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Investigating utilizations of harmonic univalent functions employing the Prabhakar fractional operator within specific categories

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Abstract

This study delves into innovative categories of harmonic functions that are established using a fractional operator. Initially, we establish fresh sets of harmonic functions by utilizing the Prabhakar fractional operator. Following that, we deduce the primary findings of our research, which encompass comprehensive and fundamental limits on coefficients for functions linked to the newly introduced categories. Furthermore, we offer various specific attributes related to the characterization of these categories. These properties encompass extreme points, convex mixtures. To sum up, we furnish a range of conclusions derived from the primary findings.

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1. Introduction

Harmonic functions play a significant role in geometric function theory, a branch of mathematics that explores the relationships between complex analysis, geometry, and function theory. In this context, harmonic functions are solutions to the Laplace's equation, which captures the

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concept of a function that maintains its average value over small regions. These functions possess fascinating properties, including the mean value property, the maximum principle, and the connection to conformal mappings. The study of harmonic functions in geometric function theory provides insights into various aspects of complex analysis and its applications in different fields.

Harmonic mappings within a specific region, denoted as $\mathfrak{D} \subseteq \mathbb{C}$ and contained within the complex number plane, can be described using unique complex-valued harmonic functions represented as $f = u+iv$. These functions consist of the pair u and v which are real harmonic functions.. The notable achievement of Clunie and Sheil-Small [6] is relevant to a category that includes complex-valued harmonic functions, maintaining both univalence and orientation preservation. This category is defined within the open unit disk $\mathbb{U}$. Their work has provided the basis for multiple investigations into distinct subsets of harmonic univalent functions, as evidenced in [2] and other relevant literature.

In any simply-connected area labeled as $\mathfrak{D}$, it has been established [6] that f can be represented as $f = h+\overline{g}$, where both h and g are analytic functions within $\mathfrak{D}$. We refer to h as the analytic part and g as the co-analytic part of f. To ensure that f demonstrates local uniqueness and preserves orientation within $\mathfrak{D}$, a condition that is both necessary and sufficient is the inequality $|h'(z)| > |g'(z)|$ within $\mathfrak{D}$ (see [6]).

Consider the set $\mathbb{H}$ containing functions expressed as $f = h + \overline{g}$. These functions are harmonic, injective, and maintain positioning within the open unit disk $\mathbb{U} = z : |z| < 1$. In this scenario, $f = h + \overline{g}$ is subject to the conditions $f(0) = h(0) = f'(0) - 1 = 0$ for normalization.

Permit us describe $\mathfrak{P}\mathfrak{F}_{\mathbb{H}}$ as a collection of functions that exhibit three distinct properties within the open unit disk $\mathbb{U}$ [1]. These functions can be formulated as $f(z) = h(z) + \overline{g(z)}$, where both $h(z)$ and $g(z)$ are analytic functions defined within the region $\mathbb{U}$. The three key characteristics of these functions are their harmonic nature, one-to-one mapping, and their ability to preserve the original orientation of objects or points within $\mathbb{U}$.

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad |b_1| < 1 \tag{1}$$

Hence,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k \quad |b_1| < 1 \tag{2}$$

Consider the set $\overline{\mathfrak{P}\mathfrak{O}\mathfrak{F}_{\mathbb{H}}}$, which is a subset of $\mathfrak{P}\mathfrak{F}_{\mathbb{H}}$. This subset comprises functions that can be represented as $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k \quad |b_1| < 1 \tag{3}$$

Clunie and Sheil-Small[6] conducted research in 1984 that delved into the mathematical class known as SH and its various geometric subcategories. They established certain limits or coefficients during their study. Since then, several other scholars, including Avcl and Zlotkiewicz[5], Silverman[21], Silverman and Silvia[22], Jahangiri[11], and Jahangiri et al.[12] have contributed to the field by publishing their own works related to SH and its subgroups. Additionally, other researchers (e.g., as mentioned in references [3] and [25]) have more recently explored harmonic functions that are utilized in specific mathematical operations. The current work introduces a novel subset of harmonic functions, particularly emphasizing the use of a fractional operator.

A rather comprehensive description, instances of employing fractional calculus in practical situations are evident in the research conducted by Oldham and Spanier, as detailed in reference [15], and in the work of Miller and Ross, as outlined in reference [14]. For additional information on the concept of fractional calculus, see [23]. See [13] for more information on the items in this work.

The concept of the complete function was first developed by Tilak R. Prabhakar in 1971 [16].The Prabhakar function $\Xi^{\rho}_{\mu,\wp}(z)$, as mentioned in [7], can also be referred to as the three-parameter Mittag-Leffler function. The generalised Mittag-Leffler function is defined by [18-20].

$$\Xi^{\rho}_{\mu,\delta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\mu k+\delta)} \frac{z^k}{k!} \tag{4}$$

In this context, we have the variables $\wp, \mu, \rho \in \mathbb{C}$ where the $\Re(\rho) > 0$. We're discussing $(\rho)_k$, which represents the Pochhammer symbol defined by a specific equation.

$$(\rho)_k = \begin{cases} 1 (\mathrm{k}=0) \\ \rho(1+\rho)......(k+\rho-1)\ (\mathrm{k} \in \mathbb{N}) \end{cases} \tag{5}$$

The generalized hypergeometric function $\tau F \eta$ is obtained through a specific procedure;

$$\begin{bmatrix} \beta_1 \cdots \beta_\tau ; \\ \beta_1 \cdots \beta_\eta ; \end{bmatrix} z = \sum_{n=0}^{\infty} \frac{(\beta_1)_n \cdots (\beta_\tau)_n}{(\wp_1)_n \cdots (\wp_\eta)_n} \frac{z^k}{k!} = {}_\tau F_\eta(\beta_1, \ldots\ldots \beta_\tau ; \wp_1 \ldots\ldots \wp_\eta ; z) \tag{6}$$

To simplify the analysis, we assume that the variable 'z' represents the numerator parameters $\beta_1 \cdots \beta_\tau$ and the denominator parameters $\wp_1 \cdots \wp_\eta$. It's crucial to emphasize that these parameters can take on complex values, as long as τ and η are either positive integers or zero (with the interpretation of an empty product as one). Equation (5) represents the specific instance of $\tau F \eta$ within the (Gauss) hypergeometric series.

In 1989, Srivastava and colleagues[23] conducted a study on the geometric characteristics of complex fractional operators. This investigation encompassed both differential and integral aspects and was further expanded to encompass two-dimensional fractional operators. Ibrahim's work in 2011[8] introduced parameters related to a set of analytical functions within a specific mathematical region. These fractional operators are applied to represent diverse analytical functions and intricate differential equations, as explained by Srivastava in 2011[24]. These operators are known as fractional algebraic differential variables and find application in exploring Ulam stability. This concept is further elucidated by the equations introduced by R.W. Ibrahim in 2012[9-10].

The notation ${}_c^d\Theta_{\beta,\wp}^{\kappa,\omega}$ is employed to symbolize the standardized the Prabhakar operator, which is complex in nature, operates within the open unit disc. When ${}_c^d\varepsilon_{\beta,\wp}^{\kappa,\omega}$ belongs to the set $\mathfrak{P}\mathfrak{F}_{\mathbb{H}}$, it becomes feasible to analyze it utilizing the structure regarding the theory related to geometric functions.

In reference [4], the series provides an expansion for the integral operator associated with the fractional differential operator ${}_c^d\varepsilon_{\beta,\wp}^{\kappa,\omega}$

$${}_c^d\Theta_{\beta,\wp}^{\omega,\kappa} f(z) = z + \sum_{k=2}^{\infty} \left[\frac{\varphi_k \Xi^{-\omega}[\omega z^\beta]}{\Gamma(k+1)\Xi_{\beta,k+1-\wp}^{-\kappa}[\omega z^\beta]} \right] z^k \tag{7}$$

It is self-evident that

$$({}_c^d\Theta_{\beta,\wp}^{\omega,\kappa} * {}_c^d\varepsilon_{\beta,\wp}^{\omega,\kappa}) f(z) = ({}_d^c\varepsilon_{\beta,\wp}^{\omega,\kappa} * {}_c^d\Theta_{\beta,\wp}^{\omega,\kappa}) f(z) = f(z)$$

When used on the harmonic function $f = h + \overline{g}$, the Prabhakar operator is formally described in the following manner:

$${}_c^d\Theta_{\beta,\wp}^{\omega,\kappa} f(z) = {}_c^d\Theta_{\beta,\wp}^{\omega,\kappa} h(z) + \overline{{}_c^d\Theta_{\beta,\wp}^{\omega,\kappa} g(z)} \tag{8}$$

Let the subgroup denoted by $\mathfrak{PF}_{\mathbb{H}}[d,c,\omega,\beta,\wp,\varepsilon]$ consist of harmonic functions $f \in \mathbb{H}$ that may be expressed in the form $f = h + \overline{g}$ such that [17]

$$\Re\left\{(1-\ell)(1-\aleph)\frac{{}_{c}^{d}\Theta_{\wp,\beta}^{\omega,\kappa}f(z)}{z}+(\ell+\aleph)\frac{({}_{c}^{d}\Theta_{\wp,\beta}^{\omega,\kappa}f(z))'}{z'}+\ell\aleph\frac{({}_{c}^{d}\Theta_{\wp,\beta}^{\omega,\kappa}f(z))''}{z''}-2\ell\aleph\right\}\geq\varepsilon \qquad (9)$$

where ${}_{c}^{d}\Theta_{\beta,\wp}^{\omega,\kappa}f(z)$ is defined (8) , $0\leq\varepsilon<1$, $\ell\geq 0$, $0\leq\aleph<1$ and $z=re^{i\theta}\in\mathbb{U}$.

Furthermore, we permit $\overline{\mathfrak{POF}_{\mathbb{H}}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$ = $\mathfrak{PF}_{\mathbb{H}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$ $\cap$ $\overline{\mathfrak{POF}_{\mathbb{H}}}$

A condition is derived that serves as a sufficient requirement for functions f defined by equation (2) to be categorized within the set $\mathfrak{PF}\mathbb{H}$ $[d,c,\omega,\beta,\wp,\varepsilon]$. Moreover, it is demonstrated that this condition for coefficients is equally important when dealing with functions within the category denoted as $\overline{\mathfrak{POF}\mathbb{H}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$. The study also investigates the endpoints of functions within the range $\overline{\mathfrak{POF}_{\mathbb{H}}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$ and establishes specific inclusion outcomes.

2. Coefficient Condition

We proceed to establish a coefficient condition that functions must satisfy in order to be part of the $\mathfrak{PF}_{\mathbb{H}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$ class.

Theorem 2.1: *A function $f \in$ $\mathfrak{PF}_{\mathbb{H}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$ if it satisfies the condition $f = h + \overline{g}$ iff*

$$\sum_{k=2}^{\infty}[1+(1+k^2)\ell\aleph-(1-k)(\ell+\aleph)]\mathfrak{E}^k\,|a_k|+\sum_{k=1}^{\infty}[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]\mathfrak{E}^k\,|b_k|\leq 1-\varepsilon \qquad (10)$$

$$0\leq\varepsilon<1,\ f\in\mathfrak{PF}_{\mathbb{H}}[d,c,\omega,\beta,\wp,\varepsilon].$$

Where,

$$\mathfrak{E}^k=\left[\frac{\varphi_k\Xi^{-\omega}[\omega z^{\beta}]}{\Gamma(k+1)\Xi_{\beta,k+1-\wp}^{-\kappa}[\omega z^{\beta}]}\right]$$

Proof: By utilising the property that $\Re(\mathfrak{w})\geq\varepsilon$ when and only when

$$\left|1-\varepsilon+\mathfrak{w}\right|\geq\left|1+\varepsilon-\mathfrak{w}\right|$$

Consider

$$\mathfrak{w} = (1-\ell)(1-\aleph)\frac{{}_c^d\Theta_{\wp,\beta}^{\omega,\kappa} f(z)}{z} + (\ell+\aleph)\frac{({}_c^d\Theta_{\wp,\beta}^{\omega,\kappa} f(z))'}{z'} + \ell\aleph\frac{({}_c^d\Theta_{\wp,\beta}^{\omega,\kappa} f(z))''}{z''} - 2\ell\aleph$$

To demonstrate this,

$$\left|1-\varepsilon+\mathfrak{w}\right| \geq \left|1+\varepsilon-\mathfrak{w}\right|$$

Currently, we possess

$$\begin{aligned}\left|1-\varepsilon+\mathfrak{w}\right| \geq 2-\varepsilon-\sum_{k=2}^{\infty}[1+(1+k^2)\ell\aleph-(1-k)(\ell+\aleph)]\mathfrak{E}^k\,|a_k|\,|z^{k-1}|\\ -\sum_{k=1}^{\infty}[1-(1+k^2)\ell\aleph-(1+k)(\ell+\aleph)]\mathfrak{E}^k\,|b_k|\,|z^{k-1}|\end{aligned}$$

and

$$\begin{aligned}\left|1+\varepsilon-\mathfrak{w}\right| \geq \varepsilon+\sum_{k=2}^{\infty}[1+(k^2+1)\ell\aleph-(1-k)(\ell+\aleph)]\mathfrak{E}^k\,|a_k|\,|z^{k-1}|\\ +\sum_{k=1}^{\infty}[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]\mathfrak{E}^k\,|b_k|\,|z^{k-1}|\end{aligned}$$

Then,

$$\left|1-\varepsilon+\mathfrak{w}\right| \geq \left|1+\varepsilon-\mathfrak{w}\right|$$
$$\left|1-\varepsilon+\mathfrak{w}\right| - \left|1+\varepsilon-\mathfrak{w}\right| \geq 0$$

$$\begin{aligned}\sum_{k=2}^{\infty}[1+(1+k^2)\ell\aleph-(1-k)(\ell+\aleph)]\mathfrak{E}^k\,|a_k| + \sum_{k=1}^{\infty}[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]\\ \mathfrak{E}^k\,|b_k| \leq 1-\varepsilon\end{aligned}$$

Thus, the demonstration has been concluded.

Theorem 2.2: *A function* $f \in \overline{\mathfrak{P}\mathfrak{O}\mathfrak{F}_{\mathbb{H}}}\,[d,c,\omega,\beta,\wp,\varepsilon]$ *if it satisfies the condition* $f = h + \overline{g} \in \overline{\mathfrak{P}\mathfrak{O}\mathfrak{F}_{\mathbb{H}}}$

$$\begin{aligned}\sum_{k=2}^{\infty}[1+(1+k^2)\ell\aleph-(1-k)(\ell+\aleph)]\mathfrak{E}^k\,|a_k|\,r^{k-1} + \sum_{k=1}^{\infty}[1-(1+k^2)\ell\aleph\\ -(1+k)(\ell+\aleph)]\mathfrak{E}^k\,|b_k|\,r^{k-1} \leq 1-\varepsilon \quad (11)\end{aligned}$$

Proof: Because $\overline{\mathfrak{P}\mathfrak{O}\mathfrak{F}_{\mathbb{H}}}[d,c,\omega,\beta,\wp,\varepsilon]$ is equal to $\mathfrak{P}\mathfrak{F}_{\mathbb{H}}[d,c,\omega,\beta,\wp,\varepsilon]$, we can establish the "if" portion of the argument. If we consider the condition "only if" and assume that the function *f(z)* falls within the set $\overline{\mathfrak{P}\mathfrak{O}\mathfrak{F}_{\mathbb{H}}}\,[d,c,\omega,\beta,\wp,\varepsilon]$ for $z = re^{i\theta}$ in the domain $\mathbb{U}$, it can be expressed as follows:

$$\Re\left[(1-\ell)(1-\aleph)\frac{{}_c^d\Theta_{\wp,\beta}^{\omega,\kappa}f(z)}{z}+(\ell+\aleph)\frac{({}_c^d\Theta_{\wp,\beta}^{\omega,\kappa}f(z))'}{z'}+\ell\aleph\frac{({}_c^d\Theta_{\wp,\beta}^{\omega,\kappa}f(z))''}{z''}-2\ell\aleph\right]$$

$$=\Re\left[1-\sum_{k=2}^{\infty}[1+(1+k^2)\ell\aleph-(1-k)(\ell+\aleph)\right]\mathfrak{E}^k a_k z^{k-1}$$
$$-\sum_{k=1}^{\infty}\left[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]\mathfrak{E}^k b_k z^{k-1}\right]\geq\varepsilon$$

The final disparity remains valid for any $z\in\mathbb{U}$. Thus, when $z=r\to 1$, we achieve the desired outcome.

$$\sum_{k=2}^{\infty}[1+(1+k^2)\ell\aleph-(1-k)(\ell+\aleph)]\mathfrak{E}^k\,|a_k|\,r^{k-1}$$
$$+\sum_{k=1}^{\infty}[1-(1+k^2)\ell\aleph-(1+k)(\ell+\aleph)]\mathfrak{E}^k\,|b_k|\,r^{k-1}\leq 1-\varepsilon$$

3. Extreme Points and Inclusion Results

In this theorem, we depict the boundary points of the set $\overline{\mathfrak{POF}_{\mathbb{H}}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$.

Theorem 3.1: *A function $f=h+\overline{g}$ belongs to the set $\overline{\mathfrak{POF}_{\mathbb{H}}}$ $[d,c,\omega,\beta,\wp,\varepsilon]$ if and only if it can be expressed in a manner where it takes on this specific structure.*

$$f(z)=M_1(z)+\sum_{k=2}^{\infty}M_kP_k(z)+\sum_{k=1}^{\infty}N_k(z)Q_k(z) \tag{12}$$

Here,

$$P_k(z)=z-\frac{1-\varepsilon}{[1+(k^2+1)\aleph\ell-(1-k)(\ell+\aleph)]}\frac{1}{\mathfrak{E}^k}z^k;\ Q_k(z)=z+\frac{1-\varepsilon}{[1-(k^2+1)\aleph\ell-(1+k)(\ell+\aleph)]}\frac{1}{\mathfrak{E}^k}(\overline{z})^k$$

The conditions $M_1\geq 0$, $N_1\geq 0$, $M_1(z)+\sum_{k=2}^{\infty}M_k+\sum_{k=1}^{\infty}N_k(z)=1$, and $M_k\geq 0$, $N_k\geq 0$ for $k=2,3,\cdots$ hold true.

Proof: If $f(z)\in\overline{\mathfrak{POF}_{\mathbb{H}}}[d,c,\omega,\beta,\wp,\varepsilon]$ by setting,

$$M_1=1-\left(\sum_{k=2}^{\infty}|P_k|+\sum_{k=1}^{\infty}|Q_k|\right)$$

$$P_k=\frac{|1+(k^2+1)\ell\aleph-(1-k)(\ell+\aleph)|}{1-\varepsilon}\mathfrak{E}^k\,|a_k|\,;\ Q_k=\frac{|1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)|}{1-\varepsilon}\mathfrak{E}^k\,|b_k|$$

$$f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|(\overline{z})^k=(1-[\sum_{k=2}^{\infty}M_k+\sum_{k=1}^{\infty}N_k])z+\sum_{k=2}^{\infty}P_k(z)M_k$$
$$+\sum_{k=1}^{\infty}Q_k(z)N_k$$
$$f(z)=M_1(z)+\sum_{k=2}^{\infty}M_kP_k(z)+\sum_{k=1}^{\infty}N_k(z)Q_k(z)$$

Conversely,If we define *f(z)* using the expression provided in equation (12), then

$$f(z)=z-\sum_{k=2}^{\infty}\frac{1-\varepsilon}{[1+(k^2+1)\ell\aleph-(1-k)(\ell+\aleph)]}\frac{1}{\mathfrak{e}^k}P_kz^k-\sum_{k=1}^{\infty}\frac{1-\varepsilon}{[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]}\frac{1}{\mathfrak{e}^k}Q_k(\overline{z})^k$$

Due to equation (11), we can reword the statement as follows:

$$\sum_{k=2}^{\infty}[1+(k^2+1)\ell\aleph-(1-k)(\ell+\aleph)]\frac{1-\varepsilon}{[1+(k^2+1)\ell\aleph-(1-k)(\ell+\aleph)]}\frac{1}{\mathfrak{e}^k}|P_k|+$$
$$\sum_{k=1}^{\infty}[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]\frac{1-\varepsilon}{[1-(k^2+1)\ell\aleph-(1+k)(\ell+\aleph)]}\frac{1}{\mathfrak{e}^k}|Q_k|$$

$$=(1-\varepsilon)\left[\sum_{k=2}^{\infty}|P_k|+\sum_{k=1}^{\infty}|Q_k|\right]=(1-M_1)(1-\varepsilon)\le(1-\varepsilon)$$

That constitutes the necessary depiction.

4. Conclusions

In conclusion,Harmonic functions are pivotal in numerous disciplines, and leveraging Prabhakar operators in fractional calculus enhances their applicability. This is particularly relevant in representing systems with non-local behavior more precisely, with practical applications in modeling anomalous diffusion, viscoelasticity, and signal processing. Our utilization of the Prabhakar operator $\Theta_{\beta,\wp}^{\omega,\kappa}f(z)$, as described in equation (7) in conjunction with a given function, led to the discovery of inclusion relationships involving the harmonic class and various other classes of harmonic analytic functions within the open disk. This study suggests the potential for identifying novel inclusion relations for additional harmonic classes of analytic functions by applying the operator $\Theta_{\beta,\wp}^{\omega,\kappa}f(z)$.

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