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Products on interval valued pythagorean fuzzy soft graphs

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Abstract

This article provides an in-depth exploration of complete Interval-valued Pythagorean fuzzy soft graphs (CIVPFSGs). It presents the definitions for the direct product, strong product, and semi-strong product operations applied to pairs of Interval-valued Pythagorean fuzzy soft graphs (IVPFSGs). Furthermore, it highlights that the strong product of two complete Interval-valued Pythagorean fuzzy soft graphs (CIVPFSGs) maintains completeness. As a result, if the direct product results in strength, it indicates the presence of strength in at least one of the two I-VPFSGs involved. Finally, the paper proceeds to examine the properties associated with the strong product and semi-strong product operations when applied to pairs of IVPFSGs.

Subject Classification: *05C90, 05C72, 62A86.*

Keywords: *Complete interval-valued Pythagorean fuzzy soft graph, Direct product, Semi-strong product, Strong product.*

1. Introduction

Zadeh [13] the notion of FS was introduced to indicate uncertainty or imprecise notions in associating a membership degree with each element, which falls within the range of 0 to 1. The idea of FSs was expanded, leading to the development of the set theory known as "IFSs". In this theory every element is defined by both its degree of member and non-member combined value falling within the range 0 to 1. Yager [8] introduced PFS and they hold greater promise in addressing decision-making challenges compared to IFS. Zhang and Xu [14] The mathematical framework for PFSs, as well as the principles of PFNs, was

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established. Molodtosov [6] The concept of SS theory was introduced as a general mathematical tool for handling uncertainties. In SS theory, the need for defining specific membership functions is eliminated, making the theory highly adaptable across a wide range of disciplines. Maji, Biwas, and Roy [4] The concept of FSSs, a more comprehensive method combining FS and SS, were presented. Presents a multitude of innovative concepts, encompassing SGs and FSGs, and their practical applications. Bhattacharya [3] FGs were discussed, and connectivity theories including fuzzy cut nodes and fuzzy brides were established. To handle such type of situations. Akram and Dudek [1] IVFGs (IVFG) were introduced. Parvathi and Karunambigai [7] explored the notion of IFSGs, with a primary focus on their structure, which, similar to crisp graphs, is heavily influenced by their arcs. Within the realm of IFGs, arcs are categorized into various operations based on their strength. These operations play a pivotal role in the analysis of complete IFSGs and constant IFGs. Mishra and Pal [5] presented by product of I-VIFGs. Akram and Davvaz [2] presented by Strong IFSGs. Yahya Mohamed and Mohamed Ali [9] introduced the notion of an IVPGs and investigates various operations associated with SIVPFGs. Shahzadi and Akram [11,12] concept of IFSGs, which encompasses various novel notions, including complete IFSGs, strong IFSGs, and the self-complement of IFSGs. Introduced by PFSGs using applications. Introduced by IVPFSGs. Sivasamy, Mohammed Jabarulla and Said [10] Introduced by SIVPFSGs is various kinds of methods an PFSGs. This paper presents the concept of complete IVPFSGs and elaborates on the operations for combining two IVPFSGs, namely the direct product, strong product, and Semi-strong product. The document also investigates and evaluates several properties associated with the strong product and semi-strong product of these IVPFSGs.

2. Preface

Definition 2.1: An SIVPFSG above the set V is provided by $\tilde{\xi} = (\xi^*, X, Y, \delta)$ i.e,

i. δ is a collection of attributes,

ii. (X, δ) is a SIVPFSS above the node set V,

iii. (Y, δ) is a SIVPFSS above the link set E,

iv. $(X(e), Y(e))$is a SIVPFSG $\forall\ e \in \delta$.

$$\alpha_{YU}^{+}\ ((\phi, \psi)) = (\alpha_{XU}^{+}\ (\phi)\ \wedge \beta_{XU}^{+}\ (\psi)) \text{ and}$$

$$\beta_{YU}^{+}\ ((\phi, \psi)) = (\beta_{XU}^{+}\ (\phi)\ \vee \beta_{XU}^{+}\ (\psi))$$

$$\alpha_{YL}^{-}\ ((\phi, \psi)) = \ (\alpha_{XL}^{-}\ (\phi)\ \wedge \alpha_{XL}^{-}\ (\psi)) \text{ and}$$

$$\beta_{YL}^{-}\ ((\phi, \psi)) = (\alpha_{XL}^{-}\ (\phi)\ \vee \beta_{XL}^{-}\ (\psi))$$

$$0 \leq\ \alpha_{YU}^{2}\ (\phi, \psi) + \beta_{YU}^{2}(\phi, \psi) \leq 1 \forall\ (\phi, \psi) \in E.$$

3. Main Results

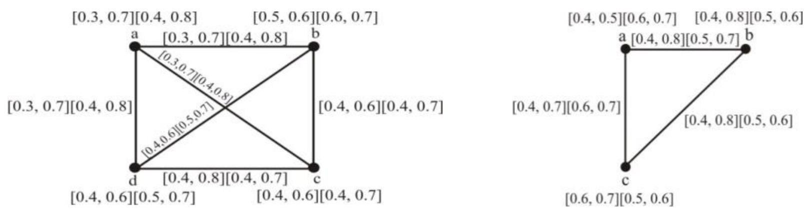
Definition 3.1: An CIVPFSG above the set V is provided by $\tilde{\xi} = (\tilde{\xi}^*, X, Y, \delta)$ i.e,

i. δ is a collection of attributes,
ii. (X, δ) is a CIVPFSS above the node set V,
iii. (Y, δ) is a CIVPFSS above the link set V,
iv. $(X(e), Y(e))$is a CIVPFSG $\forall\ e \in \delta$.

$$\alpha^{+}_{YU}\ ((\phi, \psi)) = (\alpha^{+}_{XU}\ (\phi)\ \wedge \alpha^{+}_{XU}\ (\psi)) \text{ and}$$
$$\beta^{+}_{YU}\ ((\phi, \psi)) = \left(\beta^{+}_{XU}\ (\phi)\ \vee \beta^{+}_{XU}\ (\psi)\right)$$
$$\alpha^{-}_{YL}\ ((\phi, \psi)) = (\alpha^{-}_{XL}\ (\phi)\ \wedge \alpha^{-}_{XL}\ (\psi)) \text{ and}$$
$$\beta^{-}_{YL}\ ((\phi, \psi)) = (\beta^{-}_{XL}\ (\phi)\ \vee \beta^{-}_{XL}\ (\psi))$$

$0 \leq\ \alpha^{2}_{YU}\ \ (\phi, \psi) + \beta^{2}_{YU}(\phi, \psi) \leq 1 \forall\ (\phi, \psi) \in V$. If $\tilde{\xi}^* = (X, Y)$ and $H(e) = (X(e), Y(e))$ an IVPFSG and $\tilde{\xi} = (\tilde{\xi}^*, X, Y, \delta)$ an CIVPFSG.

Example 3.2: If $\tilde{\xi}^*= (X, Y)$ be an elementary graph $X = \{a, b, c, d\}$ and $Y = \{ab, bc, cd, ad\}$. Assume that $\delta = \{e_1, e_2\}$ is an attribute, If (X, δ) be an IVPFS set V provided by $X(e_1)$ and $X(e_2)$ and (Y, δ) be an IVPFS set E provided by $Y(e_1)$ and $Y(e_2)$. It is obvious that $H(e_1) = (X(e_1), Y(e_1)$ see Figure 1 and $H(e_2) = (X(e_2), Y(e_2)$ are IVPFSG in relation to the attributes e_1 and e_2. Therefore $\tilde{\xi} = (\xi^*, X, Y, \delta)$is CIVPFSG.



$H(e_1)$ corresponding to the attribute e_1 $\qquad$ $H(e_2)$ corresponding to the attribute e_2

Figure 1
CIVPFSGs

Definition 3.3: Let X_1 and X_2 be IVPFS subsets of V_1 and V_2 and Y_1 and Y_2 be I-VPFS subsets of E_1 and E_2in their respective cases, and assuming that $V_1 \cap V_2 = \phi$. The direct product of two IVPFSGs $\tilde{\xi}_1$ and $\tilde{\xi}_2$ of the graphs ξ^*_1 and ξ^*_2 is provided by $\tilde{\xi}_1 \sqcap\ \tilde{\xi}_2$: $(X_1 \sqcap X_2, Y_1 \sqcap Y_2)$ with crisp graph $\tilde{\xi}^*$: $(V_1 \times V_2, E)$, where

1. $\left(\alpha_{X_1L} \sqcap \alpha_{X_2L}\right)(\phi, \psi) = \left(\alpha_{X_1L}(\phi)\ \wedge\ \alpha_{X_2L}(\psi)\right)$,
$\left(\alpha_{X_1U} \sqcap \alpha_{X_2U}\right)(\phi, \psi) = \left(\alpha_{X_1U}(\phi) \wedge\ \ \alpha_{X_2U}(\psi)\right)$,

$\left(\beta_{X_1L} \sqcap \beta_{X_2L}\right)(\phi,\psi) = \left(\beta_{X_1L}(\phi) \vee \beta_{X_2L}(\psi)\right)$.

$\left(\beta_{X_1U} \sqcap \beta_{X_2U}\right)(\phi,\psi) = \left(\beta_{X_1U}(\phi) \vee \beta_{X_2U}(\psi)\right), \forall\, (\phi,\psi) \in V_1 \times V_2$.

2. $\left(\alpha_{Y_1L} \sqcap \alpha_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1L}(\phi_1\phi_2) \wedge \alpha_{Y_2L}(\psi_1\psi_2))$,

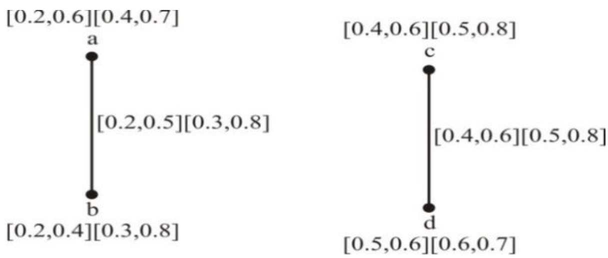
$\left(\alpha_{Y_1U} \sqcap \alpha_{Y_2U}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1U}(\phi_1\phi_2) \wedge \alpha_{Y_2U}(\psi_1\psi_2))$,

$\left(\beta_{Y_1L} \sqcap \beta_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\beta_{Y_1L}(\phi_1\phi_2) \vee \beta_{Y_2L}(\psi_1\psi_2))$,

$\left(\beta_{Y_1U} \sqcap \beta_{Y_2U}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\beta_{Y_1U}(\phi_1\phi_2) \vee \beta_{Y_2U}(\psi_1\psi_2))$.

For all $(\phi_1,\psi_1) \in E_1, (\phi_2,\psi_2) \in E_2$.

Example 3.4: Consider the two IVPFSGs on $V_1 = \{a,b\}$ and $V_2 = \{c,d\}$ given Figure 2. The direct- product of $\tilde{\xi}_1$ and $\tilde{\xi}_2$ regarding in Figure 3 is also I-VPFSG.



$H(e) = (\xi_1, \xi_2)$

Figure 2
IVPFSGs

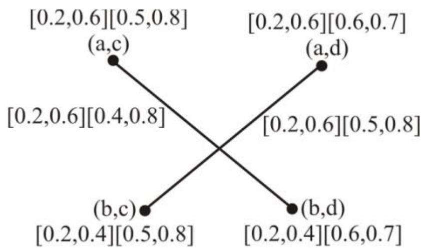


Figure 3
Direct Product

Definition 3.5: Let X_1 and X_2 be IVPFS subsets of V_1 and V_2 and Y_1 and Y_2 be I-VPFS subsets of E_1 and E_2 in their respective cases, and assuming that $V_1 \cap V_2 = \phi$. The str- product of two IVPFSGs $\tilde{\xi}_1$ and $\tilde{\xi}_2$ Concerning the graphs $\tilde{\xi}_1^*$ and $\tilde{\xi}_2^*$ is indicated by $\tilde{\xi}_1 \diamond \tilde{\xi}_2 : (X_1 \diamond X_2, Y_1 \diamond Y_2)$ with crisp graph $\tilde{\xi}^* : (V_1 \times V_2, E)$, where

1. $\left(\alpha_{X_1L} \diamond \alpha_{X_2L}\right)(\phi,\psi) = \left(\alpha_{X_1L}(\phi) \wedge \alpha_{X_2L}(\psi)\right)$,
$\left(\alpha_{X_1U} \diamond \alpha_{X_2U}\right)(\phi,\psi) = \left(\alpha_{X_1L}(\phi) \wedge \alpha_{X_2L}(\psi)\right)$,
$\left(\beta_{X_1L} \diamond \beta_{X_2L}\right)(\phi,\psi) = \left(\beta_{X_1L}(\phi) \vee \beta_{X_2L}(\psi)\right)$,
$\left(\beta_{X_1U} \diamond \beta_{X_2U}\right)(\phi,\psi) = \left(\beta_{X_1U}(\phi) \vee \beta_{X_2U}(\psi)\right), \forall (\phi,\psi) \in V_1 \times V_2$.
2. $\left(\alpha_{Y_1L} \diamond \alpha_{Y_2L}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\alpha_{Y_1L}(\phi) \wedge \alpha_{Y_2L}(\psi_1\psi_2)\right)$,
$\left(\alpha_{Y_1U} \diamond \alpha_{Y_2U}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\alpha_{Y_1U}(\phi) \wedge \alpha_{Y_2U}(\psi_1\psi_2)\right)$,
$\left(\beta_{Y_1L} \diamond \beta_{Y_2L}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\beta_{Y_1L}(\phi) \vee \beta_{Y_2L}(\psi_1\psi_2)\right)$,
$\left(\beta_{Y_1U} \diamond \beta_{Y_2U}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\beta_{Y_1U}(\phi) \vee \beta_{Y_2U}(\psi_1\psi_2)\right)$.
3. $\left(\alpha_{Y_1L} \diamond \alpha_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1L}(\phi_1\phi_2) \wedge \alpha_{Y_2L}(\psi_1\psi_2))$,
$\left(\alpha_{Y_1U} \diamond \alpha_{Y_2U}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1U}(\phi_1\phi_2) \wedge \alpha_{Y_2U}(\psi_1\psi_2))$,
$\left(\beta_{Y_1L} \diamond \beta_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\beta_{Y_1L}(\phi_1\phi_2) \vee \beta_{Y_2L}(\psi_1\psi_2))$,
$\left(\beta_{Y_1U} \diamond \beta_{Y_2U}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\beta_{Y_1U}(\phi_1\phi_2) \vee \beta_{Y_2U}(\psi_1\psi_2))$.

For all$(\phi_1,\psi_1) \in E_1(\phi_2,\psi_2) \in E_2$.

Example 3.6: Consider the two I-VPFSGs see Figure 4 $\tilde{\xi}_1 = (X_1, Y_1)$ and $\tilde{\xi}_2 = (X_2, Y_2)$ on $V_1 = \{a, b\}$ and $V_2 = \{c, d\}$ respectively.

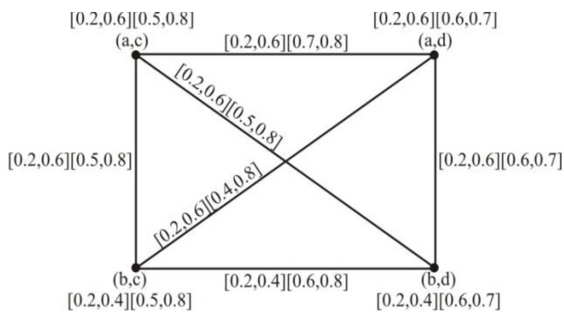


Figure 4
$\boldsymbol{G_1 \diamond G_2}$

Definition 3.7: Let X_1 and X_2 be I-VPFS subsets of V_1 and V_2 and Y_1 and Y_2 be I-VPFS subsets of E_1 and E_2in their respective cases, and assuming that $V_1 \cap V_2 = \phi$. If semi str-product of two I-VPFSGs $\tilde{\xi}_1$ and $\tilde{\xi}_2$ of the graphs $\tilde{\xi}_1^*$ and $\tilde{\xi}_2^*$ is indicated by $\tilde{\xi}_1 \otimes \tilde{\xi}_2 : (X_1 \otimes X_2, Y_1 \otimes Y_2)$ with crisp graph $\tilde{\xi}^* : (V_1 \times V_2, E)$,where

1. $\left(\alpha_{X_1L} \otimes \alpha_{X_2L}\right)(\phi,\psi) = \left(\alpha_{X_1L}(\phi) \wedge \alpha_{X_2L}(\psi)\right)$,
$\left(\alpha_{X_1U} \otimes \alpha_{X_2U}\right)(\phi,\psi) = \left(\alpha_{X_1L}(\phi) \wedge \alpha_{X_2L}(\psi)\right)$,
$\left(\beta_{X_1L} \otimes \beta_{X_2L}\right)(\phi,\psi) = \left(\beta_{X_1L}(\phi) \vee \beta_{X_2L}(\psi)\right)$,

$\left(\beta_{X_1U} \otimes \beta_{X_2U}\right)(\phi,\psi) = \left(\beta_{X_1U}(\phi) \vee \beta_{X_2U}(\psi)\right). \forall (\phi,\psi) \in V_1 \times V_2.$

2. $\left(\alpha_{Y_1L} \otimes \alpha_{Y_2L}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\alpha_{Y_1L}(\phi) \wedge \alpha_{Y_2L}(\psi_1\psi_2)\right),$
$\left(\alpha_{Y_1U} \otimes \alpha_{Y_2U}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\alpha_{Y_1U}(\phi) \wedge \alpha_{Y_2U}(\psi_1\psi_2)\right),$
$\left(\beta_{Y_1L} \otimes \beta_{Y_2L}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\beta_{Y_1L}(\phi) \vee \beta_{Y_2L}(\psi_1\psi_2)\right),$
$\left(\beta_{Y_1U} \otimes \beta_{Y_2U}\right)(\phi,\psi_1)(\phi,\psi_2) = \left(\beta_{Y_1U}(\phi) \vee \beta_{Y_2U}(\psi_1\psi_2)\right).$
3. $\left(\alpha_{Y_1L} \otimes \alpha_{Y_2L}\right)((\phi_1,\rho),(\psi_1,\rho)) = (\alpha_{Y_1L}(\phi_1\psi_1) \wedge \alpha_{Y_2L}(\rho)),$
$\left(\alpha_{Y_1U} \otimes \alpha_{Y_2U}\right)((\phi_1,\rho),(\psi_1,\rho)) = (\alpha_{Y_1U}(\phi_1\psi_1) \wedge \alpha_{Y_2U}(\rho)),$
$\left(\beta_{Y_1L} \otimes \beta_{Y_2L}\right)((\phi_1,\rho),(\psi_1,\rho)) = (\beta_{Y_1L}(\phi_1\psi_1) \vee \beta_{Y_2L}(\rho)),$
$\left(\beta_{Y_1U} \otimes \beta_{Y_2U}\right)((\phi_1,\rho),(\psi_1,\rho)) = (\beta_{Y_1U}(\phi_1\psi_1) \vee \beta_{Y_2U}(\rho)), \forall \rho \in V_2, \phi_1\psi_1 \in E_1.$
4. $\left(\alpha_{Y_1L} \otimes \alpha_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1L}(\phi_1\phi_2) \wedge \alpha_{Y_2L}(\psi_1\psi_2)),$
$\left(\alpha_{Y_1U} \otimes \alpha_{Y_2U}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1U}(\phi_1\phi_2) \wedge \alpha_{Y_2U}(\psi_1\psi_2)),$
$\left(\beta_{Y_1L} \otimes \beta_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\beta_{Y_1L}(\phi_1\phi_2) \vee \beta_{Y_2L}(\psi_1\psi_2)),$
$\left(\beta_{Y_1U} \otimes \beta_{Y_2U}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\beta_{Y_1U}(\phi_1\phi_2) \vee \beta_{Y_2U}(\psi_1\psi_2)),$

For all $(\phi_1,\psi_1) \in E_1$, $(\phi_2,\psi_2) \in E_2$.

Example 3.8: Consider the two IVPFSGs on $V_1 = \{a,b\}$ and $V_2 = \{c,d\}$ respectively see Figure 5.

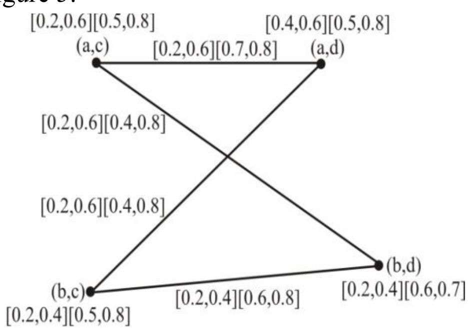


Figure 5
$\boldsymbol{\xi_1 \otimes \xi_2}$

Theorem 3.9: *If $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are SIVPFSGs then $\tilde{\xi}_1 \sqcap \tilde{\xi}_2$ is SIVPFSG.*

Proof: If $((\phi_1,\psi_1)(\phi_2,\psi_2)) \in E$, Since $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are strong we have,
$\left(\alpha_{Y_1L} \sqcap \alpha_{Y_2L}\right)(\phi_1,\psi_1)(\phi_2,\psi_2) = (\alpha_{Y_1L}(\phi_1\phi_2) \wedge \alpha_{Y_2L}(\psi_1\psi_2)),$
$= ((\alpha_{X_1L}(\phi_1)) \wedge (\alpha_{X_1L}(\phi_2)) \wedge (\alpha_{X_2L}(\psi_1)) \wedge (\alpha_{X_2L}(\psi_2))),$
$= \wedge\, (\alpha_{X_1L}(\phi_1)), (\alpha_{X_1L}(\phi_2)), (\alpha_{X_2L}(\psi_1)), (\alpha_{X_2L}(\psi_2)),$
$= ((\alpha_{X_1L}(\phi_1)) \wedge (\alpha_{X_2L}(\psi_1)) \wedge (\alpha_{X_1L}(\phi_2)) \wedge (\alpha_{X_2L}(\psi_2))),$
$= ((\alpha_{X_1L} \sqcap \alpha_{X_2L}(\phi_1,\psi_1) \wedge (\alpha_{X_1L} \sqcap \alpha_{X_2L})(\phi_2,\psi_2)).$

$$\begin{aligned}(\alpha_{Y_1U} \sqcap \alpha_{Y_2U})(\phi_1,\psi_1)(\phi_2,\psi_2) &= (\alpha_{Y_1U}(\phi_2) \wedge \alpha_{Y_2U}(\psi_1\psi_2)),\\ &= ((\alpha_{X_1U}(\phi_1)) \wedge (\alpha_{X_1U}(\phi_2)) \wedge (\alpha_{X_2U}(\psi_1)) \wedge (\alpha_{X_2U}(\psi_2))),\\ &= \wedge\, (\alpha_{X_1U}(\phi_1)), (\alpha_{X_1U}(\phi_2)), (\alpha_{X_2U}(\psi_1)), (\alpha_{X_2U}(\psi_2)),\\ &= ((\alpha_{X_1U}(\phi_1)) \wedge (\alpha_{X_2U}(\psi_1)) \wedge (\alpha_{X_1U}(\phi_2)) \wedge (\alpha_{X_2U}(\psi_2))),\\ &= ((\alpha_{X_1U} \sqcap \alpha_{X_2U}(\phi_1,\psi_1) \wedge (\alpha_{X_1U} \sqcap \alpha_{X_2U})(\phi_2,\psi_2)).\end{aligned}$$

Hence, $\tilde{\xi}_1 \sqcap \tilde{\xi}_2$ is SIVPFSG.

Theorem 3.10: *If $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are SIVPFSGs then at least $\tilde{\xi}_1$ or $\tilde{\xi}_2$ must be SIVPFSG.*

Proof: Suppose that $\tilde{\xi}_1$ and $\tilde{\xi}$ not being SIVPFSGs, there are $\phi_i, \psi_i \in E_i, i = 1,2$, such that

$$\begin{aligned}\alpha_{Y_{iL}}((\phi_i,\psi_i)) &< (\alpha_{X_{iL}}(\phi_i) \wedge \alpha_{X_{iL}}(\psi_i)),\\ \alpha_{Y_{iU}}((\phi_i,\psi_i)) &< (\alpha_{X_{iU}}(\phi_i) \wedge \alpha_{X_{iU}}(\psi_i)),\\ \beta_{Y_{iL}}((\phi_i,\psi_i)) &< (\beta_{X_{iL}}(\phi_i) \vee \beta_{X_{iL}}(\psi_i)),\\ \beta_{Y_{iU}}((\phi_i,\psi_i)) &< (\beta_{X_{iU}}(\phi_i) \vee \beta_{X_{iU}}(\psi_i)).\end{aligned}$$

Consider $((\phi_1,\psi_1)(\phi_2,\psi_2) \in E$, we have

$$\begin{aligned}(\alpha_{Y_1L} \sqcap \alpha_{Y_2L})(\phi_1,\psi_1)(\phi_2,\psi_2) &= (\alpha_{Y_1L}(\phi_1\phi_2) \wedge \alpha_{Y_2L}(\psi_1\psi_2)),\\ &< \wedge\, (\alpha_{X_1L}(\phi_1)), (\alpha_{X_1L}(\phi_2)), (\alpha_{X_2L}(\psi_1)), (\alpha_{X_2L}(\psi_2)),\end{aligned}$$

Similarly,

$$\begin{aligned}(\alpha_{Y_1U} \sqcap \alpha_{Y_2U})(\phi_1,\psi_1)(\phi_2,\psi_2) &= (\alpha_{Y_1U}(\phi_1\phi_2) \wedge \alpha_{Y_2U}(\psi_1\psi_2)),\\ &< \wedge\, (\alpha_{X_1U}(\phi_1)), (\alpha_{X_1U}(\phi_2)), (\alpha_{X_2U}(\psi_1)), (\alpha_{X_2U}(\psi_2)).\end{aligned}$$

Hence, $\tilde{\xi}_1 \sqcap \tilde{\xi}_2$ is not SIVPFSG. Which is a contradiction.
Hence the proof.

Theorem 3.11: If $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are CIVPFSGs then $\tilde{\xi}_1 \otimes \tilde{\xi}_2$ is CIVPFSG.

Proof: If $((\phi,\psi_1),(\phi,\psi_2)) \in E$, then

$$\begin{aligned}(\alpha_{Y_1L} \otimes \alpha_{Y_2L})((\phi,\psi_1),(\phi,\psi_2)) &= (\alpha_{Y_1L}(\phi) \wedge \alpha_{Y_2L}(\psi_1\psi_2)),\\ &= (\alpha_{Y_1L}(\phi) \wedge (\alpha_{Y_2L}(\psi_1) \wedge \alpha_{Y_2L}(\psi_2)),\\ &= \wedge\, (\alpha_{Y_1L}(\phi), \alpha_{Y_2L}(\psi_1), \alpha_{Y_2L}(\psi_2)),\\ &= ((\alpha_{Y_1L}(\phi) \wedge \alpha_{Y_2L}(\psi_1)) \wedge (\alpha_{Y_1L}(x) \wedge \alpha_{Y_2L}(\psi_2))),\\ &= ((\alpha_{Y_1L} \otimes \alpha_{Y_2L})(\phi,\psi_1) \wedge (\alpha_{Y_1L} \otimes \alpha_{Y_2L})(\phi,\psi_2)).\end{aligned}$$

$$\begin{aligned}(\alpha_{Y_1U} \otimes \alpha_{Y_2U})((\phi,\psi_1),(\phi,\psi_2)) &= (\alpha_{Y_1U}(\phi) \wedge \alpha_{Y_2U}(\psi_1\psi_2)),\\ &= (\alpha_{Y_1U}(\phi) \wedge (\alpha_{Y_2U}(\psi_1) \wedge \alpha_{Y_2U}(\psi_2))),\\ &= \wedge\, (\alpha_{Y_1U}(\phi), \alpha_{Y_2U}(\psi_1), \alpha_{Y_2U}(\psi_2)),\end{aligned}$$

$$= ((\alpha_{Y_1U}(\phi) \wedge \alpha_{Y_2U}(\psi_1)) \wedge (\alpha_{Y_1U}(\phi) \wedge \alpha_{Y_2U}(\psi_2))),$$
$$= ((\alpha_{Y_1U} \otimes \alpha_{Y_2U})(\phi, \psi_1) \wedge (\alpha_{Y_1U} \otimes \alpha_{Y_2U})(\phi, \psi_2)).$$

If $((\phi_1, \rho), (\phi_2, \rho)) \in E$, Since $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are complete,

$$(\alpha_{Y_1L} \otimes \alpha_{Y_2L})((\phi_1, \rho), (\phi_2, \rho)) = (\alpha_{Y_1L}(\rho) \wedge \alpha_{Y_2L}(\phi_1\phi_2)),$$
$$= (\alpha_{Y_1L}(\rho) \wedge (\alpha_{Y_2L}(\phi_1) \wedge \alpha_{Y_2L}(\psi_2))),$$
$$= \wedge \left(\alpha_{Y_1L}(\rho), \alpha_{Y_2L}(\phi_1), \alpha_{Y_2L}(\psi_2)\right),$$
$$= ((\alpha_{Y_1L}(\rho) \wedge (\alpha_{Y_2L}(\phi_1)) \wedge (\alpha_{Y_1L}(\rho) \wedge \alpha_{Y_2L}(\phi_2))),$$
$$= ((\alpha_{Y_1L} \otimes \alpha_{Y_2L})(\rho, \phi_1) \wedge (\alpha_{Y_1L} \otimes \alpha_{Y_2L})(\rho, \phi_2)).$$

$$(\alpha_{Y_1U} \otimes \alpha_{Y_2U})((\phi_1, \rho), (\phi_2, \rho)) = (\alpha_{Y_1U}(\rho) \wedge \alpha_{Y_2U}(\phi_1\phi_2)),$$
$$= (\alpha_{Y_1U}(\rho) \wedge (\alpha_{Y_2U}(\phi_1) \wedge \alpha_{Y_2U}(\phi_2))),$$
$$= \wedge (\alpha_{Y_1U}(\rho), \alpha_{Y_2U}(\phi_1), \alpha_{Y_2U}(\phi_2)),$$
$$= ((\alpha_{Y_1U}(\rho) \wedge (\alpha_{Y_2L}(\phi_1)) \wedge (\alpha_{Y_1U}(\rho) \wedge \alpha_{Y_2U}(\phi_2))),$$
$$= ((\alpha_{Y_1U} \otimes \alpha_{Y_2U})(\rho, \phi_1) \wedge (\alpha_{Y_1U} \otimes \alpha_{Y_2U})(\rho, \phi_2)).$$

If $((\phi_1, \psi_1)(\phi_2, \psi_2)) \in E$, Since $\tilde{\xi}_1$and $\tilde{\xi}_2$ are complete,

$$(\alpha_{Y_1L} \otimes \alpha_{Y_2L})\left((\phi_1, \psi_1)(\phi_2, \psi_2)\right) = (\alpha_{Y_1L}(\phi_1\phi_2) \wedge \alpha_{Y_2L}(\psi_1\psi_2)),$$
$$= ((\alpha_{Y_1L}(\phi_1) \wedge \left(\alpha_{Y_1L}(\phi_2)\right) \wedge \left(\alpha_{Y_2L}(\psi_1) \wedge \alpha_{Y_2L}(\psi_2)\right)),$$
$$= \wedge (\alpha_{Y_1L}(\phi_1), \alpha_{Y_1L}(\phi_2), \alpha_{Y_2L}(\psi_1), \alpha_{Y_2L}(\psi_2)),$$
$$= ((\alpha_{Y_1L}(\phi_1) \wedge \alpha_{Y_2L}(\psi_1)) \wedge ((\alpha_{Y_1L}(\phi_2) \wedge \alpha_{Y_2L}(\psi_2))),$$
$$= (\alpha_{Y_1L} \otimes \alpha_{Y_2L})(\phi_1, \psi_1) \wedge (\alpha_{Y_1L} \otimes \alpha_{Y_2L})(\phi_2, \psi_2).$$

$$(\alpha_{Y_1U} \otimes \alpha_{Y_2U})\left((\phi_1, \psi_1)(\phi_2, \psi_2)\right) = (\alpha_{Y_1U}(\phi_1\phi_2) \wedge \alpha_{Y_2U}(\psi_1\psi_2)),$$
$$= ((\alpha_{Y_1U}(\phi_1) \wedge \left(\alpha_{Y_1L}(\phi_2)\right) \wedge \left(\alpha_{Y_2U}(\psi_1) \wedge \alpha_{Y_2U}(\psi_2)\right)),$$
$$= \wedge (\alpha_{Y_1U}(\phi_1), \alpha_{Y_1U}(\phi_2), \alpha_{Y_2U}(\psi_1), \alpha_{Y_2U}(\psi_2)),$$
$$= ((\alpha_{Y_1U}(\phi_1) \wedge \alpha_{Y_2U}(\psi_1)) \wedge ((\alpha_{Y_1U}(\phi_2) \wedge \alpha_{Y_2U}(\psi_2))),$$
$$= (\alpha_{Y_1U} \otimes \alpha_{Y_2U})(\phi_1, \psi_1) \wedge (\alpha_{Y_1U} \otimes \alpha_{Y_2U})(\phi_2, \psi_2).$$

4. Conclusion

This paper's goal is to determine the suitability of direct product, semi-strong product, and strong product operations when applied to two IVPFSGs, demonstrating that the result of these operations remains within the realm of IVPFSGs. Consequently, if the direct product yields a strong product, it implies that at least one of the two IVPFSGs involved is strong. Similarly, if the semi-strong product results in a strong product, it signifies that at least one of the two IVPFSGs is strong as well. Furthermore, we provide evidence supporting the notion that the strong product of two complete IVPFSGs preserves completeness.

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