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Solitons on Kenmotsu manifolds

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Abstract

In this article properties of solitons on Kenmotsu manifolds with Da-homothetic deformation and Schouten-van Kampen connection are discussed.

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1. Introduction

Solov'ev studied properties of a hyperdistribution in Riemannian manifold with Schouten-van Kampen connection [2,3,4]. Then Schouten-van Kampen connection are studied by different authers with different structures [5,6,7,8].

Almost Ricci soliton is given as [14].

$$L_A G + 2Ric + 2\delta G = 0, \tag{1.1}$$

Almost η-Ricci soliton is given as [15, 21].

$$L_A G + 2Ric + 2\delta G + 2\mu\eta \otimes \eta = 0. \tag{1.2}$$

Hamilton [14] defined Yamabe flow to solve Yamabe problem and inesively studies by [16, 17, 18, 19].

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Almost Yamabe soliton is defined as [17],

$$\frac{1}{2}(L_A G) = (scal - \lambda)G. \tag{1.3}$$

2. Preliminaries

An almost contact metric structure (J, ζ, η, G) satisfies [10, 12, 13].

$$J^2(A) = -A + \eta(A)\zeta, \eta(\zeta) = 1, J\zeta = 0, \eta o J = 0, \tag{2.1}$$

$$G(JA, JB) = G(A, B) - \eta(A)\eta(B). \tag{2.2}$$

Kenmotsu manifold satisfies [11].

$$(D_A J)B = G(JA, B)\zeta - \eta(B)JA. \tag{2.3}$$

In this case,

$$D_A\zeta = A - \eta(A)\zeta \text{ and} (\nabla_A \eta)B = G(JA, JB). \tag{2.4}$$

Da-homothetic deformation satisfies [9].

$$\eta^a = a\eta, \zeta^a = \frac{1}{a}\zeta, J^a = J, G^a = aG + a(a-1)\eta \otimes \eta, \tag{2.5}$$

here $(J^a, \zeta^a, \eta^a, G^a)$ is an almost contact metric structure [7].

$\widetilde{\nabla}$ and D are related by [1, 2],

$$\widetilde{\nabla}_A B = D_A B - \eta(B)\nabla_A \zeta + (\nabla_A \eta)(B)\zeta. \tag{2.6}$$

3. Da-Homothetic deformation

By simple calculation we have,

Lemma 1: *For Kenmotsu manifold M, D^a in terms of D is*

$$D^a_A B = D_A B + (a-1)\{G(A, B)\zeta - \eta(A)\eta(B)\zeta\}. \tag{3.1}$$

By this relation, relation between K^a & K is given by theorem 1 where K^a and K are Riemannian curvature tensors with respect to Da-homothetic deformation and Levi-Civita connection.

Theorem 1: *For Kenmotsu manifold M and Da-homothetic deformation on M, the curvature tensor K^a is*

$$K^a(A,B)C = K(A,B)C + (a-1)\{2G(B,C)A - 2G(A,C)B$$
$$+2\eta(A)\eta(C)B - 2\eta(B)\eta(C)A + G(A,C)\eta(B)\zeta - G(B,C)\eta(A)\zeta\}. \quad 3.2$$

by (3.2) we have,

$$Ric^a(B,C) = Ric(B,C) + 3(a-1)G(B,C) - 3(a-1)\eta(B)\eta(C). \quad 3.3$$

Here Ric^a and Ric are Ricci tensors with respect to D^a and D respectvelly.

Replacing B and C by $v_i(i=1,2,3)$ in (3.3), as v_i is a tangent space orthonormal basis at each point on the manifold, we gives,

$$scal^a = scal + 6(a-1). \quad 3.4$$

Here $scal^a$ and scal are scalar curvature with respect to D^a and D.

Lemma 2: *For Kenmotsu manifold M and Da-homothetic deformation on M, and any vector field A, B on M, the connections* $\tilde{\nabla}$ *and D are*

$$\tilde{\nabla}^a_A B = D_A B \quad 3.5$$

Theorem 2: *For Kenmotsu manifold M with* $\tilde{\nabla}^a, \tilde{K}^a$ *and K are related by*

$$\tilde{K}^a(A,B)C = K(A,B)C + (3a-2)\{G(B,C)A - G(A,C)B\} - (a^2-3a+2)$$
$$\{\eta(A)\eta(C)B - \eta(B)\eta(C)A\} + G(A,C)\eta(B)\zeta - y(B,C)\eta(A)\zeta\}. \quad 3.6$$

By (3.6) we have,

$$\widetilde{Ric^a}(B,C) = Ric(B,C) + 2(2a^2-3a+2)\{G(B,C) - \eta(B)\eta(C)\} + (a-1)G(B,JC).$$
$$3.7$$

The Ricci tensor and scalar curvature are related by [2, 3, 4, 5, 6, 7, 8, 9, 20, 22].

$$Ric(A,B) = \left(\frac{scal}{2}+1\right)G(A,B) - \left(\frac{scal}{2}+3\right)\eta(A)\eta(B). \quad 3.8$$

With (3.8),(3.7) becomes,

$$\widetilde{Ric}^a(B,C) = \left\{\frac{scal}{2} + (4a^2-6a+5)\right\}G(B,C) - \left\{\frac{scal}{2} + (2a^2-6a+7)\right\}\eta(C)\eta(B)$$
$$+(a-1)G(B,JC). \quad 3.9$$

Replacing B and C by $v_i(i=1,2,3)$, in (3.9), as v_i is a tangent space orthonormal basis at each point on the manifold, we gives,

$$\widetilde{scal}^{a} = scal + (10a^2 - 12a + 8). \qquad 3.10$$

4. Solitons

Definition: If Ricci tensor of M is of the form,

$$\widetilde{Ric^a} = \alpha G^a + \beta\eta^a \otimes \eta^a,$$

then manifold is known as η^a -Einstein manifold.

On a Kenmotsu manifold with $\tilde{\nabla}^a$ equation(1.1) becomes,

$$\tilde{L}^a_A G^a + 2\widetilde{Ric^a} + 2\delta G^a(A,B) = 0. \qquad 4.1$$

Replacing A by ζ in (4.1)gives,

$$G^a(\tilde{\nabla}^a_A\zeta^a, B) + G^a(A, \tilde{\nabla}^a_B\zeta^a) + G^a(\nabla^a_A\zeta^a, B) + G^a(A, \nabla^a_B\zeta^a)$$
$$+2\widetilde{Ric^a}(A,B) + 2\delta G^a(A,B) = 0. \quad 4.2$$

By (2.4), (2.6), (3.1), (3.5) and (3.9), equation(4.2) gives,

$$2(2-a)\{G(A,B) - \eta(A)\eta(B)\} + 2\left[\left\{\frac{scal}{2} + (4a^2 - 6a + 5)\right\}G(A,B)\right.$$
$$-\left\{\frac{scal}{2} + (2a^2 - 6a + 7)\right\}\eta(A)\eta(B) - (a-1)G(JA,B)] + 2\delta\{aG(A,B)$$
$$+a(a-1)\eta(A)\eta(B)\} = 0. \qquad (4.3)$$

Replacing A and B by ζ in (4.3) gives,

$$2(a^2 - 1) + a^2\delta = 0 \text{ i.e. } \delta = \frac{2(1-a^2)}{a^2}.$$

Therefore, we have,

Theorem 3: *Kenmotsu manifold with* $\tilde{\nabla}^a$ *is (i) Expanding if* $1 - a^2 > 0$, *(ii) shrinking if* $1 - a^2 < 0$ *and (iii) steady if* $1 - a^2 = 0$.

By (4.3),

$$\widetilde{Ric^a}(A,B) = \left(\frac{a-2}{a} - \delta\right)G^a(A,B) + \frac{2-a}{a}\eta^a(A)\eta^a(B).$$

Therefore we have,

Theorem 4: *A Kenmotsu manifold M with* $\widetilde{\nabla^a}$ *is an* η^a *-Einstein manifold.*

On a Kenmotsu manifold with Da-homothetic deformation and Schouten-van Kampen connection equation (1.2) can be given as,

$$(\tilde{L}^a_A G^a + 2\widetilde{Ric^a} + 2\delta G^a + 2\mu\eta^a \otimes \eta^a)(A,B) = 0. \quad 4.4$$

Replacing A by ζ^a in (4.4) and with (2.4), (2.6), (3.1), (3.5), and (3.9), we have

$$\widetilde{Ric^a}(A,B) = \left(\frac{a-2}{a} - \delta\right) g^a(A,B) - \left(\frac{a-2}{a} + \mu\right)\eta^a(A)\eta^a(B). \quad 4.5$$

Therefore,

Theorem 5: *Kenmotsu manifold M with* $\widetilde{\nabla^a}$ *and almost* η*-Ricci soliton* $(\zeta,\ \delta,\ \mu,\ G)$ *is an* η^a*-Einstein manifold.*

Replacing *A* and *B* by ζ in (4.5), gives

$$a^2(\delta+\mu) + 2(a^2-1) = 0, \text{ so } \delta+\mu = \frac{2(1-a^2)}{a^2}. \quad 4.6$$

Therefore,

Theorem 6: *A Kenmotsu manifold with almost* η*-Ricci soliton and* $\tilde{\nabla}^a$ *is* $(\zeta^a,\ \delta,\ \frac{2(1-a^2)}{a^2} - \delta,\ G^a)$.

On a Kenmotsu manifold with Da-homothetic deformation and Schouten-van Kampen connection, equation (1.3) becomes,

$$\frac{1}{2}(\tilde{L}^a_A G^a(A,B) = (\widetilde{scal}^a - \lambda)G^a(A,B)). \quad 4.7$$

Replacing A by ζ^a in (4.7), gives

$$\widetilde{scal}^a = \lambda.$$

Theorem 7: *A Kenmotsu manifold M with Yamabe soliton* (ζ^a, λ, G^a) *and* $\widetilde{\nabla^a}$ *is of constant scalar curvature.*

5. Example

Let M = $(A,B,C) \in R^3 : C \neq 0$ with linearly independent vector fields at each point of M.

$$v_1 = v^C \frac{\partial}{\partial A}, v_2 = v^C \frac{\partial}{\partial B}, v_3 = \frac{\partial}{\partial C}. \quad 5.1$$

Define a Riemannian metric,

$$\begin{aligned} G_{ij} &= 1 \text{ for } i = j \\ &= 0 \text{ for } i \neq j. \end{aligned}$$

Define,

$\eta(C) = G(C, v_3)$ with any $C \in \chi(M^3)$ & (1,1) tensor field J by, 5.2

$$Jv_1 = v_2, Jv_2 = -v_1, Jv_3 = 0. \quad 5.3$$

Hence by linearty property of J and G,

$$\eta(v_3) = 1, J^2(C) = -z + \eta(C)\zeta. \quad 5.4$$

$$[v_1, v_3] = v_1, [v_2, v_3] = v_2, [v_1, v_2] = 0. \quad 5.5$$

$$D_{v_1} v_1 = -v_3, D_{v_2} v_2 = -v_3, D_{v_1} v_3 = v_1, D_{v_2} v_3 = v_2. \quad 5.6$$

Therefore $M^3(J, \zeta, \eta, G)$ is a Kenmotsu manifold with

$$K(v_1, v_2)v_2 = 2v_1, K(v_2, v_3)v_2 = -2v_3, K(v_1, v_3)v_3 = 2v_1. \quad 5.7$$

With (2.4), (2.5) is,

$$\widetilde{\nabla}_A B = D_A B + \{G(A, B)\zeta - \eta(B)A\}. \quad 5.8$$

With (5.7) & (5.8) M is flat with $\widetilde{\nabla}$.
By (3.1),

$$D^a_{v_1} v_1 = (a-2)v_3, D^a_{v_1} v_3 = v_1, D^a_{v_2} v_2 = (a-2)v_3, D^a_{v_2} v_3 = v_2. \quad 5.9$$

With (5.9), non-zero component of K^2 on M are

$$K^a(v_1, v_3)v_1 = 2(a-2)v_3, K^a(v_1, v_3)v_3 = 2v_1,$$
$$K^a(v_2, v_3)v_2 = 2(a-2)v_3, K^a(v_2, v_3)v_3 = 2v_2. \quad 5.10$$

with (3.6) and (5.9),

$$\widetilde{\nabla}^a_{v_1} v_3 = (a-1)v_1, \widetilde{\nabla}^a_{v_2} v_3 = (a-1)v_2, \widetilde{\nabla}^a_{v_2} v_2 = (a-1)v_3, \widetilde{\nabla}^a_{v_1} v_1 = (a-1)v_3. \quad 5.11$$

Therefore with (5.11),

$$\widetilde{K}^a(v_1, v_2)v_1 = 2(1-a)^2 v_2, \widetilde{K}^a(v_1, v_2)v_2 = 2(1-a)^2 v_1. \quad 5.12$$

With (5.7), (5.10) and (5.12), the non zero components of Ricci tensor Ric, $\widetilde{Ric}, Ric^a$ and $\widetilde{Ric^a}$ are

$$Ric(v_1, v_1) = 4, Ric(v_2, v_2) = 4, Ric(v_3, v_3) = 4. \quad 5.13$$

$$\widetilde{Ric}(v_1,v_1)=8,\widetilde{Ric}(v_2,v_2)=8,\widetilde{Ric}(v_3,v_3)=6. \quad 5.14$$

$$Ric^a(v_1,v_1)=8-4a,Ric^a(v_2,v_2)=8-4a,Ric^a(v_3,v_3)=4. \quad 5.15$$

$$\widetilde{Ric^a}(v_1,v_1)=a^2-6a+4,\widetilde{Ric^a}(v_2,v_2)=a^2-6a+4,\widetilde{Ric^a}(v_3,v_3)=4(1-a^2). \quad 5.16$$

For any $A,B\in\chi(M)$, write

$$A=a_1v_1+a_2v_2+a_3v_3,B=b_1v_1+b_2v_2+b_3v_3.$$

With (2.5) and (5.13) $\sim$ (5.16),gives

$$\begin{aligned}(\tilde{L}^a_{\zeta^a}G^a)(A,B)+2\widetilde{Ric^a}(A,B)+2\delta G^a(A,B)+2\mu\eta^a(A)\eta^a(B)\\ =\{(a-1)+(a^2-6a+4)-a\delta\}a_1b_1\\ +\{(a-1)+(a^2-6a+4)-a\delta\}a_2b_2\\ +\{4(1-a^2)-a^2(\delta+\mu)\}a_3b_3.\end{aligned}$$

Hence $-2a^2+7a-6+a\delta=0$ and $4(1-a^2)-\frac{a^2}{2}(\delta+\mu)=0.$

Therefore $\delta=2a+\frac{7}{2}-\frac{6}{a}$ and $\mu=-2a+\frac{1}{2}+\frac{6}{a}-\frac{4}{a^2}$

so, $\delta+\mu=-\frac{4}{a^2}+4.$

Therefore M admits an η-Ricci soliton ($\left(\zeta,2a+\frac{7}{2}-\frac{6}{a},-2a+\frac{1}{2}+\frac{6}{a}-\frac{4}{a^2},G^a\right)$.

Hence the theorem 6 is verified.

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