

On almost η -Ricci soliton of Sasakian manifolds

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Abstract

In this Paper, Properties of solitons on Sasakian manifolds with Da-homothetic deformation and Schouten-van Kampen connection are discussed.

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1. Introduction

Solov'ev studied properties of hyperdistribution in Riemannian manifold with Schouten-van Kampen connection[1, 2, 3]. Then Schouten-van Kampen connection are studied by different authors with different structures[4, 5, 6, 7]. Da -homothetic deformation is defined by taking $\theta = 0$ in (M, J, ζ, θ, g) [8].

η -Ricci soliton is defined as[13]

$$L_A g + 2Ric + 2\delta g + 2\mu\eta \otimes \eta = 0, \quad 1.1$$

2. Preliminaries

An almost contact metric structure (J, ζ, θ, g) satisfies [9, 11, 16, 17]

$$J^2(A) = -A + \theta(A)\zeta, \theta(\zeta) = 1, J\zeta = 0, \theta o J = 0, \quad 2.1$$

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$$g(JA, JB) = g(A, B) - \theta(A)\theta(B). \quad 2.2$$

Sasakian manifold satisfies [10]

$$(D_A, J)B = g(A, B)\zeta - \theta(B)A. \quad 2.3$$

Equivalently

$$D_A \zeta = -JA(D_A \theta) = -g(JA, B). \quad 2.4$$

Da-homothetic deformation is defined by taking $\theta = 0$ in (M, J, ζ, θ, g) . Hence for Da-homothetic deformation [6, 8],

$$\theta^a = a\theta, \zeta^a = \frac{1}{a}\zeta, J^a = J, g^a = ag + a(a-1)\theta \otimes \theta, \quad 2.5$$

Schouten-van Kampen connection $\tilde{\nabla}$ is defined by,

$$\tilde{\nabla}_A B = D_A B - \theta(B)\nabla_A \zeta + (\nabla_A \theta)(B)\zeta. \quad 2.6$$

3. Da-Homothetic deformation

By simple calculation we have,

Lemma 1: For Sasakian manifold M and Da-homothetic deformation on M and D^a and D are related by

$$D_A^a B = D_A B - (a-1)\{\theta(A)JB + \theta(B)JA\}. \quad 3.1$$

By this relation, the relation between K^a and K is given by Theorem 1 as follows:

Theorem 1: For Sasakian manifold M and Da-homothetic deformation on M the curvature tensor K^a is

$$\begin{aligned} K^a(A, B)W &= K(A, B)W + (a-1)\{2g(JA, B)JW - g(A, JW)JB \\ &+ g(B, JW)JA - g(A, W)\theta(B)\zeta + g(B, W)\theta(A)\zeta - 2\theta(A)\theta(W)B \\ &+ 2\theta(B)\theta(W)A\} - (a-1)^2\{\theta(A)\theta(W)B - \theta(B)\theta(W)A\}. \end{aligned} \quad 3.2$$

By (3.2) we have

$$Ric^a(B, W) = Ric(B, W) + 2(1-a)g(B, W) + (2a^2 + 2a - 4)\theta(B)\theta(W). \quad 3.3$$

Here Ric^a and Ric and Ricci tensor with connection D^a and D .

Replacing B and W by $e_i (i=1, 2, 3)$ in (3.4) as e_i is a tangent space orthonormal basis at each point on the manifold, we have,

$$scal^a = scal + 2(a-1)^2. \quad 3.4$$

Here $scal^a$ and $scal$ are scalar curvature with respect to D^a and D .

Lemma 2: For Sasakian manifold M and Da -homothetic deformation on M , the connections $\tilde{\nabla}^a$ is

$$\tilde{\nabla}_A B = D_A B - (a-1)\theta(A)JB + g(A, JB)\zeta + \theta(B)JA. \quad 3.5$$

Theorem 2: For Sasakian manifold M and Da -homothetic deformation on M and with Schouten -van Kampen connection $\tilde{K}$ is

$$\begin{aligned} \tilde{K}^a(A, B)W &= K(A, B)W + 2(a-1)g(JA, B)JW + g(A, JW)JB - g(B, JW)JA \\ &\quad + g(A, W)\theta(B)\zeta - g(B, W)\theta(A)\zeta + \theta(A)\theta(W)B - \theta(B)\theta(W)A. \end{aligned} \quad 3.6$$

By (3.6) we have

$$\tilde{Ric}^a(B, W) = Ric(B, W) + 2(1-a)g(B, W) + 2(a-2)\theta(B)\theta(W) + (a-1)g(B, JW), \quad 3.7$$

where $\tilde{Ric}^a$ and Ric are Ricci tensor with $\tilde{\nabla}$ and D .

The relation involving scalar curvature and Ricci tensor is [1, 2, 3, 4, 5, 6, 7, 8, 12, 14],

$$Ric(A, B) = \left(\frac{scal}{2} - 1\right)g(A, B) - \left(\frac{scal}{2} - 3\right)\theta(A)\theta(B). \quad 3.8$$

With (3.8),(3.7) becomes,

$$\begin{aligned} \tilde{Ric}^a(B, W) &= \left\{\frac{scal}{2} + (1-2a)\right\}g(B, W) - \left\{\frac{scal}{2} + (1-2a)\right\}\theta(W)\theta(B) \\ &\quad + (a-1)g(B, JW). \end{aligned} \quad 3.9$$

Replacing B and W by $e_i (i=1, 2, 3)$ in (3.9) as e_i is a tangent space orthonormal basis at each point on the manifold, we have,

$$\widetilde{scal}^a = scal + (-4a + 2) \quad 3.10$$

where $\widetilde{scal}^a$ and $scal$ are scalar curvature with respect to $\tilde{\nabla}$ and D .

4. Solitons

Definition 1: η -Einstein manifold is a manifold with Ricci tensor[15],

$$Ric = \alpha g + \beta \eta \otimes \eta,$$

Definition 2: If the Ricci tensor of M has the following structure,

$$\widetilde{Ric}^a = \alpha g^a + \beta \eta^a \otimes \eta^a,$$

then manifold is known as η^a -Einstein manifold.

On a Sasakian manifold with $\widetilde{\nabla}^a$, equation (1.1), becomes,

$$(\widetilde{L}_U g^a + 2\widetilde{Ric}^a + 2\delta g^a + 2\mu \theta^a \otimes \theta^a)(A, B) = 0. \quad 4.1$$

Replacing U by ζ^a in (4.1) and with (2.4), (2.6), (3.1), (3.5), and (3.9), we have

$$\widetilde{Ric}^a(A, B) = -\delta g^a(A, B) + \mu \theta^a(A) \theta^a(B). \quad 4.2$$

Therefore,

Theorem 3: A Sasakian manifold M with $\widetilde{\nabla}^a$ and almost η -Ricci soliton $(\zeta, \delta, \mu, g^a)$ is an η^a Einstein manifold.

$$a^2(\delta + \mu) = 0 \Rightarrow \delta + \mu = 0. \quad 4.3$$

Again replacing A and B by ζ in (4.2), we get Therefore,

Theorem 4: An almost η -Ricci soliton on a sasakian manifold with $\widetilde{\nabla}^a$ is $(\zeta^a, \delta, -\delta, g^a)$.

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