

## Computing the solution of hybrid fuzzy differential equations with the application of He's variational iteration method

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### Abstract

The expanded idea of higher-order fuzzy differentiability is used in this manuscript to propose the He's Variational Iteration Method (HVIM) as a solution for hybrid fuzzy differential equations (HFDE). An alternate technique that was taken from the literature was contrasted with the discrete solutions that were produced. Less time is spent thanks to the new method's lower computational cost. We assume that there exists a Hukuhara differentiable fuzzy function and its derivative. To show the effectiveness of the approach, a numerical example was shown.

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## 1. Introduction

A hybrid system is a type of dynamic system that displays both continuous and discrete dynamic behavior. It is capable of both continuous flow, which is represented by a differential equation. To set one hybrid system apart from another, say from those that combine fuzzy logic and neural networks or “electrical and mechanical drive lines”, the term “hybrid dynamic system” is often used. One benefit of a hybrid system is that it may include a larger variety of systems into its framework, which increases the flexibility with which dynamic processes can be represented.

A hybrid system's state is usually established by the discrete control mode and the values of the continuous variables. The state may change in two ways: sequentially according to a control graph, or constantly according to a flow condition.

Fuzzy differential equations are utilized as mathematical representations for various practical problems, spanning across disciplines such as science, technology, bio dynamics models [9], population models [10], and hydraulic modeling. The authors of the publication are A. Bencsik and Fodor [1]. Chang and Zadeh [3] were the first to establish the concept of derivative for functions with fuzzy values. The authors Chang and Zadeh published a paper in [3]. Dubois and Prade followed it. Dubois and Prade [5] employed the extension principle in their methodology. Puri and Ralescu have proposed alternative approaches. Later on, the idea of derivatives was used by Kaleva and Seikkala to further develop the notion of fuzzy differential equations. Several researchers have recently focused on developing theoretical and numerical solutions for fractional differential equations (FDEs). Notable contributions include the works of A. Khastan and Ivaz [2], Lakshmikantham and Mohapatra [8], who have explored various aspects of FDEs. In particular, some authors have specifically investigated second-order fuzzy differential equations, such as O. Akin [11], D.N. Georgiou and Rodriguez-Lopez [4], and Sekar and Senthilkumar [14].

Sekar and his team of researchers recently examined hybrid differential equations of fuzzy type using the single-term Haar wavelet series method and the Leapfrog method, as described in the works of Sekar and Senthilkumar [12] and Sekar and Prabhavathi [13]. The aim of this research is to employ the He's variational iteration approach for solving hybrid fuzzy differential equations, as previously explored by Sekar et al. (Sekar and Senthilkumar, [12]; Sekar and Prabhavathi, [13]). Additionally, the variational iteration method is presented for solving

hybrid fuzzy differential equations. Our method is more efficient than the methods proposed by Sekar and Senthilkumar [12] and Sekar and Prabhavathi [13], as it is simpler and requires less computational time.

## 2. He's Variational Iteration Method

This section provides a concise overview of the approach's key components. The original authors of this approach are He [6] and He [7], who presented a universal Lagrange multiplier technique. One use of variational iteration is solving differential equations

$$L[u(\eta)] + N[u(\eta)] = g(\eta)$$

The expression involves the consideration of linear and nonlinear operators, denoted as L and N, respectively. Additionally, there is an inhomogeneous term represented by the function g(t). The corrective functional is employed using this way

$$u_{n+1}(\eta) = u_n(\eta) + \int \lambda [L[u_n(s)] + N[\tilde{u}_n(s)] - g(s)] ds$$

under consideration,  $\tilde{u}_n$  denotes a limited variant,  $\delta \tilde{u}_n = 0$   $u_n$  represents the nth approximation, and where  $\lambda$  denotes a general Lagrange multiplier. Here, we start by finding the  $\lambda$  using variation method  $\delta u_{n+1}(u_n(\eta), \eta) = 0$ . We choose this multiplier so that the correction functional is stationary. Then, given any beginning function that is selective and the computed Lagrange multiplier  $u_n, n \geq 0$ , we may derive the successive approximation of the solution u. Therefore,  $u = \lim_{n \rightarrow \infty} u_n$ . It is possible to generate several approximations by using the correction functional. As a result, the limit of these successive approximations is the exact answer.

## 3. The Fuzzy Hybrid Differential Systems (FHDS)

This section focuses on analyzing the fuzzy initial value problem within the context of FHDS,

$$\left. \begin{aligned} v'(t) &= f(\eta, v(t), \lambda_k v(\eta_k)), \eta \in [\eta_k, \eta_{k+1}], \\ v(\eta_k) &= v_\eta, \end{aligned} \right\}$$

where Seikkala differentiation indicates by  $v^1$ .

The system appears to be precisely as follows

$$v'(t) = \begin{cases} v_0(\eta) = f(\eta, v_0(\eta), \lambda_0(v_0)), v_0(\eta_0) = v_0, \eta_0 \leq \eta \leq \eta_1, \\ v_1(\eta) = f(\eta, v_1(\eta), \lambda_1(v_1)), v_1(\eta_1) = v_1, \eta_1 \leq \eta \leq \eta_2, \\ \dots \\ v_k(\eta) = f(\eta, v_k(\eta), \lambda_k(v_k)), v_k(\eta_k) = v_k, \eta_k \leq \eta \leq \eta_{k+1}, \\ \dots \end{cases}$$

Since there is and can be only one solution to the previous equations for any value of  $\eta$  between  $[\eta_k, \eta_{k+1}]$  we may express the answer as the following function:

$$v(\eta) = v(\eta, \eta_0, \eta_0) \begin{cases} v_0(\eta), \eta_0 \leq \eta \leq \eta_1, \\ v_1(\eta), \eta_1 \leq \eta \leq \eta_2, \\ \dots \\ v_k(\eta), \eta_k \leq \eta \leq \eta_{k+1}, \\ \dots \end{cases}$$

The solutions of the following problem exhibit piecewise differentiability in each interval  $\eta \in [\eta_k, \eta_{k+1}]$ ,  $v_k \in E^1$ .

A pair of functions  $0 \leq r \leq 1$  may be used to represent  $x \in E^1$  using a function  $(v(r), \bar{v}(r))$ , fuzzy number representation that was developed by Wu & Ma [15].

As a result, we may use an analogous system to substitute (1).

$$\begin{cases} \underline{v}'(t) = \underline{f}(t, v, \lambda_k(v_k)) \equiv F_k(t, \underline{v}, \bar{v}), \underline{v}(t_k) = \underline{v}_k, \\ \underline{v}'(t) = \bar{f}(t, v, \lambda_k(v_k)) \equiv G_k(t, \underline{v}, \bar{v}), \underline{v}(t_k) = \underline{v}_k, \end{cases}$$

The pair  $[v(\eta; r), \bar{v}(\eta; r)]$  is a fuzzy number, equations whose parametric forms are provided by

$$\begin{cases} v'(t) = F_k(\eta, \underline{v}(\eta; r), \bar{v}(\eta; r)), \underline{v}(\eta_k; r) = \underline{v}_k(r), \\ v'(t) = G_k(\eta, \underline{v}(\eta; r), \bar{v}(\eta; r)), \underline{v}(\eta_k; r) = \underline{v}_k(r), \end{cases}$$

for  $r \in [0, 1]$ .

#### 4. Numerical Examples

In this part we demonstrate the effectiveness and precision of the provided techniques. Examine the hybrid fuzzy initial value problem presented by Sekar and Prabhavathi in their paper.

$$\left. \begin{aligned} v'(\eta) &= v(\eta) + m(\eta)\lambda_k v(\eta_k), \eta \in [\eta_k, \eta_{k+1}], \eta_k = k, \\ v(\eta; r) &= [(0.75 + 0.25r)e^\eta, (1.125 - 0.125r)e^\eta], 0 \leq r \leq 1, \end{aligned} \right\} \quad (1)$$

$$\text{Where } m(\eta) = \begin{cases} 2(\eta \bmod 1) & \text{if } \eta \bmod 1 \leq 0.5 \\ 2(1 - \eta \bmod 1) & \text{if } \eta \bmod 1 > 0.5 \end{cases}$$

$$\lambda_k(\mu) = \begin{cases} \hat{0}, & \text{if } k = 0 \\ \mu, & \text{if } k \in \{1, 2, \dots\} \end{cases}$$

The hybrid fuzzy initial value problem (IVP) can be expressed as a set of fuzzy IVPs:

$$\left. \begin{aligned} v'_0(\eta) &= v_0(\eta), \eta \in [0, 1], \\ v(0; r) &= [(0.75 + 0.25r)e^\eta, (1.125 - 0.125r)e^\eta], 0 \leq r \leq 1, \\ v'_i(\eta) &= v_i(\eta) + m(\eta)v_{i-1}(\eta), \eta \in [\eta_i, \eta_{i+1}], v_i(\eta) = v_{i-1}(\eta_i), i = 1, 2, \dots \end{aligned} \right\}$$

In (12)  $v(\eta) + m(\eta)\lambda_k v(\eta_k)$  is continuous function of  $\eta$ ,  $v$  and  $\lambda_k v(\eta_k)$ . Therefore, by Example 6.1 of Sekar et al., for each  $k = 0, 1, 2, \dots$  the fuzzy IVP

$$\left. \begin{aligned} v'(\eta) &= v(\eta) + m(\eta)\lambda_k v(\eta_k), \eta \in [\eta_k, \eta_{k+1}], \eta_k = k, \\ v(\eta_k) &= v_{\eta_k}, \end{aligned} \right\}$$

has a unique solution  $[\eta_k, \eta_{k+1}]$ .

**Table 1**  
Error calculations

| t   | Leapfrog Error |               | HVIM Error    |               |
|-----|----------------|---------------|---------------|---------------|
|     | $Y_1(t_i; r)$  | $Y_2(t_i; r)$ | $Y_1(t_i; r)$ | $Y_2(t_i; r)$ |
| 0.1 | 1E-09          | 1E-09         | 1.01E-12      | 1.11E-12      |
| 0.2 | 2E-09          | 2E-09         | 2.01E-12      | 2.11E-12      |
| 0.3 | 3E-09          | 3E-09         | 3.01E-12      | 3.11E-12      |
| 0.4 | 4E-09          | 4E-09         | 4.01E-12      | 4.11E-12      |
| 0.5 | 5E-09          | 5E-09         | 5.01E-12      | 5.11E-12      |
| 0.6 | 6E-09          | 6E-09         | 6.01E-12      | 6.11E-12      |

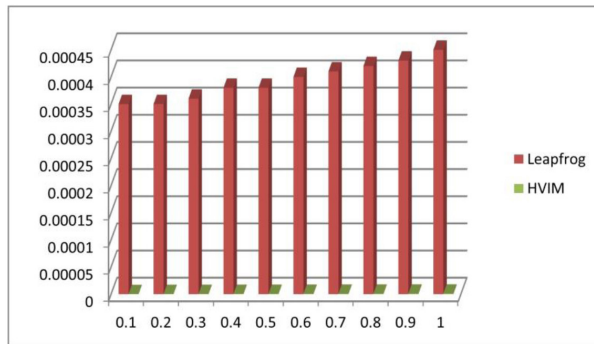


Figure 1

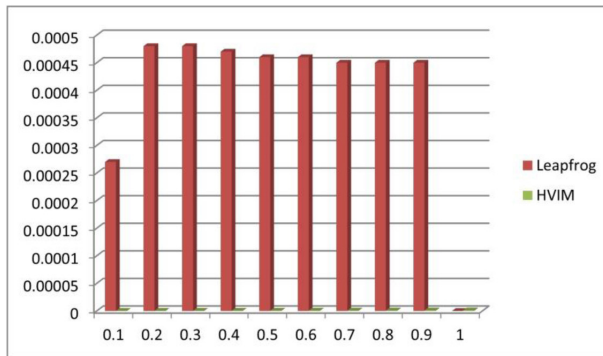
Error estimation of Example at  $Y_1(t; r)$ 

Figure 2

Error estimation of Example at  $Y_2(t; r)$ 

#### 4. Conclusions

The results obtained from the error computations of the hybrid fuzzy differential equations demonstrate the effectiveness of He [7] variational iteration method in finding solutions. Based on the data presented in Table 1 and Figures 1 and 2 for Example 5, it is evident that the He's variational iteration method consistently exhibits lower absolute error than the Leapfrog method outlined by Sekar et al. over most time intervals. The study conducted by Sekar and Prabhavathi in [12] provides both approximate solutions with a minor margin of error. It utilizes a smaller amount of memory and requires less computational time compared to the approach described in Sekar and Prabhavathi's study from [12]. An

illustrative example was provided to showcase the technique's validity and application.

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