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2-point right Radau inequalities for differentiable s-convex functions

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Abstract

Convexity is one of the fundamental principles of analysis. In the last decades, many important inequalities are established for different classes of convex functions. In this paper, we propose to study a 2-point right Radau type integral inequalities for functions whose first derivatives are s-convex in the second sense. The cases where the first derivatives are bounded as well as Hölderian are also discussed. Applications to quadrature formulas and special means are provided.

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1. Introduction

Integral inequalities are a potent analytical tool that are crucial to the study and development of many problems in several fields of science and engineering. This concept has a closed relationship with theory of convexity, see for examples [1, 3, 4, 5, 8] and their references.

As you may remember, a function $\mathcal{M} : \mathbf{I} \to \mathbb{R}$ is considered convex if, for every *h*, *k*, in $\mathbf{I}$ and every *e* in $[0,1]$ (refer to [6]), we have

$$\mathcal{M}(eh+(1-e)k) \le e\mathcal{M}(h)+(1-e)\mathcal{M}(k).$$

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The concept of convexity has been generalized in diverse manners. One of them is the so-called s-convex function or Breckner convexity defined as follows:

A nonnegative function $\mathcal{M}: I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if

$$\mathcal{M}(eh+(1-e)k) \leq e^s \mathcal{M}(h)+(1-e)^s \mathcal{M}(k)$$

holds for all $h,k \in \mathbf{I}$ and $e \in 0,1]$ (see [2]).

In this paper we propose to study the 2-point right Radau formula (see [7]) given as follows:

$$\int_h^k \mathcal{M}(z)\,dz \simeq \frac{k-h}{4}\left(3\mathcal{M}\left(\frac{2h+k}{3}\right)+\mathcal{M}(k)\right)+E(\mathcal{M}),$$

where $E(\mathcal{M})$ is the associated approximation error.

First, we establish a novel integral identity. We create new 2-point right Radau type integral inequalities for functions whose first derivatives are s-convex in the second sense based on this identity. We also cover the hölderian functions and the circumstances when the first derivatives are bounded. There are applications for quadrature formula and special means given.

2. Main results

Lemma 1: *Let $\mathcal{M}: \mathbf{I} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbf{I}^{\circ}$, $h,k \in \mathbf{I}^{\circ}$ with $h<k$ and $\mathcal{M}' \in L^1[h,k]$, then the following equality holds*

$$\begin{aligned}&\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_k^k \mathcal{M}(z)dz\\ &= \frac{k-h}{9}\left(\int_0^1 2e\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)de+\int_0^1 (8e-5)\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)de\right).\end{aligned}$$

Proof: Let

$$I_1=\int_0^1 2e\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)de$$

and

$$I_2=\int_0^1 (8e-5)\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)de.$$

Integrating by parts I_1, we get

$$\begin{aligned}I_1 &= \tfrac{6}{k-h} e\mathcal{M}\left((1-e)h + e\tfrac{2h+k}{3}\right)\Big|_{e=0}^{e=1} - \tfrac{6}{k-h}\int_0^1 \mathcal{M}\left((1-e)h + e\tfrac{2h+k}{3}\right)de \\ &= \tfrac{6}{k-h}\mathcal{M}\left(\tfrac{2h+k}{3}\right) - \tfrac{18}{(k-h)^2}\int_h^{\frac{2h+k}{3}} \mathcal{M}(z)dz.\end{aligned} \tag{2.1}$$

Similarly, we get

$$\begin{aligned}I_2 &= \tfrac{3}{2(k-h)}(8e-5)\mathcal{M}\left((1-e)\tfrac{2h+k}{3} + ek\right)\Big|_{e=0}^{e=1} - \tfrac{12}{k-h}\int_0^1 \mathcal{M}\left((1-e)\tfrac{2h+k}{3} + ek\right)de \\ &= \tfrac{9}{2(k-h)}\mathcal{M}(k) + \tfrac{15}{2(k-h)}\mathcal{M}\left(\tfrac{2h+k}{3}\right) - \tfrac{18}{(k-h)^2}\int_{\frac{2h+k}{3}}^{k} \mathcal{M}(z)dz.\end{aligned} \tag{2.2}$$

The required outcome is obtained by summing (2.1) and (2.2), multiplying the resultant equality by $\frac{k-h}{9}$.

Theorem 1: *Let $\mathcal{M}$ be as in Lemma 1. If $|\mathcal{M}'|$ is s-convex on $[h,k]$, then we have*

$$\begin{aligned}&\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right) + \tfrac{1}{2}\mathcal{M}(k) - \tfrac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right| \\ &\le \tfrac{k-h}{9}\left(\tfrac{2}{(s+1)(s+2)}|\mathcal{M}'(h)| + \tfrac{8^{s+1}(7s+4)+2\times 3^{s+2}}{8^{s+1}(s+1)(s+2)}\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right| + \tfrac{8^{s+1}(3s-2)+2\times 5^{s+2}}{8^{s+1}(s+1)(s+2)}|\mathcal{M}'(k)|\right),\end{aligned}$$

where $h \ge 0$ and s is a fixed number in (0, 1].

Proof: From Lemma 1, modulus and s-convexity of $|\mathcal{M}'|$, we have

$$\begin{aligned}&\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right) + \tfrac{1}{2}\mathcal{M}(k) - \tfrac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right| \\ &\le \tfrac{k-h}{9}\left(\int_0^1 2e\left|\mathcal{M}'\left((1-e)h + e\tfrac{2h+k}{3}\right)\right|de + \int_0^1 |8e-5|\left|\mathcal{M}'\left((1-e)\tfrac{2h+k}{3} + ek\right)\right|de\right) \\ &= \tfrac{k-h}{9}\left(\int_0^1 2e\left|\mathcal{M}'\left((1-e)h + e\tfrac{2h+k}{3}\right)\right|de + \int_0^{\frac{5}{8}} (5-8e)\left|\mathcal{M}'\left((1-e)\tfrac{2h+k}{3} + ek\right)\right|de\right. \\ &\left.+\int_{\frac{5}{8}}^1 (8e-5)\left|\mathcal{M}'\left((1-e)\tfrac{2h+k}{3} + ek\right)\right|de\right) \\ &\le \tfrac{k-h}{9}\left(\int_0^1 2e\left((1-e)^s|\mathcal{M}'(h)| + e^s\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|\right)de\right. \\ &+\int_0^{\frac{5}{8}} (5-8e)\left((1-e)^s\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right| + e^s|\mathcal{M}'(k)|\right)de\end{aligned}$$

$$+\int_{\frac{5}{8}}^{1}(8e-5)\left((1-e)^{s}\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|+e^{s}\left|\mathcal{M}'(k)\right|\right)de\Big)$$
$$=\tfrac{k-h}{9}\Bigg(\left|\mathcal{M}'(h)\right|\int_{0}^{1}2e(1-e)^{s}\,de+\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|\int_{0}^{1}2e^{s+1}\,de$$
$$+\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|\int_{0}^{\frac{5}{8}}(5-8e)(1-e)^{s}\,de+\left|\mathcal{M}'(k)\right|\int_{0}^{\frac{5}{8}}(5-8e)e^{s}de$$
$$+\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|\int_{\frac{5}{8}}^{1}(8e-5)(1-e)^{s}\,de+\left|\mathcal{M}'(k)\right|\int_{\frac{5}{8}}^{1}(8e-5)e^{s}de\Bigg)$$
$$=\tfrac{k-h}{9}\left(\tfrac{2}{(s+1)(s+2)}\left|\mathcal{M}'(h)\right|+\tfrac{8^{s+1}(7s+4)+2\times3^{s+2}}{8^{s+1}(s+1)(s+2)}\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|+\tfrac{8^{s+1}(3s-2)+2\times5^{s+2}}{8^{s+1}(s+1)(s+2)}\left|\mathcal{M}'(k)\right|\right),$$

where we have used

$$\int_{0}^{1}2e(1-e)^{s}\,de=\tfrac{2}{(s+1)(s+2)}, \tag{2.3}$$

$$\int_{0}^{1}2e^{s+1}de=\tfrac{2}{s+2}, \tag{2.4}$$

$$\int_{0}^{\frac{5}{8}}(5-8e)(1-e)^{s}\,de=\tfrac{8^{s+1}(5s+2)+3^{s+2}}{8^{s+1}(s+1)(s+2)}, \tag{2.5}$$

$$\int_{0}^{\frac{5}{8}}(5-8e)e^{s}de=\tfrac{5^{s+2}}{8^{s+1}(s+1)(s+2)}, \tag{2.6}$$

$$\int_{\frac{5}{8}}^{1}(8e-5)(1-e)^{s}\,de=\tfrac{3^{s+2}}{8^{s+1}(s+1)(s+2)} \tag{2.7}$$

and

$$\int_{\frac{5}{8}}^{1}(8e-5)e^{s}de=\tfrac{8^{s+1}(3s-2)+5^{s+2}}{8^{s+1}(s+1)(s+2)}. \tag{2.8}$$

The proof is completed.

Corollary 1: *Under the assumptions of Theorem 1, and if* $|\mathcal{M}'|$ *is a convex function, then we have*

$$\begin{aligned}&\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right)+\tfrac{1}{2}\mathcal{M}(k)-\tfrac{2}{k-h}\int_h^k\mathcal{M}(z)dz\right|\\ \leq\ &\tfrac{k-h}{9}\left(\tfrac{1}{3}\left|\mathcal{M}'(h)\right|+\tfrac{379}{192}\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|+\tfrac{157}{192}\left|\mathcal{M}'(k)\right|\right).\end{aligned}$$

Theorem 2: *Assume that all assumptions of Theorem 1 are valid. If* $|\mathcal{M}'|^q$ *is s-convex, then we have*

$$\begin{aligned}&\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right)+\tfrac{1}{2}\mathcal{M}(k)-\tfrac{2}{k-h}\int_h^k\mathcal{M}(z)dz\right|\\ &\leq\tfrac{k-h}{9}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}}\left(2\left(\frac{|\mathcal{M}'(h)|^q+\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q+|\mathcal{M}'(k)|^q}{s+1}\right)^{\frac{1}{q}}\right),\end{aligned}$$

where s is a fixed number in (0, 1] and $q,p>1$ *with* $\frac{1}{q}+\frac{1}{p}=1.$

Proof: Applying the absolute value in two sides of equality of Lemma 1, then Hölder's inequality, and *s*-convexity of $|\mathcal{M}'|^q$, we obtain

$$\begin{aligned}&\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right)+\tfrac{1}{2}\mathcal{M}(k)-\tfrac{2}{k-h}\int_h^k\mathcal{M}(z)dz\right|\\ &\leq\tfrac{k-h}{9}\left(\left(\int_0^1(2e)^p de\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{M}'\left((1-e)h+e\tfrac{2h+k}{3}\right)\right|^q de\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1|8e-5|^p de\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{M}'\left((1-e)\tfrac{2h+k}{3}+ek\right)\right|^q de\right)^{\frac{1}{q}}\right)\\ &\leq\tfrac{k-h}{9}\left(\left(\int_0^1(2e)^p de\right)^{\frac{1}{p}}\left(\int_0^1\left((1-e)^s|\mathcal{M}'(h)|^q+e^s\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|^q\right)de\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^{\frac{5}{8}}(5-8e)^p de+\int_{\frac{5}{8}}^1(8e-5)^p de\right)^{\frac{1}{p}}\times\left(\int_0^1\left((1-e)^s\left|\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right|^q+e^s|\mathcal{M}'(k)|^q\right)de\right)\right)^{\frac{1}{q}}\\ &=\frac{k-h}{9(p+1)^{\frac{1}{p}}}\left(2\left(\frac{|\mathcal{M}'(h)|^q+\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q+|\mathcal{M}'(k)|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

The proof is finished.

Corollary 2: *Under the assumptions of Theorem 2, and if* $|\mathcal{M}'|^q$ *is a convex function, then we have*

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_k^k \mathcal{M}(z)dz\right|$$

$$\leq \frac{k-h}{9(p+1)^{\frac{1}{p}}}\left(2\left(\frac{|\mathcal{M}'(h)|^q+\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q+|\mathcal{M}'(k)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Theorem 3: *Assume that all assumptions of Theorem 1 are valid. If* $|\mathcal{M}'|^q$ *is s-convex, then we have*

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right|$$

$$\leq \frac{k-h}{9}\left(\left(\frac{2}{(s+1)(s+2)}|\mathcal{M}'(h)|^q+\frac{2}{s+2}\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{17}{8}\left(\frac{8^{s+1}(5s+2)+2\times 3^{s+2}}{17\times 8^s(s+1)(s+2)}\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q+\frac{8^{s+1}(3s-2)+2\times 5^{s+2}}{17\times 8^s(s+1)(s+2)}|\mathcal{M}'(k)|^q\right)^{\frac{1}{q}}\right),$$

where s is a fixed number in (0, 1] and $q\geq 1$.

Proof: Applying the absolute value in two sides of equality of Lemma 1, then power mean inequality, s-convexity of $|\mathcal{M}'|$ and using (2.3)-(2.8), we have

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right|$$

$$\leq \frac{k-h}{9}\left(\left(\int_0^1 2ede\right)^{1-\frac{1}{q}}\left(\int_0^1 2e\left|\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)\right|^q de\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1|8e-5|de\right)^{1-\frac{1}{q}}\left(\int_0^1|8e-5|\left|\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)\right|^q de\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{k-h}{9}\left(\left(\int_0^1 2e\left((1-e)^s|\mathcal{M}'(h)|^q+e^s\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\right)de\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{5}{8}}(5-8e)de+\int_{\frac{5}{8}}^1(8e-5)de\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |8e-5|\left((1-e)^S\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q + e^s\,|\mathcal{M}'(k)|^q\right)de\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{k-h}{9}\left(\left(\int_0^1 2e\left((1-e)^S\,|\mathcal{M}'(h)|^q + e^s\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\right)de\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{5}{8}}(5-8e)\left((1-e)^s\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q + e^s\,|\mathcal{M}'(k)|^q\right)de\right.$$

$$\left.\left.+\int_{\frac{5}{8}}^1(8e-5)\left((1-e)^S\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q + e^s\,|\mathcal{M}'(k)|^q\right)de\right)^{\frac{1}{q}}\right)$$

$$=\frac{k-h}{9}\left(\left(|\mathcal{M}'(h)|^q\int_0^1 2e(1-e)^s\,de + \left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\int_0^1 2e^{s+1}de\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\left(\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\int_0^{\frac{5}{8}}(5-8e)(1-e)^s\,de + |\mathcal{M}'(k)|^q\int_0^{\frac{5}{8}}(5-8e)\,e^s de\right.$$

$$\left.\left.+\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\int_{\frac{5}{8}}^1(8e-5)(1-e)^s\,de + |\mathcal{M}'(k)|^q\int_{\frac{5}{8}}^1(8e-5)e^s de\right)^{\frac{1}{q}}\right)$$

$$=\frac{k-h}{9}\left(\left(\frac{2}{(s+1)(s+2)}|\mathcal{M}'(h)|^q + \frac{2}{s+2}\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\frac{17}{8}\left(\frac{8^{s+1}(5s+2)+2\times 3^{s+2}}{17\times 8^s(s+1)(s+2)}\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q + \frac{8^{s+1}(3s-2)+2\times 5^{s+2}}{17\times 8^s(s+1)(s+2)}|\mathcal{M}'(k)|^q\right)^{\frac{1}{q}}\right).$$

Corollary 3: *Under the assumptions of Theorem 3, and if* $|\mathcal{M}'|^q$ *is convex, then we have*

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right|$$

$$\le\frac{k-h}{9}\left(\left(\frac{|\mathcal{M}'(h)|^q+2\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\frac{17}{8}\left(\frac{251\left|\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|^q+157\left|\mathcal{M}'(k)\right|^q}{408}\right)^{\frac{1}{q}}\right).$$

3. Further results

Theorem 4: *As in Lemma 1, let $\mathcal{M}$ be. If constants $-\infty < m < M < +\infty$ exist such that, for every $x \in h,k]$, $m \le \mathcal{M}'(x) \le M$, then we have*

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right| \le \frac{25(k-h)(M-m)}{144}.$$

Proof: Per Lemma 1, we possess

$$\begin{aligned}
&\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k \mathcal{M}(z)dz\\
=\ & \frac{k-h}{9}\left(\int_0^1 2e\left(\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)de\right.\\
&\left.+\int_0^1 (8e-5)\left(\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)de\right)\\
=\ & \frac{k-h}{9}\left(\int_0^1 2e\left(\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)-\frac{m+M}{2}\right)de+\frac{m+M}{2}\int_0^1 2ede\right.\\
&\left.+\int_0^1 (8e-5)\left(\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)-\frac{m+M}{2}\right)de+\frac{m+M}{2}\int_0^1 (8e-5)de\right)\\
=\ & \frac{k-h}{9}\left(\int_0^1 2e\left(\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)-\frac{m+M}{2}\right)de\right.\\
&\left.+\int_0^1 (8e-5)\left(\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)-\frac{m+M}{2}\right)de\right), \qquad (3.1)
\end{aligned}$$

where we have taken into consideration that

$$\int_0^1 2ede+\int_0^1 (8e-5)de=\int_0^1 (10e-5)de=0.$$

Using the absolute value on (3.1)'s two sides yields

$$\begin{aligned}
&\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right|\\
=\ & \frac{k-h}{9}\left(\int_0^1 2e\left|\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)-\frac{m+M}{2}\right|de\right.\\
&\left.+\int_0^1 |8e-5|\left|\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)-\frac{m+M}{2}\right|de\right). \qquad (3.2)
\end{aligned}$$

Given that for any x in $[h,k]$, $m \le \mathcal{M}'(x) \le M$, we get

$$\left|\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)-\frac{m+M}{2}\right| \le \frac{M-m}{2} \qquad (3.3)$$

and

$$\left|\mathcal{M}'\left((1-e)\tfrac{2h+k}{3}+ek\right)-\tfrac{m+M}{2}\right|\le \tfrac{M-m}{2}. \tag{3.4}$$

Using (3.3)-(3.4) in (3.2), we get

$$\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right)+\tfrac{1}{2}\mathcal{M}(k)-\tfrac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right|$$
$$\le \frac{(k-h)(M-m)}{18}\left(\int_0^1 2ede+\int_0^1|8e-5|de\right)=\frac{25(k-h)(M-m)}{144}.$$

The evidence is finished.

Theorem 5: *As in Lemma 1, let* $\mathcal{M}$ *be. If* $\mathcal{M}'$ *is r-L-Hölderian function on* $[h,k]$ *(i.e. there exist* $L>0$ *and* $0<r\le 1$ *such that* $|\mathcal{M}'(x)-\mathcal{M}'(y)|\le L\,|x-y|^r$*), then we have*

$$\left|\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right)+\tfrac{1}{2}\mathcal{M}(k)-\tfrac{2}{k-h}\int_h^k \mathcal{M}(z)dz\right|$$
$$\le \frac{1}{3}\left(\frac{k-h}{3}\right)^{r+1}\left(\frac{2^{2r+2}(r+1)(r+4)+2^{3r+1}(3r-2)+5^{r+2}}{2^{2r+2}(r+1)(r+2)}\right)L.$$

Proof: From Lemma 1, we have

$$\tfrac{3}{2}\mathcal{M}\left(\tfrac{2h+k}{3}\right)+\tfrac{1}{2}\mathcal{M}(k)-\tfrac{2}{k-h}\int_h^k \mathcal{M}(z)dz$$
$$=\tfrac{k-h}{9}\left(\int_0^1 2e\left(\mathcal{M}'\left((1-e)h+e\tfrac{2h+k}{3}\right)-\mathcal{M}'(h)+\mathcal{M}'(h)\right)de\right.$$
$$\left.+\int_0^1(8e-5)\left(\mathcal{M}'\left((1-e)\tfrac{2h+k}{3}+ek\right)-\mathcal{M}'\left(\tfrac{2h+k}{3}\right)+\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right)de\right)$$
$$=\tfrac{k-h}{9}\left(\int_0^1 2e\left(\mathcal{M}'\left((1-e)h+e\tfrac{2h+k}{3}\right)-\mathcal{M}'(h)\right)de+\mathcal{M}'(h)\int_0^1 2ede\right.$$
$$\left.+\int_0^1(8e-5)\left(\mathcal{M}'\left((1-e)\tfrac{2h+k}{3}+ek\right)-\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right)de+\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\int_0^1(8e-5)de\right)$$
$$=\tfrac{k-h}{9}\left(\int_0^1 2e\left(\mathcal{M}'\left((1-e)h+e\tfrac{2h+k}{3}\right)-\mathcal{M}'(h)\right)de\right.$$
$$\left.+\int_0^1(8e-5)\left(\mathcal{M}'\left((1-e)\tfrac{2h+k}{3}+ek\right)-\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right)de+\mathcal{M}'(h)-\mathcal{M}'\left(\tfrac{2h+k}{3}\right)\right). \tag{3.5}$$

Using the absolute value on (3.5)’s two sides yields

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_n^k\mathcal{M}(z)dz\right|$$
$$\leq\frac{k-h}{9}\left(\int_0^1 2e\left|\mathcal{M}'\left((1-e)h+e\frac{2h+k}{3}\right)-\mathcal{M}'(h)\right|de\right.$$
$$\left.+\int_0^1|8e-5|\left|\mathcal{M}'\left((1-e)\frac{2h+k}{3}+ek\right)-\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|de+\left|\mathcal{M}'(h)-\mathcal{M}'\left(\frac{2h+k}{3}\right)\right|\right). \quad (3.6)$$

Since $\mathcal{M}'$ is an r-L-Hölderian function on $[h, k]$, from (3.6) we have

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k\mathcal{M}(z)dz\right|$$
$$\leq\frac{1}{3}\left(\frac{k-h}{3}\right)^{r+1}L\left(\int_0^1 2e^{r+1}de+2^r\int_0^1|8e-5|e^r de+1\right)$$
$$=\frac{L}{3}\left(\frac{k-h}{3}\right)^{r+1}\left(1+\int_0^1 2e^{r+1}de+2^r\int_0^{\frac{5}{8}}(5-8e)e^r de+2^r\int_{\frac{5}{8}}^1(8e-5)e^r de\right)$$
$$=\frac{1}{3}\left(\frac{k-h}{3}\right)^{r+1}\left(\frac{2^{2r+2}(r+1)(r+4)+2^{3r+1}(3r-2)+5^{r+2}}{2^{2r+2}(r+1)(r+2)}\right)L.$$

The proof is over.

Corollary 4: *If $\mathcal{M}'$ is the L-Lipschitzian function and we make the assumptions of Theorem 5, we obtain*

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2h+k}{3}\right)+\frac{1}{2}\mathcal{M}(k)-\frac{2}{k-h}\int_h^k\mathcal{M}(z)dz\right|\leq\frac{301}{2592}(k-h)^2L.$$

4. Applications

Radau quadrature formula

Let Υ be the partition of the interval $[h,k]$ into the points $h = x_0 < x_1 < \ldots < x_n = k$. Then, take into consideration the quadrature formula.

$$\int_h^k\mathcal{M}(u)\,du=\lambda(\mathcal{M},\Upsilon)+R(\mathcal{M},\Upsilon),$$

with

$$\lambda(\mathcal{M},\Upsilon)=\sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{4}\left(3\mathcal{M}\left(\frac{2x_i+x_{i+1}}{3}\right)+\mathcal{M}(x_{i+1})\right)$$

and the corresponding approximation error is shown by $R(\mathcal{M},\Upsilon)$.

Proposition 1: Let $\mathcal{M}$ be as in Theorem 1 and $n\in\mathbb{N}$. If $|\mathcal{M}'|$ is convex, then we have

$$|R(\mathcal{M},\Upsilon)|\leq\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{18}\left(\frac{1}{3}|\mathcal{M}'(x_i)|+\frac{379}{192}\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+\frac{157}{192}\left|\mathcal{M}'(x_{i+1})\right|\right).$$

Proof: Using the subintervals $[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$ of the partition Υ and Corollary 1, we get

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2x_i+x_{i+1}}{3}\right)+\frac{1}{2}\mathcal{M}(x_{i+1})-\frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\mathcal{M}(z)dz\right|$$
$$\leq\frac{x_{i+1}-x_i}{9}\left(\frac{1}{3}|\mathcal{M}'(x_i)|+\frac{379}{192}\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|+\frac{157}{192}\left|\mathcal{M}'(x_{i+1})\right|\right).$$

The required outcome is produced by multiplying both sides of the aforementioned inequality by $\frac{x_{i+1}-x_i}{2}$, adding the resulting inequalities for all $i=0,1,...,n-1$, and applying the triangle inequality.

Proposition 2: *Let $\mathcal{M}$ be as in Theorem 2 and $n\in\mathbb{N}$. If $|\mathcal{M}'|$ is convex, then we have*

$$\left|R(\mathcal{M},\Upsilon)\right|\leq\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^2}{18(p+1)^{\frac{1}{p}}}\left(2\left(\frac{|\mathcal{M}'(x_i)|^q+\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q+|\mathcal{M}'(x_{i+1})|^q}{2}\right)^{\frac{1}{q}}\right).$$

Proof: Using the subintervals $[x_i,x_{i+1}]$ $(i=0,1,...,n-1)$ of the partition Υ and Corollary 2, we get

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2x_i+x_{i+1}}{3}\right)+\frac{1}{2}\mathcal{M}(x_{i+1})-\frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\mathcal{M}(z)dz\right|$$
$$\leq\frac{x_{i+1}-x_i}{9(p+1)^{\frac{1}{p}}}\left(2\left(\frac{|f'(x_i)|^q+\left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{5^{p+1}+3^{p+1}}{8}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q+|f'(x_{i+1})|^q}{2}\right)^{\frac{1}{q}}\right).$$

The required outcome is produced by multiplying both sides of the aforementioned inequality by $\frac{x_{i+1}-x_i}{2}$, adding the resulting inequalities for all $i=0,1,...,n-1$, and applying the triangle inequality.

Proposition 3: *Let $\mathcal{M}$ be as in Theorem 3 and $n \in \mathbb{N}$. If $|\mathcal{M}'|$ is convex, then we have*

$$|R(\mathcal{M},\Upsilon)| \le \sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i)^2}{18}\left(\left(\frac{1}{3}|\mathcal{M}'(x_i)|^q + \frac{2}{3}\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q\right)^{\frac{1}{q}} + \frac{17}{8}\left(\frac{251}{408}\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q + \frac{157}{408}|\mathcal{M}'(x_{i+1})|^q\right)^{\frac{1}{q}}\right).$$

Proof: Using the subintervals $[x_i, x_{i+1}]$ $(i=0,1,\dots,n-1)$ of the partition Υ and Corollary 3, we get

$$\left|\frac{3}{2}\mathcal{M}\left(\frac{2x_i+x_{i+1}}{3}\right) + \frac{1}{2}\mathcal{M}(x_{i+1}) - \frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}\mathcal{M}(z)dz\right|$$
$$\le \frac{x_{i+1}-x_i}{9}\left(\left(\frac{1}{3}|\mathcal{M}'(x_i)|^q + \frac{2}{3}\left|\mathcal{M}'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q\right)^{\frac{1}{q}} + \frac{17}{8}\left(\frac{251}{408}\left|f'\left(\frac{2x_i+x_{i+1}}{3}\right)\right|^q + \frac{157}{408}|\mathcal{M}'(x_{i+1})|^q\right)^{\frac{1}{q}}\right).$$

The required outcome is produced by multiplying both sides of the aforementioned inequality by $\frac{x_{i+1}-x_i}{2}$, adding the resulting inequalities for all $i=0,1,\dots,n-1$, and applying the triangle inequality.

Application to special means

For each real number h, k, and l, we have:

The arithmetic mean: $A(h,k) = \frac{a+b}{2}$ and $A(h,k,l) = \frac{h+k+l}{3}$

The p-logarithmic mean: $L_p(h,k) = \left(\frac{k^{p+1}-h^{p+1}}{(p+1)(k-h)}\right)^{\frac{1}{p}}$, $h,k>0, h\ne k$ and $p \in \mathbb{R}\setminus\{-1,0\}$.

Proposition 4: *If h,k are in $\mathbb{R}$ and $0<h<k$, then we have*

$$\left|3A^{s+1}(h,h,k) + k^{s+1} - 4L_{s+1}^{s+1}(h,k)\right|$$
$$\le \frac{2(k-h)}{9(s+2)}\left(2h^s + \frac{2^{3s+2}(7s+4)+3^{s+2}}{2^{3s+2}}\left(\frac{2h+k}{3}\right)^s + \frac{2^{3s+2}(3s-2)+5^{s+2}}{2^{3s+2}}k^s\right).$$

Proof: When Theorem 1 is applied to the function $\mathcal{M}(x) = x^{s+1}$, the claim is derived.

Proposition 5: *If* h, k *are in* $\mathbb{R}$ *and* $0 < h < k$, *then we have*

$$\left|3A^2(h,h,k) + k^2 - 4L_2^2(h,k)\right| \le \frac{25(k-h)^2}{36}.$$

Proof: When Theorem 4 is applied to the function $\mathcal{M}(x) = x^2$, the claim is derived.

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