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A notion of *Dg*-orthogonality and *Dg*-angle in normed spaces

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Abstract

The purpose of this article is to formulate the *Dg*-orthogonality using a 2-norm in normed spaces. We will prove the relations between the *Dg*-orthogonality and the *g*-orthogonality. Moreover, we study the *Dg*-angle and its properties in normed spaces. We also prove the relations between the *g*-angle and the *Dg*-angle.

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1. Introduction

Orthogonality and angles between two vectors are two important concepts in $(A,\langle\cdot,\cdot\rangle)$ and $(A,\|\cdot\|)$. $\langle c,d\rangle = 0$ implies c and d are said to be orthogonal (symbolized by $c \perp d$). The orthogonal $(\perp)$ fulfiles the traits as follows [3].

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(a) $c \perp d$ implies $c = 0$ (*nondegeneracy*).

(b) $c \perp d$ implies $d \perp c$ (*symmetry*).

(c) $c \perp d$ implies $\alpha c \perp \beta d$ for every $\alpha, \beta \in \mathbb{R}$ (*homogeneity*).

(d) $c \perp d$ and $c \perp d'$ implies $c \perp (d + d')$ (*additivity*).

(e) For every c and d in A there is $\gamma \in \mathbb{R}$ so that $c \perp (\gamma c + d)$ (*resolvability*).

(f) $c_n \to c$ and $d_n \to d$ (in norm) and $c_n \perp d_n$ for any $n \in \mathbb{N}$ implies $c \perp d$ (*continuity*).

Then the angle $\varphi(c,d)$ where c and d in A is $\varphi(c,d)$:= arccos $\frac{\langle c,d\rangle}{\|c\|\|d\|}$. In here, $\| c \|$:= $\langle c,c\rangle^{\frac{1}{2}}$ denotes the induced norm in A. The angle $\varphi(c,d)$ fulfiles the traits as follows. (see [5]):

(a) c and d are of the same direction if and only if $\varphi(c,d) = 0$; c and d are of opposite direction if and only if $\varphi(c,d) = \pi$ (*parallelism*).

(b) $\varphi(c,d) = \varphi(d,c)$ (*symmetry*).

(c) For every c and d in A

$$\varphi(\alpha c, \beta d) = \begin{cases} \varphi(c,d), & \alpha\beta > 0 \\ \pi - \varphi(c,d), & \alpha\beta < 0 \end{cases}$$

(*homogeneity*).

(d) $c_n \to c$ and $d_n \to d$ (in norm) implies $\varphi(c_n, d_n) \to \varphi(c,d)$ (*continuity*).

Now let $(A, \|\cdot\|)$ be a normed space. In this space, many researchers intrduced several orthogonality and angle during last decades -- see, for instance, [1, 2, 8, 11, 14, 15]. Next, the function $g : A^2 \to \mathbb{R}$ formulated by the form

$$g(c,d) := \frac{\|c\|[\kappa_+(c,d) + \kappa_-(c,d)]}{2},$$

where

$$\kappa_\pm(c,d) := \lim_{\tau \to \pm 0}\left[\frac{\|c+\tau d\|}{\tau} - \frac{\|c\|}{\tau}\right]$$

and $c, d \in A$. Then, we can check that the mapping g fulfiles the properties:

(1) $g(c,c) = \| c \|^2$;

(2) $g(\omega c, \theta d) = \omega\theta g(c,d)$ for every $\theta, \omega \in \mathbb{R}$;

(3) $\| c \|^2 + g(c,d) = g(c, c+d)$;

(4) $\| d \| \| c \| \geq \| g(c,d) \|$.

In addition g is linear on d implies g is named a *semi-inner product* on A. Then, Milićić [12, 13] defined *g-orthogonality* on A ($c \perp_g d$ if only if $g(c,d) = 0$). Next Birkhoff [4] defined B-orthogonality on A ($c \perp_B d$ if only if $\| c + \alpha d \| \geq \| c \|$ for any $\alpha \in \mathbb{R}$). As a note in $(A, \langle \cdot, \cdot \rangle)$, $c \eta d$ if and only if $\langle c, d \rangle = 0$ holds for each $\eta \in \{\perp_g, \perp_B\}$. In 1983, by using the 2-norms, Dimminie [5] developed the notion of D-orthogonality in A. A 2-norm is defined to be the function $\| \cdot, \cdot \| : A \times A \to \mathbb{R}$ which fulfills the properties:

(1) c and d are linearly dependent if and only if $\| c, d \|$ has the value 0;
(2) $\| c, d \| = \| d, c \|$;
(3) $\| \omega c, d \| = |\omega| \| c, d \|$ for any $\omega \in \mathbb{R}$;
(4) $\| c, c' + d \| \leq \| c, c' \| + \| c, d \|$ for any $c, c', d \in A$.

Next $(A, \| \cdot, \cdot \|)$ is named a 2-*normed space*. This space was first discussed by Gähler [9] in 1960's. Further research can be seen in [10, 16, 18, 19].

Recently, Nur and Gunawan in [17] defined a 2-norm on $(A, \| \cdot \|)$ by

$$\| c, d \|_g := \sup_{\| w_j \| \leq 1, j=1,2} \begin{vmatrix} g(w_1, c) & g(w_2, c) \\ g(w_1, d) & g(w_2, d) \end{vmatrix}. \tag{1}$$

Note that in $(A, \langle \cdot, \cdot \rangle)$, the 2-norm $\| \cdot, \cdot \|_g$ is equivalent to standard 2-norm $\| \cdot, \cdot \|_s$ [17].

Using the 2-norm $\| \cdot, \cdot \|_g$, we will formulate the Dg-orthogonality in A and its properties. Next, we will investigate the connection between the Dg-orthogonality with the g-orthogonality. Moreover, we will also formulate Dg-angle on A and its traits .

2. Main Results

We will devide this part into two subsections, namely the Dg-Orthogonality and the Dg-angle in $(A, \| \cdot \|)$.

2.1 *The Dg-Orthogonality*

By using the *D*-orthogonality [5] and the 2-norm $\| c,d \|_g$ in (1), we formulate *Dg*-orthogonality in $(A, \| \cdot \|)$.

Definition 2.1: If $\| c \| \times \| d \| = \| c,d \|_g$ for every $c,d \in A$ then *c* and *d* are said to be *Dg*-orthogonal. In this case we symbolize $c \perp_{Dg} d$.

Remark 2.2: If $(A, \langle \cdot,\cdot \rangle)$ be an inner product space, then the 2-norm $\| \cdot,\cdot \|_g$ is equivalent to standard 2-norm $\| \cdot,\cdot \|_s$ in $(A, \langle \cdot,\cdot \rangle)$ [17], that is

$$\begin{aligned} \| c,d \|_g &= \sup_{\|w_j\| \le 1,\, j=1,2} \begin{vmatrix} g(w_1,c) & g(w_2,c) \\ g(w_1,d) & g(w_2,d) \end{vmatrix} \\ &= \begin{vmatrix} \langle c,c \rangle & \langle c,d \rangle \\ \langle d,c \rangle & \langle d,d \rangle \end{vmatrix}^{\frac{1}{2}} \\ &= \sqrt{\| c \|_s^2 \times \| d \|_s^2 - \langle c,d \rangle^2}. \end{aligned}$$

As a consequence, $(\perp_{Dg})$ coincides with $(\perp)$. Moreover, we obtain Hadamard inequality $\| \cdot,\cdot \|_g \le \| \cdot \| \times \| \cdot \|$.

By using a 2-norm and the *D*-orthogonality traits in [6], we have

Proposition 2.3: The *Dg*-orthogonality satisfies four properties:

(1) $c \perp_{Dg} c$ implies $c = 0$ (nondegeneracy).

(2) $c \perp_{Dg} d$ implies $d \perp_{Dg} c$ (symmetry).

(3) If $c \perp_{Dg} d$ then $\alpha c \perp_{Dg} \beta d$ for any α and β in $\mathbb{R}$ (homogeneity).

(4) For any *c* and *d* in *A* there is $\gamma \in \mathbb{R}$ so that $c \perp_{Dg} (\gamma c + d)$ (resolvability).

Next, we have the connection between $\perp_{Dg}$ and $\perp_g$ as follows.

Theorem 2.4: *Let* $\| c,d \|_g \le \sqrt{g(c,c)} \times \sqrt{g(d,d)}$ *for any c and d in A.* $c \perp_g d$ *implies* $c \perp_{Dg} d$.

Proof: Suppose that $\| c,d \|_g \le \sqrt{g(c,c)} \times \sqrt{g(d,d)}$ for any *c* and *d* in *A*. For $c = 0$ or $d = 0$ then $\| c,d \|_g = \sqrt{g(c,c)} \times \sqrt{g(d,d)}$. Next, for $c \ne 0$ and $d \ne 0$. If $w_1 = \frac{c}{\sqrt{g(c,c)}}$ and $w_2 = \frac{d}{\sqrt{g(d,d)}}$, then $\| w_1 \| = 1$ and $\| w_2 \| = 1$. Next, using the traits of *g* and determinants, we have

$$\begin{vmatrix} g(w_1,c) & g(w_2,c) \\ g(w_1,d) & g(w_2,d) \end{vmatrix} = \frac{1}{\sqrt{g(c,c)}\times\sqrt{g(d,d)}} \begin{vmatrix} g(c,c) & g(d,c) \\ g(c,d) & g(d,d) \end{vmatrix}$$
$$= \frac{1}{\sqrt{g(c,c)}\times\sqrt{g(d,d)}}[g(c,c)\times g(d,d) - g(c,d)\times g(d,c)].$$

Because $g(c,d)=0$, we have $\|c,d\|_g \geq \sqrt{g(c,c)} \times \sqrt{g(d,d)}$. Hence, $\|c,d\|_g = \sqrt{g(c,c)} \times \sqrt{g(d,d)}$. Using properties g, we have $c \perp_{Dg} d$.

Remark 2.5: The contrary of above theorem is not applicable. For instance, suppose that $A = \mathbb{R}^2$ with norm $\|c\| = \|(c_1,c_2)\| = \max\{|c_1|,|c_2|\}$. Take $e_1 = (1,0)$ and also $e_2 = (0,1)$. One may check that $\|e_1\| = 1$, $\|e_2\| = 1$ and $\|e_1,e_2\|_g = 1$. Because $\|c,d\| = \|(c_1,c_2),(d_1,d_2)\| = |c_1d_2 - d_1c_2| \quad \|e_1,e_2\|$ then $\|c,d\|_g = |c_1d_2 - d_1c_2|$. Next, if $c=(1,0)$ and $d=(2,-1)$ then $\|c,d\|_g = \|c\| \times \|d\| = 2$. Hence, $c \perp_{Dg} d$. Next, we can observe that

$$\kappa_{\pm}(c,d) = \lim_{\tau\to\pm 0}\left[\frac{\|c+\tau d\|}{\tau} - \frac{\|c\|}{\tau}\right] = 2.$$

Hence, $g(c,d) = \frac{1}{2}\|c\|[\kappa_+(c,d) + \kappa_-(c,d)] = 2$. So, $\|c,d\|_g = \|c\| \times \|d\|$ but $g(c,d) = 2$.

Moreover, for $\|c,d\|_g \leq \sqrt{g(c,c)} \times \sqrt{g(d,d)}$ for any c and d in A, we have the following lemma about the relation between the Dg-orthogonality and Birkhoff orthogonality $(\perp_B)$.

Lemma 2.6: *Suppose that $(A,\|\cdot\|)$ is a normed space. $c \perp_{Dg} d$ implies $c \perp_B d$.*

Proof: Using properties 2-norm we obtain for every $\alpha \in \mathbb{R}$

$$\|c,d\|_g = \|c+\alpha d, d\|_g \leq \sqrt{g(c+\alpha d, c+\alpha d)} \times \sqrt{g(d,d)}.$$

Moreover, because $c \perp_{Dg} d$ then $\sqrt{g(c,c)} \leq \sqrt{g(c+\alpha d,c+\alpha d)}$. Hence $c \perp_B d$.

If $(A,\|\cdot\|)$ is a smooth normed space, that is $\kappa_-(c,d) = \kappa_+(c,d)$ for every c and d in A, then we have the following result.

Lemma 2.7: **[7]** *Suppose that $(A,\|\cdot\|)$ is the smooth normed space. $c \perp_B d$ implies $c \perp_g d$.*

Theorem 2.8: *Suppose that $(A,\|\cdot\|)$ is the smooth normed space and $\|c,d\|_g \leq \sqrt{g(c,c)} \times \sqrt{g(d,d)}$ for every $c,d \in A$. $c \perp_g d$ if and only if $c \perp_{Dg} d$.*

Proof: Suppose that $(A,\|\cdot\|)$ is a smooth normed space and $\|c,d\|_g \leq \sqrt{g(c,c)} \times \sqrt{g(d,d)}$ for every $c,d \in A$. If $c \perp_g d$ then $c \perp_{Dg} d$ by Theorem 2.4. Conversely, using properties of 2-norm, we obtain

$$\|c,d\|_g = \|c+\alpha d,d\|_g \leq \sqrt{g(c+\alpha d,c+\alpha d)} \times \sqrt{g(d,d)}$$

for any $\alpha \in \mathbb{R}$. Thus $\sqrt{g(c,c)} \leq \sqrt{g(c+\alpha d,c+\alpha d)}$, so that $c \perp_B d$. Using Lemma 2.7, we obtain $c \perp_g d$.

Remark 2.9: By using the three results above, we have $c \perp_B d$ if and only if $c \perp_{Dg} d$.

Next, using Theorem 2.8 then Lemma 2.6, we have the corollarry as follows.

Corollary 2.10: *Suppose that* $(A,\|\cdot\|)$ *is a smooth normed space. Let* $\|c,d\|_g \leq \sqrt{g(c,c)} \times \sqrt{g(d,d)}$ *for any c and d in A. Then*

(1) If $c \perp_{Dg} d$ and $c \perp_{Dg} e$, then $c \perp_{Dg} (d+e)$.
(2) If $c_n \to c$ and $d_n \to d$ (in norm) and $c_n \perp_{Dg} d_n$ for every $n \in \mathbb{N}$, then $c \perp_{Dg} d$.

2.2 *The Dg-angle*

Let $\|c,d\|_g \leq \|c,d\|_g \leq \sqrt{g(c,c)} \times \sqrt{g(d,d)} = \|c\| \times \|d\|$ for any $c,d \in A$. Then using the above $\perp_{Dg}$ and the D-angle in [11], we can write the *Dg-angle* between *a* and *b* in *A*, symbolized by $\varphi_{Dg}(c,d)$, with formula

$$\varphi_{Dg}(c,d) := \arcsin \frac{\|c,d\|_g}{\|c\| \times \|d\|}.$$

Remark 2.11: $\varphi_{Dg}(c,d) = \frac{1}{2}\pi$ if and only if $c \perp_{Dg} d$. In $(A,\langle\cdot,\cdot\rangle)$, we observe that

$$\sin \varphi_{Dg}(c,d) = \frac{\|c,d\|_g}{\|c\| \times \|d\|} = \frac{\|a,b\|_s}{\|c\| \times \|d\|} = \sqrt{1 - \frac{\langle c,d\rangle^2}{\|c\|^2 \times \|d\|^2}} = \sin \varphi(c,d).$$

Hence,

$$\varphi_{Dg} = \begin{cases} \varphi, \text{ if } \varphi \text{ is acute} \\ \pi - \varphi, \text{if } \varphi \text{ is obstuse.} \end{cases}$$

Using the 2-norm $\|\cdot,\cdot\|_g$ dan the properties of the D-angle [11], we have

Proposition 2.12: The Dg-angle $\varphi_{Dg}(\cdot,\cdot)$ fulfills four properties:

(1) $\varphi_{Dg}(c,d)=0$ if and only if c and d are linearly dependent (part of paralelism).

(2) $\varphi_{Dg}(c,d)=\varphi_{Dg}(d,c)$ for any $c,d\in A$ (symmetry).

(3) $\varphi_{Dg}(\alpha c,\beta d)=\varphi_{Dg}(c,d)$ for any $c,d\in A$ (part of homogeneity).

(4) $c_n\to c$ and $d_n\to d$ (in norm) implies $\varphi_{Dg}(c_n,d_n)\to\varphi_{Dg}(c,d)$ (continuity).

Remark 2.13: Because $\varphi_{Dg}(\alpha c,\beta d)=\varphi_{Dg}(c,d)$ then we can say that the Dg -angle between vector c and vector d is also the Dg -angle between line c and line d in A.

Next, because $\varphi_g(\cdot,\cdot):=\arccos\frac{g(\cdot,\cdot)}{\|\cdot\|\times\|\cdot\|}$ in [11] and $\|\cdot,\cdot\|_g\le\|\cdot\|\times\|\cdot\|$, we have some relations between the g-angle and the Dg-angle as follows.

Theorem 2.14:

(a) $\varphi_g(c,d)=\frac{\pi}{2}$ *implies* $\varphi_{D\ g}(c,d)=\frac{\pi}{2}..$

(b) $\varphi_{Dg}(c,d)=0$ *implies* $\varphi_g(c,d)=0$ *or* $\varphi_g(c,d)=\pi$.

Proof:

(a) $\varphi_g(c,d)=\frac{\pi}{2}$ implies $c\perp_g d$ by [11]. By using Theorem 2.4, we obtain $c\perp_{Dg} d$. Hence $\varphi_{Dg}(c,d)=\frac{\pi}{2}$.

(b) Suppose that $\varphi_{Dg}(c,d)=0$. By definition the Dg -angle, we have $\|c,d\|_g=0$. As consequence, we have $c=kd$ for any $k\in\mathbb{R}$. By the g-angle in [11], we have $\cos\varphi_g(c,d)=\pm1$. Therefore, $\varphi_g(c,d)=0$ or $\varphi_g(c,d)=\pi$.

Suppose that $(A,\|\cdot\|)$ is a smooth normed space. Using Theorem 2.8, we have concluded as follows.

Corollary 2.15: *Suppose that* $(A,\|\cdot\|)$ *is a smooth normed space.* $\varphi_{Dg}(c,d)=\frac{\pi}{2}$ *if and only if* $\varphi_g(c,d)=\frac{\pi}{2}$.

3. Concluding Remark

The concept of orthogonality and angles are two important concepts in a normed space. This paper have formulated notion of *Dg*-orthogonality and have constructed definitions of *Dg*-angles. We also have discussed their properties.

The *Dg* orthogonality satisfies nodegeneracy property, symmetry property, homogeneity property and resolvability property. Next, we have showed some relations between the *Dg*-orthogonality and the *g*-orthogonality. In addition, in a smooth normed space, the *Dg*-orthogonality fulfills additivity and continuity property. Next, the *Dg*-angle fulfills the property: parts of parallelism, symmetry, parts of homogeneity, and continuity. We also have showed the relations between the *Dg*-angle and the *g*-angle.

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