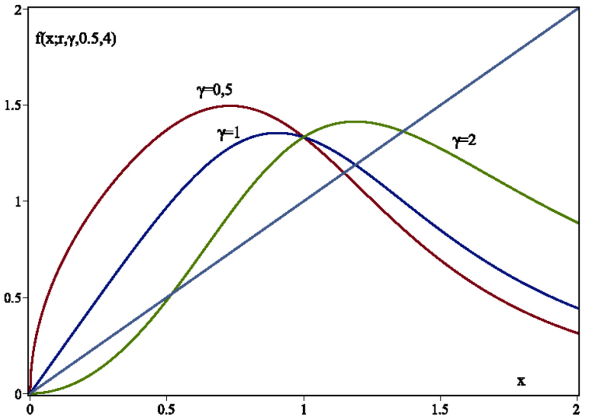


Figure 3

The graph of f when $r=2,\ \delta=0.5$ and $m=4$ the color green corresponds at f with Allee effect $\gamma=2>1,$ the color blue and reed corresponds at f without Allee effect.

$$S_{ch}=\{(r,\gamma,\delta,m)\in R_0 : f(c)=x_1^*, \gamma>1, \Phi(x)<r<\Psi(x)\},$$

with x_1 being Allee's fixed point, $c=\sqrt[m]{\frac{\gamma}{\delta(m-\gamma)}}$ is the critical point of f, and $\Phi(x)$ is the fold bifurcation.

4.1 *Snap back repeller bifurcation*

This subsection presents the relationship between the Allee fixed point and the homoclinic orbit. We recall that a trajectory, or homoclinic orbit, of a fixed point is that tending towards this point in the forward process and in certain reverse processes. When the homoclinic orbit is associated with the expanding fixed point, the fixed point is called a snap-back repeller (SBR) [7].

Definition 4.3: [7] An expanding fixed point of $f: X\to X$ denoted as x_p becomes a snap-back repeller (SBR) if there exists a point $x_0\in U\setminus\{x_p\}$, where $f^m(x_0)=x_p$ for a suitable integer m.

Proposition 4.1: Consider $f:\mathbb{R}_+\to\mathbb{R}_+$ to be the function given by the equation (1) with Allee effect. The Allee fixed point x_1 satisfies the Equation (4.1) if and only if x_1 is a SBR of f.

Proof: We suppose that x_1 verifies the Equation (4.1), i.e., $f^2(c)=x_1$ for all $\gamma>1$ and $r>\Phi(x)$. Let c the critical point defined in Definition 4, a

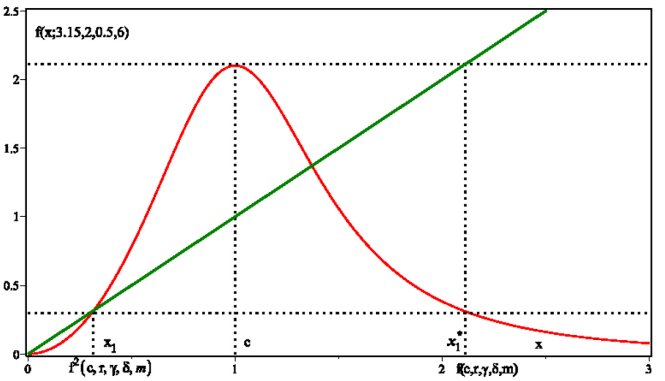


Figure 4

The graph of f when the first critical homoclinic orbit occurs at the Allee fixed point, i.e., $f^2(c) = x_1$.

critical homoclinic orbit occurs at this point (see Figure 4). From Remark 3.1, we know that the fixed point x_1 is expanding. Moreover f satisfies the conditions $(A_1)-(A_5)$ given in Definition 4 then we have:

$$0 < x_1 < c < x_2 \text{ and } x_1 < c < x_1^*,$$

where $x_1^* = \max\{f^{-1}(x_1)\}$. This means $x_1^* \in [f^2(c), f(c)]$ and that x_1 is an SBR of f.

To show the sufficient condition, we reason by contradiction. Suppose that x_1 does not verify the Equation (4.1), then two cases can occur:

$$f^2(c) < x_1 \text{ or } f^2(c) > x_1, \textit{i.e.}, f(c) > x_1^* \text{ or } f(c) < x_1^*.$$

In the first case, i.e., if $f(c) > x_1^*$. The monotonicity property of f implies that $x_1^* \in [c, f(c)]$ and $0 < f(x) < x_1$ for all $x \in]x_1^*, f(c)[$ (see Figure 5). Furthermore, for all $x \in [0, x_1]$, we have $f'(x) > 0$ and $f(x) < x$.

This shows that $x = 0$ is the unique fixed point in $]x_1^*, +\infty]$. Hence, $f^n(x)$ is a decreasing sequence on $]x_1^*, +\infty[$ and converges to 0. Consequently, x_1 cannot be an SBR of f.

In the second case, i.e., if $f(c) < x_1^*$, then according to the monotonicity and unimodality of f, we have $x_1^* \notin [f^2(c), f(c)]$ and the inequalities below:

$$0 < x_1 < c < f^2(c) < f(c) < x_1^* \text{ (see Figure 6).}$$

Moreover, f is decreasing on $[c, +\infty[$, in particular for $x \in [f(c), x_1^*[$, we have $f'(x) < 0$. Taking into account Proposition 3.2, As a function of the variation of parameter r, the function f can exhibit different dynamic

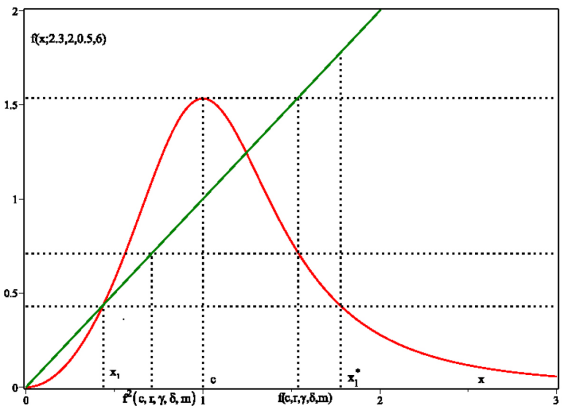


Figure 5

The graph of f **when** $f^2(c) > x_1$.

behavior: fixed point stability x_2, period doubling bifurcation, or chaotic dynamics see[12]. For any $x \in]f(c), x_1^*[$, we can see that . If $x \in [x_1^*, +\infty[$ according to the monotonicity of f, we get :

$$0 < f(x) < x_1.$$

Moreover, we have $f'(x) > 0$ for all $x \in [0, x_1[$, which shows that the trivial fixed point is the unique fixed point in the range $]x_1^*, +\infty[$ and in this region $f^n(x)$ is a decreasing sequence converge towards the origin. The origin is thus an attractor ,i.e., $\lim_{n \to +\infty} f^n(x) = 0$ for all $x \in]x_1^*, +\infty[$. Consequently x_1 is note an SBR of f.

4.2 *Essential extinction*

In this situation, f admits at least three fixed points. It means that almost all initial densities will result in extinction in finite time and occur when the maximum size of a growing population crosses a critical threshold, see [13, 14]. This region is also called the Allee effect region, see [11, 12].

Definition 4.4: [12] Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be the function given by the equation (1) with Allee effect. We define Essential extinction region by:

$$R_{ess} = \{(r, \gamma, \delta, m) \in R_0 : f^2(c) < x_1, \gamma > 1, \Phi(x) < r < \Psi(x)\}, \qquad (4.2)$$

where x_1 is the Allee fixed point.

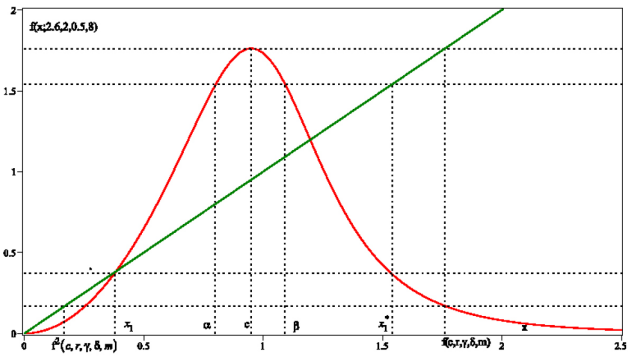


Figure 6

The graph of f when $f^2(c) < x_1$.

Proposition 4.2: Consider $f : \mathbb{R}_+ \to \mathbb{R}_+$ to be the function given by the equation (1) with Allee effect. Let

$$\alpha = \min\{f^{-1}(x_1^*)\}, \beta = \max\{f^{-1}(x_1^*)\},$$

and define $I =]\alpha, \beta[\subset]x1, x^*1[$. If $(r, \gamma, \delta, m) \in R_{ess}$, then

$$\lim_{x \to +\infty} f^n(x) = 0 \text{ for all } x \in I.$$

Proof: Let $(r, \gamma, \delta, m) \in R_{ess}$, then $f^2(c) < x_1$, note that this last inequality is identically equivalent to $f(c) > x_1^*$, with $x_1^* = \max\{f^{-1}(x_1)\}$. Using the unimodality of f, we deduce the following inequality:

$$x_1 < \alpha < c < \beta < x_1^*, \text{(see Figure 7)}.$$

In addition, from the monotonicity of f, we have $f'(x) > 0$ for all $x \in [0, c[$ and $f'(x) < 0$ for all $x \in]c, +\infty[$. However, taking into account condition c of Definition 4, it follows that

$$f(x) > x \text{ for all } x \in]\alpha, \beta[.$$

Moreover, we have $x_1^* < f(x)$. Using the monotonicity of f, we note that:

$$0 < f^2(x) < x_1 \text{ for all } x \in]\alpha, \beta[. \tag{4.3}$$

Now in this region $(]0, x_1[)$, $f^n(x)$ represents a decreasing monotone sequence which is limited below by 0. Due to the continuity of f, it get that the sequence $f^n(x)$ converges towards $\inf_{x \in \mathbb{R}+} \{f^n(x)\} = 0$. Hence,

$$\lim_{n \to +\infty} f^n(x) = 0 \text{ for all } x \in]\alpha, \beta[, \text{(see Figure 7)}. \tag{4.4}$$

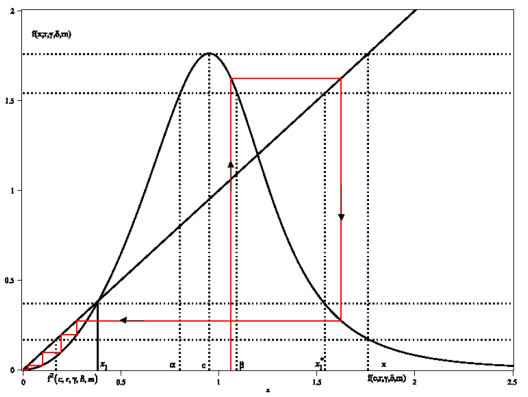


Figure 7

The graph of f when $f^2(c) < x_1$ shows that for any initial x_0 point ready in the interval $]\alpha, \beta[$, its orbit approaches the fixed point zero.

5. Conclusion

In this work, we presente a generalization of the Beverton-Holt model defined by Equation (1), a model introduced in 2019 by E. Liz and S. Buedo-Fernandez in [9], whose global stability of the existing fixed point they studied. The novelty of this work lies in the Allee effect and its influence on the dynamic properties of the model in the parameter set R_0. The conditions for the existence and stability of fixed points have been determined. An extinction phenomenon has been observed in this model through the presence of a zone called the essential extinction region.

References

[1] W. C. Allee, Animal Aggregations, University of Chicago Press, Chicago (1931).

[2] L .J . S. Allen, An Introduction to Mathematical Biology, Prentice-Hall, Upper Saddle River, NJ (2007).

[3] M. Bohner, S. Streipert, Optimal harvesting policy for the Beverton-Holt model, *Mathematical Biosciences and Engineering*, **13**:4, 673-695 (2016).

[4] F.Bozkurt, A. Yousef, Flip bifurcation and stability analysis of a fractional-order population dynamics with Allee effect, *J. Interdiscip. Math.* 22(6), 1009–1029 (2019).

[5] F. Courchamp, L. Berec , J. Gascoigne, Allee Effects in Ecology and Conversation, Oxford University (2008).

[6] S. Elaydi, E. Kwessi, G. Livadiotis, Hierarchical competition models with the Allee effect III:Multispecies, *J. Biol. Dyn*.(12), 271-287 (2018).

[7] L. Gardini, I. Sushko, V. Avrutin, M. Schanz, Critical homoclinic orbits lead to snap-back repellers, *Chaos Solitons and Fractals*, (44), 433-449 (2011).

[8] E.Liz, A new flexible discrete-time model for stable populations, Discrete Contin. Dyn. Syst. (23), 2487-2498 (2018).

[9] E. Liz, S Buedo-Fernandez, A new formula to get sharp global stability criteria for one dimensional discrete-time models, Qual. Theory Dyn. Syst, published online, 1-12 (2019).

[10] J.Maynard Smith, M. Slatkin, The stability of predator-prey systems, Ecology, (54). 384-391 (1973).

[11] J.L. Rocha, A.K Taha, Allee effect bifurcation in generalized logistic maps, *Int. J. Bifurcation and Chaos*, (29), 1-19 (2019).

[12] J.L. Rocha, A-K Taha, Bifurcation analysis of the γ -Ricker populationmodel using the Lambert W function, *Int. J . Bifurcation and Chaos,* 30:(7), 2050108 (2020).

[13] S. Schreiber, Chaos and population disappearances in simple ecological models, *J. Math. Biol,* (42), 239-260 (2001).

[14] S. Schreiber, Allee effects, extinctions, and chaotic transients in simple population models, *Theor. Popul. Biol,* (64), 201-209 (2003).

[15] J. G, Shepherd, A versatile new stock-recruitment relationship for fisheries, and the construction of sustainable yield resources, *J. Conserv. Int. Explor. Mer,* (40), 67-75 (1982).

[16] J. A. Walter, K. L. Grayson, D. M. Johnson, Variation in Allee effects: Evidence, unknowns, and directions forward, Popul. Ecol, (59), 99-107 (2017).

Received January, 2023