*Journal of Interdisciplinary Mathematics*
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 5, pp. 1039–1052
DOI : 10.47974/JIM-1931

# Modular chromatic number on inflated graphs of some tree graphs

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## Abstract

For any graph $G(V, E)$, $V(G)$ is the vertex set and $E(G)$ is the edge set. Let the vertex coloring be called modular coloring $m_c(G)$ of $G$ if any two adjacent vertices receive different color sums, that is, $S(u) \neq S(v)$ for $u, v \in E(G)$ here $S(u) = \sum_{v \in N(u)} C(v_{ij})$. In this paper, the structural properties of an inflated tree-related graph were discussed, and in addition, the modular chromatic number for the same was determined.

***Subject Classification:*** *05C15, 05C76, O5C63.*

***Keywords:*** *$k_{1,n}$ - Star graph, $k_{1,n,n}$-double star graph, $B_{m,n}$ - bi-star graph, $C_n$ - coconut tree graph, Banana tree graph, Perfect binary tree, $p_n \odot k_{1,m}$, $k_{1,n} \odot k_{1,m}$, Inflated graphs, Modular coloring, Modular chromatic number.*

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## 1. Introduction

Graph theory plays a vital role in applied mathematics; coloring is most important in networking, scheduling, etc. Modular coloring has a broad range of applications in networking, like avoiding more connections at a single point to avoid connecting time; an inflated graph is helpful to expand the single node to multiple nodes depending on its adjacency. Using modular coloring, we can minimize the number of connections for the maximum number of vertices involved. The graph $G$ with the vertex set $V(G)$ and the edge set $E\,(G)$ considered in this paper is simple, non-trivial, undirected, and connected. Let $G$ be an isolated vertex-free graph. In [1] the modular coloring is a vertex coloring $C\,:\,\mathcal{V}(G) \rightarrow\, \mathbb{Z}_n, n \geq 2$ (not a perfect vertex coloring). The coloring is called a modular sum k-coloring for $\mathbb{Z}_k$, if $\mathcal{S}(\alpha) \neq\, \mathcal{S}(\beta)$ for $\alpha, \beta \in E(G)$, let the color sum is defined by $\mathcal{S}(\alpha) = \sum_{u \in N(\alpha)} c(u)$. Let $N(\alpha)$ is the neighborhoods of $\alpha$ (set of all adjacent vertices to $\alpha$), for all $\alpha \in G$. The modular chromatic number (MCN) $m_c(G)$ of $G$ is the minimum $k$ for which $G$ has a modular sum $k$-coloring. In [2] The inflation or inflated graph $IG$ of a graph $G$ without isolated vertices is obtained as follows; each vertex $x_i$ of degree $d(x_i)$ of $G$ is replaced by a clique $X_i \cong\, \mathrm{K}d(x_i)$ and each edge $X_iX_j$ of $G$ is replaced by an edge $uv$ in such a way that $u \in X_i$, $v \in X_j$, and two different edges of $G$ are replaced by non-adjacent edges of $IG$. Here after in this paper $i$: $[0, n]$ denotes $i$ varies from $0\ to\ n$ (the positive integer). In [5, 6, 7], the fundamental definitions of coconut tree graph, banana tree graph, and binary tree graph are given. In [1] F. Okamoto, E. Salehi, and P. Zhang introduced modular coloring in 2010. In [8] P. Sumathi, S. Tamilselvi, found the modular chromatic number of certain cyclic graphs. In [9], we investigated the modular chromatic number of the inflated graphs of the wheel, gear, fan, friendship, and flower graphs. In [10], also we found the MCN of the corona product of a generalized Jahangir graph. In this paper, we determined the modular chromatic number of an inflated graph of tree- related graphs.

## 2. Main result

*Inflated graph of graph $G$*

In [2] A non- trivial connected graph $G$ is said to be inflated graph $I(G)$ if $\mathcal{V}$: $x_i\ \rightarrow$ clique$(X_i) \cong k_{d(x_i)}$. Each edge $x_ix_j$ of $G$ is retrieved in to an edge $v_iv_j$ in such a way that $v_i \in X_i$, $v_j \in X_j$, and any two adjacent edges of $G$ be converted to non-adjacent edges of $I(G)$ by including an edge.

**Theorem 3.1:** *Let $I(k_{1,n})$ denote the inflated graph of the star graph of $|V, E| = (2n, nC_2 + n)$, where $n \geq 3$, then $m_c(I(k_{1,n})) = n$.*

**Proof:** Let $I\big(k_{1,n}\big)$ be an inflated graph of stars on $n \geq 3$, and $\mathcal{V}(I(k_{1,n})) = \{v_0, v_1, \dots, v_{n-1}\} \cup \{u_0, u_1, \dots, u_{n-1}\}$. Let $\left|\mathcal{V}(I(k_{1,n}))\right| = 2n$ and $\left|E\big(I(k_{1,n})\big)\right| = nC_2 + n$. These two cases encompass all possible scenarios for the theorem and provide a comprehensive understanding of its implications.

**Case 1:** When n is odd, and $i$ is varies from $0\ to\ n-1$

Every complete graph of order $n$ has a modular chromatic number $n$. The inflated graph of star $I(k_{1,n})$ has a complete graph $k_n$ as an induced sub graph, that is $m_c(I(k_{1,n})) \geq n$.

Let the coloring for the graph be defined by $C: \mathcal{V} \rightarrow \mathbb{Z}_{n-1}^{+}$ as follows,

$$C(u_i) = i, \text{ and } C(v_i) = (2i+1)(mod\ n).$$

Let the modular coloring of the inflated graph of star $I(k_{1,n})$ is

$$\mathcal{S}(u_i) = (i+1)(mod\ n) \text{ and } \mathcal{S}(v_i) = i,$$

It gives the modular n- coloring; every adjacent vertex receives a different color sum. Thus $m_c\left(I(k_{1,n})\right) = n.$

**Case 2:** When n is even, and $i$ is varies from $0\ to\ n-1$

The inflated graph of star $I(k_{1,n})$ has a complete graph as an induced sub graph, every complete graph of order n has a modular chromatic number n. That is $m_c\left(I(k_{1,n})\right) \geq n$.

Let the coloring for the graph is defined by $C: \mathcal{V} \rightarrow \mathbb{Z}_{n-1}^{+}$ as follows,

$$C(u_i) = i, \text{ and } C(v_i) = \tfrac{1}{2}(n+4i+2)(mod\ n).$$

Let the modular coloring of inflated graph of star $I(k_{1,n})$ is

$$\mathcal{S}(u_i) = (1+i)(mod\ n), \text{ and } \mathcal{S}(v_i) = i.$$

It gives the modular $n$ – coloring; every adjacent vertex receives a different color sum. Thus $m_c\left(I(k_{1,n})\right) = n.$

**Theorem 3.2:** *Let $I(k_{1,n,n})$ denote the inflated graph of the double star graph of order $4n$ and size $nC_2 + 3n$, ( $n \geq 3$), then $m_c(I(k_{1,n})) = n$.*

***Proof:*** In a star graph each pendant vertex join by a pendant edge is named as the double star graph $K_{1,n,n}$[3]. The graph $K_{1,n,n}$ has $|V, E|$= ( $2n+1$, $2n$). The vertex set of $K_{1,n,n}$ is defined as $\{u\} \cup \{v_i\text{: i ranges from zero to n}\} \cup \{w_i\text{: i ranges from one to n}\}$. An inflated graph of double star graph on $n \geq 3$ is denoted by $I(k_{1,n,n})$, and the vertex set is defined as $\{u_i \cup v_i \cup w_i \cup x_i\ for\ all\ i\ is\ from\ zero\ to\ n-1\}$, and cardinality of vertex and edge set are $4n$ and $nC_2 + 3n$ respectively. Inflated graph of double star $I(k_{1,n,n})$ has complete graph $k_n$ as induced sub graph, every complete graph of order n has modular chromatic number n. That is $m_c(I(k_{1,n,n})) \geq n$.

Let the coloring for the graph be defined by $C: \mathcal{V} \rightarrow \mathbb{Z}_{n-1}^{+}$as follows:

When $i$ is varies from $0\ to\ n-1$

$$C(u_i) = 0\ \forall i\ , C(v_i) = i(mod\ n),$$

$$C(w_i) = i + 1(mod\ n),\ C(x_i) = 2\ \forall i.$$

Let the modular coloring of the inflated graph of the double star $I(k_{1,n,n})$ be

$$\mathcal{S}(u_i) = i,\ \mathcal{S}(v_i) = i + 1(mod\ n), \text{ and}$$
$$\mathcal{S}(w_i) = i + 2(mod\ n),\ \mathcal{S}(x_i) = i + 1(mod\ n).$$

It gives the modular $n$ – coloring; every adjacent vertex receives a different color sum. Thus $m_c\left(I(k_{1,n,n})\right) = n$.

**Theorem 3.3:** For any integer $s, t \geq 2$, $m_c(I(B_{s,t})) = \max\{s + 1, t + 1\}$.

**Proof:** In [4]. A graph G is said to be a bi-star graph $B_{m,n}$ if an edge adjacent with $m, n$ pendant vertices. The vertex set of $B_{m,n}$ is $\mathcal{V}(B_{m,n}) = \{u\ \cup v\ \cup u_i\ \cup v_j : i = 1,2, \dots, m, j = 1, \dots, n\}$, and $E(B_{m,n}) = \{uu_1, uu_2, \dots, uu_n, uv, vv_1, vv_2, \dots, vv_n\}$. Let $I(B_{s,t})$ be an inflated graph of bi-stars on $s, t \geq 2$. Let vertices of $I(B_{s,t})$ denoted by $\mathcal{V}(I(B_{s,t})) = \{(v_i \cup u_i) : \text{for some } i \text{ is from } 0\ to\ (s + 1)\} \cup \{(v_i' \cup u_i') : i\ is\ from\ 0\ to\ (t + 1)\}$. And the edges of $I(B_{s,t})$ denoted by $E(I(B_{s,t}))$=$\{e_1, e_2, \dots, e_{sC_2}\} \cup \{x_0, x_1, \dots, x_{s-1}\} \cup \{e'_1, e'_2, \dots, e'_{tC_2}\} \cup \{x'_1, \dots, x'_{t-1}\}$.

Here, the graph $I(B_{s,t})$ is isomorphic to $I(B_{t,s})$. Inflated graph of bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as an induced sub graph, we know that every complete graph of order $n$ has a modular chromatic number $n$. By addressing the following three cases, we can prove the theorem easily

**Case 1:** When $s$ and $t$ are odd,

**Subcase 1.1** $\boldsymbol{s = t}$**,**

Since the inflated graph of bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as an induced sub graph, we know that every complete graph of order $n$ has a modular chromatic number $n$. That gives the MCN of $I(B_{s,t})$ is at least $(s + 1)$, $when\ s = t$.

Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_s^+$ as follows,

For all $j$ ranges from 0 to $s$; $C(v_i) = \begin{cases} 0 & \text{if } i \text{ is } 0 \\ \frac{1}{2}(s) & \text{otherwise} \end{cases}$,

For all $j$ ranges from 1 to $s - 1$; $C(u_j) = \begin{cases} 0 & \text{if } j \text{ is } \frac{1}{2}(s) \\ j & \text{otherwise} \end{cases}$,

For every $i$ and $j$, $C(v'_i) = 0$, and $C(u'_j) = j$.

Let the modular coloring of the inflated graph of bi-star $I(B_{s,t})$ be

For all $j$ ranges from 0 to $s$; $\mathcal{S}(v_i) = \begin{cases} \frac{1}{2}(s) & \text{if } i \text{ is } 0 \\ 0 & \text{if } i \text{ is } \frac{1}{2}(s) \\ i & \text{otherwise} \end{cases}$,

For all $j$ ranges from 1 to $-1$ $\mathcal{S}(u_j) = \begin{cases} 0 & \text{if } j \text{ is } 0 \\ \frac{s}{2} & \text{otherwise} \end{cases}$,

$\mathcal{S}(v'_i) = i$,for every $i$ $\mathcal{S}(u'_j) = \begin{cases} \frac{s}{2} & \text{if } j \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}$.

**Subcase 1.2** $\boldsymbol{s \neq t}$ **when** $\boldsymbol{s < t}$

We know that every complete graph of order $n$ has a modular chromatic number $n$. Since Inflated graph of bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as an induced sub graph. That gives the modular coloring of $I(B_{s,t})$ is at least $(t+1)$, when $s < t$.

Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_t^+$ as follows,

When $i$ & $j$ is varies from 0 to $s$;

$$C(v_i) = \begin{cases} 0 & \text{if } i \text{ is } 0 \\ \frac{t}{2} & \text{otherwise} \end{cases}, C(u_j) = \begin{cases} 0 & \text{if } j \text{ is } s \\ j & \text{otherwise} \end{cases},$$

When $i$ is varies from 0 to $t$; $C(v'_i) = 0$, and When $j$ is varies from 1 to $t$; $C(u'_j) = j$.

Let the modular coloring of the inflated graph of bi-star $I(B_{s,t})$ be

$$\mathcal{S}(v_i) = \begin{cases} \frac{t}{2} & \text{if } i \text{ is } 0 \\ i & \text{if } i: [1, \frac{s}{2}) \\ 0 & \text{if } i \text{ is } \frac{s}{2} \\ i & \text{if} (\frac{s}{2}, s] \end{cases}, \mathcal{S}(u_j) = \frac{t}{2}, if\ j: [1, s],$$

When $i$ & $j$ is varies from 1 to $t$; $\mathcal{S}(v'_i) = i$, and $\mathcal{S}(u'_j) = 0$.

**Subcase 1.3** $\boldsymbol{s \neq t}$ **where** $\boldsymbol{s > t}$

Here the inflated graph of bi-star $I(B_{s,t})$ is isomorphic to $I(B_{t,s})$, so that we can follow the result from the previous case.

**Case 2:** When $s$ and $t$ is even,

**Subcase 2.1** $\boldsymbol{s = t}$**,**

Since Inflated graph of Bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as induced sub graph, we know that every complete graph of order $n$ has

MNC is $n$. That gives the modular coloring of $I(B_{s,t})$ is at least $(s+1)$, $when\ s = t$.

Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_s^+$ as follows,

$$C(v_i) = i, \text{for all } i \text{ ranges from 0 to } s,$$

When $i$ & $j$ varies from 1 to $s$; $C(u_j) = (1+2j)\ mod\ (s+1)$;

$C(v'_i) = (1+i) mod\ (s+1)$; $C(u'_j) = 2i\ mod\ (s+1),\ if\ j{:}\,[1, s]$.

Let the modular coloring of the inflated graph of Bi-star $I(B_{s,t})$ be

When $i$ varies from 0 to $s$;

$$\mathcal{S}(v_i) = (1+i)\ mod\ (s+1); \mathcal{S}(v'_i) = \begin{cases} s & \text{if } i \text{ is } 0 \\ i-1 & \text{otherwise} \end{cases}$$

When $j$ varies from 1 to $s$; $\mathcal{S}(u_j) = j$; $\mathcal{S}(u'_j) = \begin{cases} 0 & \text{if } j \text{ is } s \\ 1+j & \text{otherwise} \end{cases}$.

**Subcase 2.2 $s \neq t$ when $s < t$**

We know that every complete graph of order $n$ has a modular chromatic number $n$. Since the inflated graph of bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as an induced sub graph, That gives the modular coloring of $I(B_{s,t})$ is at least $(t+1)$, when $s < t$.

Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_t^+$ as follows,

$$C(v_i) = 0, \text{for all } i \text{ ranges from 0 to } s\ ,$$
$$C(u_j) = 1 + j, \text{for all } j \text{ ranges from 1 to } s,$$

When $i$ & $j$ varies from 1 to $t$; $C(v'_i) = 1 + i\ mod\ (t+1)$; $C(u'_j) = 2i\ mod\ (t+1)$.

Let the modular coloring of inflated graph of Bi-star $I(B_{s,t})$ be

When $i$ & $j$ varies from 0 to $s$; $\mathcal{S}(v_i) = 1 + i$; $\mathcal{S}(u_j) = 0$,

When $i$ varies from 0 to $t$; $\mathcal{S}(v'_i) = \begin{cases} t & \text{if } i \text{ is } 0 \\ -1+i & \text{otherwise} \end{cases}$, and When $j$ varies from 1 to $t$ $\mathcal{S}(u'_j) = \begin{cases} 0 & \text{if } j \text{ is } t \\ 1+j & \text{otherwise} \end{cases}$.

**Subcase 2.3 $s \neq t$ where $s > t$**

Here the inflated graph of bi-star $I(B_{s,t})$ is isomorphic to $I(B_{s,t})$, so that we can follow the result from the previous case.

**Case 3:** When $s$ is odd and $t$ is even,

**Subcase 3.1 when $s < t$,**

We know that every complete graph of order $n$ has a modular chromatic number $n$. Since the inflated graph of bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as an induced sub graph, That gives the modular coloring of $I(B_{s,t})$ is at least $(t+1)$, when $s < t$. Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_t^+$ as follows,

$$C(v_i) = 0, \text{for all } i \text{ ranges from 0 to } s,$$

When $i$ & $j$ varies from 1 $to$ $t$; $C(u_j) = 1 + j$; $C(u'_j) = j - 1$, and

When $i$ varies from 0 $to$ $t$; $C(v'_i) = \begin{cases} 1 & \text{if } i \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}$.

Let the modular coloring of the inflated graph of bi-star $I(B_{s,t})$ be

When $i$ & $j$ varies from 0 $to$ $t$; $\mathcal{S}(v_i) = 1 + i$; $\mathcal{S}(u_j) = \mathcal{S}(u'_j) = 0$, and $\mathcal{S}(v'_i) = i$ for all $i$ ranges from 0 to $s$.

**Subcase 3.2 when $s > t$**

We know that every complete graph of order $n$ has a modular chromatic number $n$. Since the inflated graph of bi-star $I(B_{s,t})$ has two complete graphs $K_{s+1}$ and $k_{t+1}$ as an induced sub graph, That gives the modular coloring of $I(B_{s,t})$ is at least $(s+1)$, when $s > t$.Let the coloring for the graph is defined by $C: V \to \mathbb{Z}_s^+$ as follows,

$$C(v_i) = i,\ C(u_j) = 1 + 2j, mod(s+1),$$

$$C(v'_i) = \begin{cases} 1 & \text{if } i \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}, \text{ and } C(u'_j) = \begin{cases} 0 & \text{if } j \text{ is } 0 \\ -1 + j & \text{otherwise} \end{cases}.$$

Let the modular coloring of inflated graph of Bi-star $I(B_{s,t})$ be

$\mathcal{S}(v_i) = 1 + i, mod(s+1)$, $\mathcal{S}(u_j) = j, if\ j: [1, s]$, $\mathcal{S}(v'_i) = i, if\ i: [1, t]$, and $\mathcal{S}(u'_j) = \begin{cases} 1 & \text{if } j \text{ is } 0 \\ 0 & \text{otherwise} \end{cases}$.

**Case 4**: When $s$ is even and $t$ is odd,

Here, $I(B_{s,t})$ is isomorphic to $I(B_{s,t})$ it is clear that, in case-3, and case-4 graphs are isomorphic. Proof of the case-4 is similar to the case-3.

That is every adjacent vertex receives different color sum. Therefore $m_c\left(I(B_{s,t})\right) = \begin{cases} s+1 & s \geq t \\ t+1 & \text{otherwise} \end{cases}$. Thus the modular chromatic number of an Inflated graph of bi-star is $s + 1$ $or$ $t + 1$ for all $, t \geq 2$ .

**Theorem 3.4:** Let $I(Ct_{s,t})$ denote the inflated graph of the coconut tree graph of order $2(s + t - 1)$ and size $(s + 1)C_2 + 3t - 2$ where $t \geq 3$ then $m_c(I(Ct_{s,t})) = s + 1$

***Proof:*** Let $I(Ct_{s,t})$ be an inflated graph of coconut tree on $n \geq 3$, $\mathcal{V}(I(Ct_{s,t})) = \{v_0, v_1, \dots, v_s\} \cup \{u_0, u_1, \dots, u_s\} \cup \{w_1, w_2, \dots, w_{2(t\ -2)}\}$. Let $\left|\mathcal{V}\left(I(Ct_{s,t})\right)\right| = 2(s + t - 1)$ and $\left|E\left(I(Ct_{s,t})\right)\right| = (s + 1)C_2 + 3s - 2$ . The discussion of the below cases deals the proof of the theorem,

**Case 1:** When $s$ is odd,

Inflated graph of coconut tree $I(Ct_{s,t})$ has complete graph $k_{s+1}$ as induced sub graph, since every complete graph of order n has modular chromatic number $n$. That is $m_c\left(I(Ct_{s,t})\right) \geq (s+1)$.

Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_{s+1}^{+}$as follows,

When $i$ varies from 0 $to$ $s$; $C(v_i) = i$; $C(u_i) = \frac{1}{2}(s) + i + \frac{3}{2}(mod\ s+1)$, and

$$C(w_i) = \begin{cases} \frac{1}{2}(s) + 2i + \frac{1}{2}(mod\ s+1) \text{ for } i \text{ is } 0 \ \&\ 1 \\ 0\ if\ i = 3, \dots, 2t-4, \text{ for all odd } i \\ 1\ if\ i = 3, \dots, 2t-4, \text{ for all even } i \end{cases}$$

Let the modular coloring of the inflated graph of coconut tree $I(Ct_{s,t})$ be

When $i$ varies from 0 $to$ $s$; $\mathcal{S}(v_i) = 1 + i\ mod\ (s+1)$ , and $\mathcal{S}(u_i) = i$.

**Subcase 1.1:** When t is odd and $t = 3$

$$\mathcal{S}(w_i) = \begin{cases} 1 \text{ for } i \text{ is } 1 \\ 3 \text{ for } i \text{ is } 3 \\ 0\ if\ i = 3, \dots, 2t-4, \quad \text{for all even } i \\ 2\ if\ i = 3, \dots, 2t-4, \quad \text{for all odd } i \end{cases}$$

**Subcase 1.2:** When t is odd and $t > 3$

$$\mathcal{S}(w_i) = \begin{cases} \frac{s+5}{2}, \text{ if } i \text{ is } 1 \\ 0\ if\ i = 3, \dots, 2t-4, \text{ for all even } i \\ 2\ if\ i = 3, \dots, 2t-4, \text{ for all odd } i \end{cases}$$

It gives the modular - $s+1$coloring every adjacent vertex receive different color sum. Therefore $\left(I(Ct_{s,t})\right) = s+1$.

**Case 2:** When $s$ is even,

Inflated graph of star $I(Ct_{s,t})$ has complete graph $k_{s+1}$ as an induced sub graph, every complete graph of order n has a modular chromatic number n. That is $m_c\left(I(Ct_{s,t})\right) \geq s+1$.

Let the coloring for the graph be defined by $C: V \to \mathbb{Z}_{s+1}^{+}$ as follows,

$C(v_i) = i,$ for all $i$ ranges from 0 to $s$, $C(u_i) = (1+2i)(mod\ s+1)$ if $i$: $[0, t-1]$, and

$$C(w_i) = \begin{cases} 0\ \ if\ \ i: [1, 2(t\ -2)], \text{ for all odd } i \\ 1\ if\ \ i: [1, 2(t\ -2)], \text{ forall even } i \end{cases}.$$

Let the modular coloring of the inflated graph of star $I(Ct_{s,t})$ be

$\mathcal{S}(v_i) = \{(1+i)(mod\ t),$ form 0 $to$ $t-1$ , $\mathcal{S}(u_i) = i, if\ \ i: [0, t-1]$, and

$$\mathcal{S}(w_i) = \begin{cases} 2 \ \ if \ \ i: [1,2(t \ \ -2)], \ forall \ odd \ i \\ 0 \ \ if \ \ i: [1,2(t \ \ -2)], \ forall \ even \ i \end{cases}$$

It gives the modular $s+1$ - coloring every adjacent vertex receives different color sum. Thus $m_c\left(I(Ct_{s,t})\right) = s+1$.

**Theorem 3.5:** *For any integer* $s, t \geq 2$, *the modular chromatic number of banana tree is* $m_c\left(I(Bt_{s,t})\right) = \begin{cases} s, & s \geq t \\ t, & s < t \end{cases}$.

***Proof:*** Let $I(Bt_{s,t})$ be an inflated graph of banana tree on $s, t \geq 2$. Let vertices of $I(Bt_{s,t})$ denoted by $\mathcal{V}\left(I(Bt_{s,t})\right) = \{u_1, \dots, u_s\} \cup \{v_1, \dots, v_t\} \cup \{v_{t+1}, \dots, v_{2s}\} \cup \{w_1^1, \dots, w_t^1\} \cup \{w_1^2, \dots, w_t^2\} \dots \cup \{w_1^s, \dots, w_t^s\} \cup \{x_1^1, \dots, x_{t-1}^1\} \cup \{x_1^2, \dots, x_{t-1}^2\} \dots \cup \{x_1^s, \dots, x_{t-1}^s\}$. And the edges of $I(Bt_{s,t})$ denoted by $E\left(I(Bt_{s,t})\right)$= $\{e_1, e_2, \dots, e_{sC_2}\} \cup \{f_1, \dots, f_s\} \cup \{f_{t+1}, \dots, f_{2s}\} \cup \{g_1^1, \dots, g_{tC2}^1\} \cup \{g_1^2, \dots, g_{tC2}^2\} \dots \cup \{g_1^m, \dots, g_{tC2}^m\}$. Let $\left|\mathcal{V}\left(I(Bt_{s,t})\right)\right| = 2s(t+1)$ and $\left|E\left(I(Bt_{s,t})\right)\right| = sC_2 + s(tC_2) + s(t+2)$. Inflated graph of banana tree $I(Bt_{s,t})$ has two type of complete graphs $K_s$ and $k_t$ as induced sub graph, we know that every complete graph of order $n$ has a modular chromatic number $n$. Since the inflated graph of banana tree $I(Bt_{s,t})$ has two type of complete graph $K_s$ and $k_t$ as an induced sub graph, we know that every complete graph of order $n$ has a modular chromatic number $n$. That gives $m_c\left(I(Bt_{s,t})\right) \geq s, \ where \ s > t$. Let the coloring for the graph is defined by $C: V \rightarrow z_s^+ \ or \ z_t^+$ if $s$ is greater than or equal to $t$ then we use $s$ colors to color the graph or if $s < t$ we use $t$ colors to color the graph as follows,

$$C(u_i) = 0, \text{for all } i \text{ ranges from 0 to } s,$$

$$C(v_j) = \begin{cases} j-1 & \text{for all } j \text{ ranges from 1 to } s \\ 1 & \text{if } j \text{ is } s+1 \\ 0 & \text{for all } i \text{ ranges from } s+2 \ to \ 2s \end{cases},$$

$$C(w_k^l) = 0, if \ k: [1, t] \ \& \ l: [1, s] \text{ and}$$

$$C(x_k^l) = \begin{cases} k, & if \ k: [1, t-1] \ \& \ l: [2, s] \\ k+1, & if \ k: [1, t-1] \ \& \ l = 1 \end{cases}.$$

Let the modular coloring of the inflated graph of banana tree $I(Bt_{s,t})$ be

$$\mathcal{S}(u_i) = i - 1 \text{ if } i: [1, s],$$

$$\mathcal{S}(v_j) = \begin{cases} 1 & if \ s = 1 \\ 0 & \text{for all } j \text{ ranges from 2 to } s \\ j-1 & \text{for all } j \text{ ranges from } s+1 \ to \ 2s \end{cases},$$

$$\mathcal{S}(w_k^l) = \begin{cases} k, & \text{if } k: [1, t] \ \& \ l \ is \ 1 \\ k-1, & if \ k: [1, t] \ \& \ l: \text{otherwise} \end{cases},$$

and $\mathcal{S}(x_k^l) = 0$, for all $k/\{t\}$ & for all $l$.

This admits the modular max$\{s, t\}$ - coloring, that is every adjacent vertex receives different color sum. Thus the modular chromatic number of an inflated graph of banana tree is max{s, t}.

**Theorem 3.6:** *Let $I(Bit_h)$ denote the inflated graph of the perfect binary tree graph of order $2(2^h - 2)$ where $n \geq 3$, $h \geq 2$ then $m_c(I(Bit_h)) = 3$.*

***Proof:*** Let $I(Bit_h)$ be an inflated graph of perfect binary tree on $n \geq 3, h \geq 2$. Let vertices of $I(Bit_n)$ denoted by $\mathcal{V}(I(Bit_h)) = \{u_1, \dots, u_{2(2^h-2)}\}$. Let $|\mathcal{V}(I(Bt_h))| = 2(2^h - 2)$. Inflated graph of perfect binary tree $I(Bit_h)$ has a complete graphs $K_3$ as an induced sub graph, we know that every complete graph of order $n$ has a modular chromatic number $n$. Since Inflated graph of perfect binary tree $I(Bit_h)$ has $K_3$ (maximum clique) as an induced sub graph that gives $m_c(I(Bit_h)) \geq 3$,

Let the coloring for the graph be defined by $C: \mathcal{V} \to \{0,1,2\}$ to color the graph we use 3 colors as follows,

$C(u_i^h) = 0$ if $h$ is odd and $\forall i$,

When $h$ is even and $h \equiv 0(mod\ 4)$ $C(u_i^h) = \begin{cases} 0\ if\ i \equiv 1(mod3) \\ 1\ if\ i \equiv 2(mod3) \\ 2\ if\ i \equiv 0(mod3) \end{cases}$,

When $h$ is even and is $h \equiv 2(mod\ 4)$ $C(u_i^h) = \begin{cases} 0\ if\ i \equiv 0(mod3) \\ 1\ if\ i \equiv 1(mod3) \\ 2\ if\ i \equiv 2(mod3) \end{cases}$

Let the modular coloring of the inflated graph of perfect binary tree $I(Bit_h)$ be

$$\mathcal{S}(u_i^1) = \begin{cases} 1; \text{if } i = 1 \\ 2; \text{if } i = 2 \end{cases}, \mathcal{S}(u_i^h) = 0; \text{if } h \text{ is even, and}$$

$$\mathcal{S}(u_i^h) = \begin{cases} 1; \text{if } i \text{ is odd and } h \geq 3, \text{where } h \text{ is odd} \\ 2; \text{if } i \text{ is even and } h \geq 3, \text{where } h \text{ is odd} \end{cases}.$$

This admits the modular 3- coloring that is every level of vertex is adjacent with previous and next level of its vertex, here every alternative level (even) modular coloring is 0 and odd level modular coloring is 1 and 2 for alternative i's so every adjacent vertices receives different color sum. Therefore $m_c(I(Bit_h)) = 3$.

**Theorem 3.7:** *Let $I(p_t \odot \overline{k_{1,s}})$ denote an inflated graph of corona product of path and star compliment of order $2(t(s+2) - 1)$ and size $(t-2)(s+1)C_2 + 2(s+1)C_2 + t(s+2) - 1$ where $s, t \geq 2$, then $m_c\left(I(p_t \odot \overline{k_{1,s}})\right) = s + 3$.*

***Proof:*** Let $I(p_t \odot \overline{k_{1,s}})$ be an inflated graph of corona product of path and star compliment on $s, t \geq 2$, and $\mathcal{V}\left(I(p_t \odot \overline{k_{1,s}})\right) = \{u_1, u_2, \dots, u_{(s+2)}\} \cup$

$\{v_1, v_2, \dots, v_{(s+2)}\} \cup \{w_1^1, \dots, w_{s+3}^1\} \cup \{w_1^2, \dots, w_{s+3}^2\} \dots \cup \{w_1^{t-2}, \dots, w_{s+3}^{t-2}\} \cup \{x_1^1, \dots, x_{s+1}^1\} \cup \{x_1^2, \dots, x_{s+1}^2\} \dots \cup \{x_1^{t-2}, \dots, x_{s+1}^{t-2}\} \cup \{y_1, y_2, \dots, y_{(s+2)}\} \cup \{z_1, z_2, \dots, z_{(s+1)}\}$. Let $\left|\mathcal{V}\left(I\left(p_t \odot \overline{k_{1,s}}\right)\right)\right| = 2(t(s+2)-1)$ and $\left|E\left(I\left(p_t \odot \overline{k_{1,s}}\right)\right)\right| = (t-2)(s+1)C_2 + 2(s+1)C_2 + t(s+2) - 1$. The discussion of the below cases deals the proof of the theorem,

**Case 1:** When $t$ is even,

Inflated graph of $\left(p_t \odot \overline{k_{1,s}}\right)$ has a complete graph as an induced sub graph, every complete graph of order n has modular chromatic number $n$. That is $m_c\left(I\left(p_t \odot \overline{k_{1,s}}\right)\right) \geq (s+3)$.

Let the coloring for the graph be defined by $C: V \to \{0,1,\dots,s+3\}$ as follows,

When $i$ is ranges from 1 to $s+1$; $C(u_i^1) = i-1$; $C(u_i^t) = i+1$

When $j$ is ranges from 1 to $s+2$; $C(v_j^1) = \begin{cases} 1, & \text{if } j \text{ is } 1 \\ 0, & \text{otherwise} \end{cases}$; $C(v_j^t) = 0$.

When $k$ is even and k ranges from 2 to t−2 ,

$C(u_i^k) = 1+i, if\ i: [1, s+1]$ ; $C(v_j^k) = 0$, for all i ranges from 1 to $s+3$.

When $k$ is odd and k ranges from 2 to t−1

$C(u_i^k) = i$, for all $i$ ranges from 1 to $s+2$, $C(v_j^k) = \begin{cases} 0, \text{if } j: [1, s+2] \\ 1, \text{otherwise} \end{cases}$.

Let the modular coloring of inflated graph of $\left(p_t \odot \overline{k_{1,s}}\right)$ be

$\mathcal{S}(u_i^k) = 0$, for all $i$ ranges from 1 to $s+3$ & $k\ is$ 1 & $t$,

When $k$ is ranges from 2 to $t-1$; $\mathcal{S}(v_j^k) = \begin{cases} j, & \text{if } j: [1, s+2] \\ 0, & \text{if } j\ is\ s+3 \end{cases}$,

When $j$ is ranges from 1 to $s+2$; $\mathcal{S}(v_j^1) = j-1$; $\mathcal{S}(v_j^t) = j$.

**Case 2:** When $m$ is odd,

$C(u_i^1) = -1+i, \text{if } i: [1, s+1]$, $C(v_j^1) = \begin{cases} 1, \text{if } j\ is\ 1 \\ 0, \text{if} j: [2, s+2] \end{cases}$,

$C(u_i^t) = \begin{cases} i, \text{if } i: [1, s] \\ i+i, \text{if } i \text{ is } s+1 \end{cases}$, $C(v_j^t) = \begin{cases} 0 \text{ if } i: [1, s+1] \\ 1, \text{if } j\ is\ s+2 \end{cases}$.

When $k$ is even, and $k$ is varies from 2 to $t-2$

$C(u_i^k) = \{1+i$, for all $i$ varies from 1 to $s+1$

$C(v_j^k) = \{0$, for all $j$ varies from 1 to $s+3$.

When $k$ is odd and $k$ is varies from 2 to $t-2$

$C(u_i^k) = i, if\ i: [1, s+1]$, $C(v_j^k) = \begin{cases} 0, \text{if } i: [1, s+2] \\ 1, \text{ if } j \text{ is } s+3 \end{cases}$.

Let the modular coloring of inflated graph of $\left(p_t \odot \overline{k_{1,s}}\right)$ be

$\mathcal{S}(u_i^k) = 0$, for all $i$ ranges from 1 to $s+3$ and $k$ is 1 &$t$

When $j$ varies from 1 to $s+2$; $\mathcal{S}(v_j^1) = -1+j$;$\mathcal{S}(v_j^t) = j$,

When $k$ is varies from 2 to $t-1$;$\mathcal{S}(v_j^k) = \begin{cases} j, if\ j: [1, s+2] \\ 1, if\ \ j\ is\ s+3 \end{cases}$,

It gives the modular - $s+3$ coloring every adjacent vertex receive different color sum. Therefore $\left(I(p_t \odot \overline{k_{1,s}})\right) = s+3$.

**Theorem 3.8:** *Let $I(k_{1,t} \odot \overline{k_{1,s}})$ denote the inflated graph of corona product of star and star compliment of order $2((s+1)+t(s+2))$ and size $(s+t+1)C_2 + t(s+2)C_2 + (t+1)(s+1)$ where $s,t \geq 2$ then $m_c\left(I(k_{1,t} \odot \overline{k_{1,s}})\right) = t+s+1$.*

***Proof:*** Let $I(k_{1,t} \odot \overline{k_{1,s}})$ be an inflated graph of corona product of star and star compliment on $s,t \geq 2$, and $\mathcal{V}\left(I(k_{1,t} \odot \overline{k_{1,s}})\right) = \{v_0, v_1, \dots, v_{2(s+2)}\} \cup \{u_1^1, \dots, u_{s+3}^1\} \cup \{u_1^2, \dots, u_{s+3}^2\} \dots \cup \{u_1^{n-2}, \dots, u_{s+3}^{n-2}\} \cup \{w_1, w_2, \dots, w_{t(s+1)}\}$. Let $\left|\mathcal{V}\left(I(k_{1,t} \odot \overline{k_{1,s}})\right)\right| = 2((s+1)+t(s+2))$ and $\left|EI(k_{1,t} \odot \overline{k_{1,s}})\right| = (s+t+1)C_2 + t(s+2)C_2 + (t+1)(s+1)$. The discussion of the below cases deals the proof of the theorem,

**Case 1:** When $t+s+1$ is odd,

Inflated graph of star corona star complement has two types of complete graphs $k_{t+s+1}$ and $k_{s+1}$ as an induced sub graph; every complete graph of order n has a modular chromatic number n. That is $m_c\left(I(k_{1,t} \odot \overline{k_{1,s}})\right) \geq t+s+1$.

Let the coloring for the graph be defined by $C: \mathcal{V} \to z_{t+s+1}^+$ as follows,

$C(u_i) = 1+i$, for all $i$ ranges from 1 to $t+s+1$,

$C(v_i) = 0$, for all $i$ ranges from 1 to $s+1$

$$C(w_j^k) = \begin{cases} 1, & \text{if } k \text{ is } 1, j \text{ is } 2 \\ 0, \text{if } k \text{ is } 1,\ j \text{ is } 1 \ \&\ j: [3, s+2], & \\ 0, & \text{if } j: [1, s+2] \ \&\ \text{for all } k \text{ ranges from 1 to } t \end{cases}$$

$$C(x_j^l) = \begin{cases} l + (j-1), \text{for all } j \text{ ranges from 1 to } s+1 \ \&\ \text{for all l ranges from 1 to } t \\ 2, \quad \text{if } l \text{ is } 1, \quad j \text{ is } 1 \\ j, \text{if } l \text{ is } 1 \ \&\ \text{for all } j \text{ ranges from 1 to } s+1 \end{cases}$$

Let the modular coloring of the inflated graph of $(k_{1,t} \odot \overline{k_{1,s}})$ be

$\mathcal{S}(u_i) = (t+s+1)+(1-i)$, for all $i$ ranges from 1 to $t+s+1$,

$\mathcal{S}(v_i) = (t+s+1)-i$, for all $i$ ranges from 1 to $s+1$,

When j is ranges from 1 to s+2 we get,

$$\mathcal{S}(w_j^k) = \begin{cases} j, & \text{if\& } k \text{ is } 1 \ \&\ 2 \\ k+j-2, & \text{if } \&\ \text{for all } k \text{ ranges from 3 to } t \end{cases}, \text{ and}$$

$$\mathcal{S}(x_j^l) = \begin{cases} 0, & \text{if } j: [1, s+1] \text{ \& for all } l \text{ ranges from 2 to } t \\ 1, & \text{where } l \text{ is } 1, j \text{ is } 1 \\ 0, & \text{where } l \text{ is } 1, \text{ for all } j \text{ ranges from 2 to } s+1 \end{cases}.$$

It gives the modular - $t + s + 1$coloring every adjacent vertex receive different color sum.

Therefore $\left(I\left(k_{1,t} \odot \overline{k_{1,s}}\right)\right) = t + s + 1.$

**Case 2:** When $t + s + 1$ is even,

Inflated graph of star corona star complement has two type of complete graphs $k_{t+s+1}$ and $k_{s+1}$ as an induced sub graph, every complete graph of order n has a modular chromatic number n. That is $m_c\left(I\left(k_{1,t} \odot \overline{k_{1,s}}\right)\right) \geq t + s + 1$.

Let the coloring for the graph be defined by $C: \mathcal{V} \to z_{t+s+1}^{+}$ as follows,

$$C(u_i) = i - 1, \text{for all } i \text{ ranges from 1 to } t + s + 1,$$

$$C(v_i) = 0, \text{for all } i \text{ ranges from 1 to } s + 1,$$

$$C(w_j^k) = 0, \text{if } i: [1, s+2] \text{\& for all } k \text{ ranges from 1 to } t,$$

$$C(x_j^l) = l + (j - 1), \text{if } j: [1, s+1] \text{ \& } l \text{ is for all values ranges from 1 to } t$$

Let the modular coloring of the inflated graph of $\left(k_{1,t} \odot \overline{k_{1,s}}\right)$ be

$$\mathcal{S}(u_i) = \frac{3}{2}(t + s + 1) + (1 - i), \text{for all } i \text{ ranges from 1 to } t + s + 1],$$

$$\mathcal{S}(v_i) = (t + s + 1) - i, \text{for all } i \text{ ranges from 1 to } s + 1,$$

$$\mathcal{S}(w_j^k) = \begin{cases} 0, & \text{if } j \text{ is 1 \& } k \text{ is for all values ranges from 1 to } t \\ (j-1) + k, & \text{if } j: [2, s+2] \text{ \& } k \text{ is for all values ranges from 1 to } t \end{cases}, \text{ and}$$

$$\mathcal{S}(x_j^l) = 0, \text{if } j: [1, s+1] \text{ \& } l \text{ is for all values ranges from 1 to } t$$

It gives the modular $t + s + 1$ - coloring every adjacent vertices receive different color sum. Thus $m_c\left(I\left(k_{1,t} \odot \overline{k_{1,s}}\right)\right) = t + s + 1$.

## 3. Conclusion

In this article, we have explored the concept of modular coloring on inflated graphs and derived some standard results for important tree-related graphs. It is obvious to conclude that the modular chromatic index of a tree-related graph is the size of the maximum clique that occurs in inflated tree-related graphs. Additionally, further research could investigate the application of modular coloring to other types of graphs beyond tree-related graphs. This could potentially provide new insights and results in the field of graph theory.

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*Received March, 2024*