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Exploring the properties and applications of complex fuzzy cut vertices and bridges

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Abstract

Within the context of complex fuzzy graphs, this study presents and investigates the ideas of complex fuzzy cut vertices and complex fuzzy bridges. By adding degrees of membership or truth in edges, complex fuzzy graphs expand upon standard graph theory and provide a more complex representation of connections. By finding vertices whose removal will gradually affect the network's connectedness due to various degrees of membership, we redefine cut vertices as complex fuzzy cut vertices. In a similar vein, we expand the concept of bridges to include complex fuzzy bridges and identify edges whose removal would cause progressive structural changes in the graph. This paper explores the characteristics, methods, and mathematical underpinnings of these ideas.

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Keywords: *Complex fuzzy graph, Strong arc, Fuzzy cut vertices, Fuzzy bridges.*

1. Introduction

The conceptualization of fuzzy sets initiated by Zadeh [12] in 1965, wherein the degree of membership of an object could take any real value within the confines of the interval [0,1]. In 2002, Daniel Ramot and others [3] introduced innovative supplement of this notion, termed complex fuzzy sets, wherein the scope of membership is not confined to [0,1], but rather encompasses the unit circle within the complex plane. In 1975, Rosenfeld [7]

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introduced the concept of fuzzy graphs, leading to the subsequent development of various fuzzy analogues to fundamental ideas from classical graph theory, including trees, cycles, connectivity, cut nodes, and bridges. The notations of fuzzy relations and fuzzy graphs were discussed by R.T. Yeh et.al [11]. Fuzzy trees were described by Sunitha and Vijayakumar [9] primarily through the utilization of their unique maximum spanning tree (MST), establishing a necessary criterion for identifying a node as a fuzzy cut node. The same researchers [8] covered a wide range of topics in fuzzy graph theory, including centre problems, blocks in fuzzy graphs, and the characteristics of self-complementary fuzzy graphs. They also used the idea of strongest paths to explain the concept of blocks in fuzzy graphs . In addition, strong arcs , fuzzy end nodes, and geodesics in fuzzy graphs were introduced by Bhutani and Rosenfeld [2], who also provided exact definitions for strong pathways and strong arcs. These contributions further enhanced the subject. Naveed Yaqoob et al.[6] merged the goals of graph theory and complex intuitionistic fuzzy sets, giving rise to the creation of complex intuitionistic fuzzy graphs in 2019. Thirunavukarasu et.al[10] conducted a thorough examination of the energy of complex fuzzy graphs. Azhagendran N and Mohamed Ismayil A [1] used examples to show how to do basic operations on complex fuzzy graphs, such as intersection, composition, union, and cartesian product in polar form. Additionally [5], they clarified the weak and strong isomorphisms between two complex fuzzy graphs and demonstrated that isomorphism on complex fuzzy graphs is an equivalency relation. For an exploration of crisp graphs, one may refer to the book "Graph Theory" authored by Harary [4].

When it comes to complex fuzzy networks, where edges have degrees of membership or truth, the conventional ideas of bridges and cut vertices become more difficult. These ideas are essential to comprehending the structure and connection of networks. This environment is seen from a new angle via complex fuzzy cut vertices and complex fuzzy bridges, which explain the continuous and intricate interactions in the graph.

According to conventional graph theory, cut vertices are important vertices whose removal results in a graph having more linked components. By adding degrees of membership, complex fuzzy cut vertices (CFCV) in complex fuzzy graphs expand on this idea. When a complex fuzzy cut vertex is gradually removed, the connection between various regions of the graph decreases, which represents the relationships' steady deterioration. This idea encapsulates the core of complex relationships, in which ties may weaken gradually as opposed to suddenly.

According to traditional graph theory, bridges are edges that add to the number of linked components in a graph when they are removed. In the context of complex fuzzy graphs, complex fuzzy bridges (CFB) redefine this idea. A complex fuzzy bridge represents an edge with degrees of membership, and the graph gradually splits into additional components when someone removes it. Complex fuzzy bridges reflect the ongoing fading of linkages, reflecting the complexities of connection strengths that exist in the real world, in contrast to classic bridges that establish clear boundaries.

2. Preliminaries

Definition 1: Complex fuzzy graph

A Complex fuzzy graph (CFG) is defined to be a pair $G_c = (X, Y)$ where X is a complex fuzzy set on set of all vertices V , it is denoted by σ_c and Y is a complex fuzzy set on set of all edges $E \subseteq VxV$, it is interpreted by μ_c such that $\mu_Y(a,b)e^{i\alpha_Y(a,b)} \leq \min\{\mu_X(a), \mu_X(b)\}e^{i\min\{\alpha_X(a),\alpha_X(b)\}}$ for all $a, b \in V$.

Definition 2: Order of a CFG

Let $G_c = (X, Y)$ be a CFG, the order of a CFG is described by $O(G_c) = \sum_{a\in V} \mu_X(a)\, e^{i\sum_{a\in V}\alpha_X(a)})$, the degree of a node in G_c is described by $d(x) = \sum_{(a,b)\in E} \mu_Y(a,b)\, e^{i\sum_{(a,b)\in E}\alpha_Y(a,b)})$.

Definition 3: Complex fuzzy bridge (CFB)

Let $G_c = (\sigma_c, \mu_c)$ be a CFG. Let a, b be two different vertices, we obtained partial CFG $G_c{}'$ by deleting the edge (ab) in G_c. We say that (ab) a CFB if $\mu_c'^{\infty}(x,y) < \mu_c^{\infty}(x,y)$ for some $x \, and \, y$ in σ_c' that is in G_c the strength of connectedness between certain vertices in a graph decreases when an edge is deleted.

Definition 4: Strong arc

An arc (m,n) is called strong if $\mu_c(m,n) < \mu_c^{\infty}(m,n)$.

3. Complex fuzzy cut vertices and complex fuzzy bridges

Theorem 1: *Let $G_c = (\sigma_c, \mu_c)$ be a CFG. Thus, the subsequent statements are comparable.*

a) (x, w) *is a CFB*

b) $\mu_c'^{\infty}(x,w) < \mu_c^{\infty}(x,w)$

c) (x, w) *is not the weakest edge of any cycle*

Proof:

$(b) \Rightarrow (a)$: This implication is self-evident from the definition.

$(a) \Rightarrow (b)$: Within a cycle C, if (xw) does not constitute a strong arc that is part of a path P, we can transform it into another path, denoted as P', where (xw) is not an edge. In this case, we utilize the remaining part of cycle C as a path from $'x' \, to \, 'w'$. Consequently, (xw) does not qualify as a CFB. $(c) \Rightarrow (b)$: We establish this by contradiction. If $\mu_c'^{\infty}(x,w) < \mu_c^{\infty}(x,w)$, it implies the existence of a path P that does not include an arc (xw) with energy equivalent to or higher than $\mu_c(x,w)$. When an arc (xw) is combined with the path P, it forms a cycle in G, indicating that (xw) is not a strong arc, hence, it cannot be the least strong edge (i.e., the weakest edge).

Definition 6: Complex fuzzy cut vertex (CFCV)

Let $G_c = (\sigma_c, \mu_c)$ be a CFG. Let u be any vertex and if we delete the vertex u in G_c, we get partial complex fuzzy subgraph $G'_c = (\sigma_c', \mu_c')$ such that $\sigma_c'(u) = 0, \sigma_c = \sigma_c'$ for remaining vertices and $\mu_c'(ua) = 0$ for all 'a', $\mu_c' = \mu_c$ for remaining edges. We say that u is a CFCV, if $\mu_c'^{\infty}(x,y) < \mu_c^{\infty}(x,y)$ for some $x, y \in V$ *and* $x \neq u \neq y$. The strength of connection between certain other vertices decreases when a vertex (u) is deleted; this effect is known as CFCV. That partial complex fuzzy subgraph has no CFCV then it is called non-separable or a block.

Definition 7: Maximum spanning tree (MST)

The maximum spanning tree T_c of G_c is a tree where the strength of the unique strongest x-y route in T_c for every $x, y \in G_c$ is represented by $\mu_c^{\infty}(x,y)$.

Theorem 2: *A vertex is a CFCV ⇔ every maximum spanning tree has* u *as an internal vertex.*

Proof: In a CFG represented as $G_c = (\sigma_c, \mu_c)$, let u be a CFCV. This implies the existence of x and y, distinct from u, such that u is included in every strongest $x - y$ path. This $x - y$ path is a component of each MST in CFG G_c, thus establishing u as an internal vertex in every MST.

Conversely, when u is an internal vertex in every MST, we consider an MST denoted as T_c, where xu and uy are two edges within it. In this tree, the path x, u, v represents the strongest $x - y$ path. Assume that u is not a CFCV.

Corollary 1: *An arc* (xy) *within a CFG* $G_c = (\sigma_c, \mu_c)$ *is classified as a CFB if and only if every MST of* G_c *contains the arc* (xy).

Proof: If an arc (xy) is indeed a CFB, it is inherently part of the unique strongest $x - y$ path. Consequently, all MSTs of G_c are bound to incorporate the arc (xy).

Conversely, suppose that every spanning tree of G_c includes an arc (xy), while also assuming that (xy) does not qualify as a CFB. In this scenario, (xy) may be the weakest arc within certain cycles in G_c, with the condition that $\mu_c^{\infty}(x,y) \geq \mu_c(x,y)$. However, it becomes apparent that (xy) is not present in any of the MSTs of G_c.

Theorem 3: *In a CFG denoted as* $G_c = (\sigma_c, \mu_c)$, *and given a cycle* $G_c^* = (\sigma_c^*, \mu_c^*)$, *a vertex in* G_c *is considered a CFCV if and only if it is a shared vertex between two CFBs.*

Proof: Let us first assume that u is a CFCV within CFG G_c. This implies the existence of x and y, such that u is present in all strongest $x - y$ paths. Since $G_c^* = (\sigma_c^*, \mu_c^*)$ is a cycle, there exists only one strongest $x - y$ path that includes u, and additionally, all arcs within this cycle are categorized as CFGs. Hence, u serves as a vertex that is common to two CFGs.

Conversely, consider u as a vertex common to two CFGs, namely, xu and uy. In this context, both arcs are not classified as weak within G_c. Furthermore, the arcs xu and uy do not appear within the $x - y$ path that possesses the minimal strength, given by $\mu_c(xu)$ *and* $\mu_c(uy)$. Consequently, u qualifies as a CFCV.

Theorem 4: *A vertex u is a CFCV if it is a common vertex in two or more distinct CFBs.*

Proof: Consider two CFBs denoted as a_1u and ua_2. When there are vertices a and b such that both a_1u and ua_2 are present, it follows that a_1u is part of every strongest path from a to b. This implies that u can only be a CFCV if it is distinct from both a and b. If either a or b is equivalent to u, then a_1u becomes part of every strongest path from a to u, or every strongest path from u to b contains ua_2.

Now, assume that u is not a CFCV. In this scenario, there must exist at least one strongest path between any pair of vertices that does not include u. When we consider vertices a and b, there must be at least one strongest path, denoted as P, that does not contain the vertex u. This path, combined with a_1u and ua_2, forms a cycle.

We discuss in two cases:

1. First, suppose that $a_1, u,\ a_2$ is not the strongest path in the cycle. In this case, at least one of the arcs, or possibly both, must be the weakest arcs within the cycle. This situation contradicts the assertion that a_1u and ua_2 are classified as CFBs.
2. Second, let us consider the scenario where $a_1, u,\ a_2$ is indeed the strongest path within the cycle, and it connects with a_2. In this case, the strength of path P is denoted as $\mu_c^\infty(a_1, a_2)$. Furthermore, $\mu_c^\infty(a_1, a_2)$ represents the minimum value of $\mu_c(a_1, u)$ and $\mu_c(u, a_2)$. The arcs within the path P are strong, as indicated by $\mu_c(a_1, u)$ and $\mu_c(u, a_2)$, which implies that both arcs are the weakest within the cycle. This conclusion again leads to a contradiction.

Hence, if u is a common vertex in at least two CFBs, it must be a CFCV.

Theorem 5: *A vertex is a CFCV if and only if it disconnects the graph when removed.*

Proof: Suppose u is a CFCV in CFG $G_c = (\sigma_c, \mu_c)$. There are vertices (apart from u) x and y such that $\mu_c'^\infty(x, y) < \mu_c^\infty(x, y)$ when u is removed. This implies that removing u disconnects the graph, and thus, u is a CFCV.

Conversely, assume that removing vertex u disconnects the graph. This means there exist vertices x and y such that there is no path between x and y when u is removed. In other words, $\mu_c'^\infty(x, y) = 0$ when u is removed. Since

$\mu_c'^{\infty}(x,y) < \mu_c^{\infty}(x,y)$ implies u is a CFCV, we can conclude that u is indeed a CFCV.

Theorem 6: *In a CFG, a simple cycle has all its CFCVs.*

Proof: Suppose u is a CFCV in CFG $G_c = (\sigma_c, \mu_c)$. There exist vertices x and y (distinct from u) such that $\mu_c'^{\infty}(x,y) < \mu_c^{\infty}(x,y)$ when u is removed. Now, consider the strongest $x - y$ path in G_c. Since u is a CFCV, it must be part of this path. Therefore, there is a path from x to u and a path from u to y. When we remove u from G_c, these paths are separated, and there is no direct path from x to y. This implies that there must be a simple cycle in G_c that contains u, connecting x and y. Hence, every CFCV exists in a simple cycle.

Theorem 7: *If a CFB (xy) is part of a cycle, then (xy) cannot be a weak arc in that cycle.*

Proof: Assume that (xy) is a CFB in CFG $G_c = (\sigma_c, \mu_c)$. and that (xy) is part of a cycle in G_c. We want to show that (xy) cannot be a weak arc in that cycle. Since (xy) is part of the cycle, there must exist other vertices and arcs within the cycle. Let us consider an arbitrary vertex a within the cycle that is not $x \text{ or } y$. Now, we analyse the strength of the path from $x \text{ to } a$ within this cycle. Since (xy) is a CFB, it must be part of the strongest $x - y$ path in G_c. Therefore, the path from $x \text{ to } y$ in G_c must include (xy). Now, consider the path from x to a within the cycle. It is a sub path of the path from $x \text{ to } y$ in G_c. Since (xy) is part of the path from $x \text{ to } y$ in G_c, it is also part of the path from $x \text{ to } a$ within the cycle. As a result, (xy) is not a weak arc within the cycle because it contributes to the strength of the path from x to a. Therefore, (xy) cannot be a weak arc in the cycle, as desired.

Corollary 2: *Removing a CFCV disconnects the graph.*

Proof: By Theorem, a vertex is a CFCV if and only if it disconnects the graph when removed. Therefore, removing a CFCV will indeed disconnect the graph.

Corollary 3: *Every vertex in a simple cycle is a CFCV.*

Proof: By Theorem, every CFCV exists in a simple cycle. Therefore, if a vertex is part of a simple cycle, it must be a CFCV.

Corollary 4: *If (xy) is a weak arc in a cycle, then (xy) cannot be a CFB.*

Proof: Suppose (xy) is a weak arc in a cycle in CFG $G_c = (\sigma_c, \mu_c)$. We want to show that (xy) cannot be a CFB. If (xy) is a weak arc within the cycle, it means that the path containing (xy) has a strength less than or equal to the strengths of other paths within the cycle.

Let us assume, for the sake of contradiction, that (xy) is a CFB. This implies that (xy) is part of the unique strongest $x - y$ path in G_c. However, if (xy) were a CFB, the path containing (xy) within the cycle should also be the strongest $x - y$ path, which would contradict our initial assumption. Therefore, (xy) cannot be a CFB if it is a weak arc in the cycle.

4. Application

As extensions of classical graph theory ideas to the domain of CFGs, CFCVs and CFBs find use in a variety of domains where interactions display different degrees of strength. These are a few uses. In a social network, people whose absence could potentially cause a disruption in information flow or community formation can be identified using complex fuzzy cut vertices and bridges. Fuzzy bridge analysis can identify weak links that, if eliminated, might split the network apart into smaller parts. Complex fuzzy bridges may represent important routes with different levels of significance in transportation networks. Finding and eliminating these bridges can provide information about how resilient the system is and how disruptions affect it. Intricate fuzzy cut vertices could identify communication hubs that, in the event of their loss, could cut off various network segments. This is particularly important in systems of communication where the signal intensity varies. Complex fuzzy cut vertices and bridges can be used to highlight proteins in biological networks, such as protein-protein interaction networks, whose interactions are essential for preserving network integrity or regulating information flow. Complex fuzzy bridges may represent species interactions that have the potential to change the dynamics of an ecosystem if they are disturbed. Species with strong linkages that could be affected by the removal of complex fuzzy cut vertices could be indicative of ecosystems that are unstable.

To find crucial communication hubs and weak links, let's look at an example of a communication network shown as a CFG with nodes and edges that each have membership values.

Consider the CFG $G_c = (\sigma_c, \mu_c)$, where $\sigma_c = \{(a, 0.7e^{i0.8\pi})$, $(b, 0.6e^{i0.7\pi})$, $(c, 0.8e^{i\pi}), (d, 0.4e^{i1.1\pi}), (d, 0.5e^{i0.5\pi})\}$ and $\mu_c = \{(ab, 0.5e^{i0.6\pi})$, $(ac, 0.6e^{i0.8\pi})$, $(bd, 0.6e^{i0.9\pi}), (cd, 0.7e^{i\pi}), (ce, 0.4e^{i0.4\pi})\}$

The goal is to find vulnerable linkages and important communication hubs that, if compromised, might impair the connectivity and functionality of the entire network by identifying complex fuzzy bridges and complex fuzzy cut vertices. The communication network is represented graphically by the CFG, where nodes stand in for communication hubs and edges for communication lines. Each node is linked to a certain amplitude and phase value. Through this study, the communication corporation may improve the overall performance and efficiency of the communication network by strengthening the network's resilience, optimizing communication links, and ensuring uninterrupted and dependable communication services.

5. Conclusion

In summary, complex fuzzy cut vertices and complex fuzzy bridges are pivotal components in graph theory. These elements play a crucial role in altering graph connectivity and resilience. Complex fuzzy cut vertices act as disconnectors, impacting the strength of connections, while complex fuzzy bridges contribute to the formation of simple cycles. Understanding these properties enhances our grasp of graph theory and has practical implications in various fields.

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