

The Raducanu-Orhan operator associated with subclass of meromorphic bi-univalent functions

Reem O. Rasheed

Department of Physics

College of Education Tuzkhurmatu

University of Tikrit

Tikrit

Iraq

Abstract

In this study, we create a subclass of meromorphic bi-univalent functions related to the Raducanu-Orha operator. $\Sigma_{\delta\mu}^*(k, \beta)$. Additionally, estimates of the general coefficients and estimates of the beginning coefficients $|c_0|$, $|c_1|$ and $|c_2|$ and obtain important properties of subclasses to the Raducanu-Orha operator defined on the open unite disk $\mathbb{U}$.

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1. Introduction

Suppose the class of functions is of the form $\mathcal{F}(0) = 0 = \mathcal{F}'(0) - 1$, and let

$$\mathcal{A} = \{ \mathcal{F}: \mathbb{U} \rightarrow \mathbb{C}: \mathcal{F} \text{ is analytic in } \mathbb{U} \}.$$
$$\mathcal{F}(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (1)$$

and allow $\mathcal{S}$ to be a subclass of $\mathcal{A}$ made up of all univalent $\mathcal{F}$ functions in

$$\mathbb{U} = \{z: z \in \mathbb{C}, |z| < 1\}.$$

Koebe quarter theorem [2]. Consequently, each function $\mathcal{F} \in \mathcal{A}$ has an inverse $\mathcal{F}^{-1}$ defined by

$$\mathcal{F}^{-1}(w) = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_3^2 - 5b_2 b_3 + b_4)w^4 + \dots \quad (2)$$

A function $\mathcal{F} \in \mathcal{A}$ is called to be bi-univalent in $\mathbb{U}$, if $\mathcal{F}$ and $\mathcal{F}^{-1}$ are univalent in $\mathbb{U}$. In [6] reality, aglar [3], Frasin and Aouf [2] followed the work by Srivastava et al. [10] investigation of a number of bi-univalent function class Σ subclass in recent years [11]. This study includes the class of meromorphic functions defined on as an extension of the idea of bi-univalent.

$$\mathbb{U}^* = \{z: z \in \mathbb{C}, 1 < |z| < \infty\}.$$

Let the group of meromorphic univalent functions of h with the following form be denoted by the symbol Σ :

$$h(z) = z + c_0 + \sum_{n=1}^{\infty} c_n z^{-n}, \quad (z \in \mathbb{U}) \quad (3)$$

The following extension (see [8],[9]) describes the existence of an inverse $h^{-1} = g$ fulfilling function for any function $h \in \Sigma^*$.

$$h^{-1}(w) = l(w) = w + \mathcal{D}_0 + \sum_{n=1}^{\infty} \frac{\mathcal{D}_n}{w^n}, (\mathcal{M} < |w| < \infty). \quad (4)$$

Simple equations demonstrates that

$$\begin{aligned} w = h(l(w)) &= (c_0 + \mathcal{D}_0) + w + \frac{c_1 + \mathcal{D}_1}{w} + \frac{\mathcal{D}_2 - c_1 \mathcal{D}_0 + c_2}{w^2} \\ &+ \frac{\mathcal{D}_3 - c_1 \mathcal{D}_1 + c_1 \mathcal{D}_0^2 - 2c_2 \mathcal{D}_0 + c_3}{w^3} + \dots \end{aligned} \quad (5)$$

These values were added to (4), and the result is

$$\begin{aligned} g(w) = h^{-1}(w) &= w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} \\ &+ \dots, (\mathcal{M} < |w| < \infty). \end{aligned} \quad (6)$$

Numerous intriguing meromorphic univalent function subclasses have had the coefficient problem explored (see [1,4]).

According to Raducanu and Orhan's [5,7] differential operator for

$$\mathcal{F} \in \mathcal{A}, 0 \leq \mu \leq \delta \text{ and } k \in \mathbb{N} = \{1, 2, 3, \dots\}, D_{\delta\mu}^k \mathcal{F}(z) = D_{\delta\mu}(D_{\delta\mu}^{n-1} \mathcal{F}(z))$$

Operating $D_{\delta\mu}$ successively on the function $\mathcal{F}$, it becomes

$$D_{\delta\mu}^n \mathcal{F}(z) = z + \sum_{n=2}^{\infty} Q_k(\delta, \mu, \lambda) a_n z^n$$

where $q(z) = z + \sum_{n=2}^{\infty} Q_k(\delta, \mu, \lambda) z^n$

and $Q_k(\delta, \mu, \lambda) = [1 + (\delta\mu n + \delta - \mu)(n - 1)]^\lambda$.

Simultaneously, the Raducanu-Orhan differential operator can be described as follows for the value of $h \in \Sigma^*$ given in (3):

$$D_{\delta\mu}^0 h(z) = h(z),$$

$$D_{\delta\mu}^1 h(z) = D_{\delta\mu} h(z) = \delta\mu z^2 h''(z) + (\delta - \mu) z h'(z) + (1 - \delta + \mu) h(z) \quad (7)$$

$$D_{\delta\mu}^k h(z) = D_{\delta\mu}(D_{\delta\mu}^{n-1} h(z)) \quad (8)$$

Then from (7) and (8) we get,

$$\begin{aligned} D_{\delta\mu}^k h(z) &= z + \sum_{n=0}^{\infty} = [1 + (\delta\mu n + \delta - \mu)(n - 1)]^\lambda \frac{c_n}{z^n}; \\ &(\lambda \in \mathfrak{N}_0 = \{0, 1, 2, 3, \dots\}) \end{aligned} \quad (9)$$

2. Coefficient bounded for functions in $\Sigma_{\delta\mu}^*(k, \beta)$

This section defines subclass of the meromorphic bi-univalent and provides general coefficient estimates for its functions.

Definition 1: If the following criterion is met, the meromorphic bi-univalent function $h(z)$ provided in (3) supposedly belongs to the class $\Sigma_{\delta\mu}^*(k, \beta)$:

$$\left| \arg \left(\frac{z \left(D_{\delta\mu}^k h(z) \right)' }{D_{\delta\mu}^k h(z)} \right) \right| < \frac{\beta\pi}{2} \quad (10)$$

and

$$\left| \arg \left(\frac{w \left(D_{\delta\mu}^k \mathcal{G}(w) \right)' }{D_{\delta\mu}^k \mathcal{G}(w)} \right) \right| < \frac{\beta\pi}{2} \quad (11)$$

Where $0 < \beta \leq 1$, $0 \leq \mu \leq \delta$, $k \in \mathbb{N}_0$ and $z, w \in \mathbb{U}^*$ the function $h^{-1}(w) = \mathcal{G}(w)$ is given by (6).

For theorems that ensure, we provide bounds for the coefficient estimates $|c_0|$, $|c_1|$ and $|c_2|$ for h and $\mathcal{G}$ for the subclass $\Sigma_{\delta\mu}^*(k, \beta)$ is derived.

Theorem 1: Let $h(z)$ given in (3) belong to the class $\Sigma_{\delta\mu}^*(k, \beta)$ where $0 < \beta \leq 1$, $0 \leq \mu \leq \delta$, $k \in \mathbb{N}_0$ and $z, w \in \mathbb{U}^*$. Then

$$|c_0| \leq \frac{2\beta}{(1-(\delta-\mu))^n} \quad (12)$$

and

$$|c_1| \leq \frac{\sqrt{5}\beta^2}{(1-2(\delta-\mu))^n} \quad (13)$$

$$|c_2| \leq \frac{2\beta}{3(1-(2\delta\mu+\delta-\mu))^n} \left[2 \left\{ \frac{(6(1-(\delta-\mu))^{3n-1})\beta^2+3\beta-2}{3} \right\} + 3 - 2\beta \right] \quad (14)$$

Proof: Let the function $\mathcal{G}(w)$ given by (6), there has two functions v and τ such that

$$\frac{z \left(D_{\delta\mu}^k h(z) \right)' }{D_{\delta\mu}^k h(z)} = (v(z))^\beta \quad (15)$$

and

$$\frac{w \left(D_{\delta\mu}^k \mathcal{G}(w) \right)' }{D_{\delta\mu}^k \mathcal{G}(w)} = (\tau(w))^\beta \quad (16)$$

where $v(z)$ and $\tau(w)$ fulfil the inequality $\Re(v(z)) > 0$ and $\Re(\tau(w)) > 0$ respectively. Additionally, the functions $v(z)$ and $\tau(w)$ take the following forms:

$$\frac{z \left(D_{\delta\mu}^k h(z) \right)' }{D_{\delta\mu}^k h(z)} = 1 - \frac{(1-(\delta-\mu))^\lambda c_0}{z} + \frac{(1-(\delta-\mu))^{2n} c_0^2 - 2c_1}{z^2}$$

$$- \frac{(1 - (\delta - \mu))^{3n} c_0^3 - 3(1 - (\delta - \mu))^n c_0 c_1 + 3c_2}{z^3} + \dots \quad (17)$$

and

$$\begin{aligned} \frac{w(D_{\delta\mu}^k \vartheta(w))}{D_{\delta\mu}^k \vartheta(w)} &= 1 + \frac{(1 - (\delta - \mu))^\lambda c_0}{w} + \frac{(1 - (\delta - \mu))^{2n} c_0^2 + 2c_1}{w^2} \\ &\quad + \frac{(1 - (\delta - \mu))^{3n} c_0^3 + 3(1 - (\delta - \mu))^n c_0 c_1 + 3(1 - (2\delta\mu + \delta - \mu))^n c_2}{w^3} + \dots \quad (18) \end{aligned}$$

A simple calculation is shown

$$(v(z))^\beta = 1 + \frac{\beta v_1}{z} + \frac{\frac{1}{2}\beta(\beta-1)v_1^2 + \beta v_2}{z^2} + \frac{\frac{1}{6}\beta(\beta-1)(\beta-2)v_1^3 + \beta(\beta-1)v_1 v_2 + \beta v_3}{z^3} + \dots \quad (19)$$

and

$$(\tau(w))^\beta = 1 + \frac{\beta \tau_1}{w} + \frac{\frac{1}{2}\beta(\beta-1)\tau_1^2 + \beta \tau_2}{w^2} + \frac{\frac{1}{6}\beta(\beta-1)(\beta-2)\tau_1^3 + \beta(\beta-1)\tau_1 \tau_2 + \beta \tau_3}{w^3} + \dots \quad (20)$$

Putting (18), (20) in (16) and (19), (20) in (17), we have

$$-(1 - (\delta - \mu))^\lambda c_0 = \beta v_1, \quad (21)$$

$$(1 - (\delta - \mu))^{2n} c_0^2 - 2c_1 = \frac{1}{2}\beta(\beta - 1)v_1^2 + \beta v_2, \quad (22)$$

$$\begin{aligned} -(1 - (\delta - \mu))^{3n} c_0^3 - 3(1 - (\delta - \mu))^n c_0 c_1 + 3c_2 \\ = \frac{1}{6}\beta(\beta - 1)(\beta - 2)v_1^3 + \beta(\beta - 1)v_1 v_2 + \beta v_3, \end{aligned} \quad (23)$$

$$-(1 - (\delta - \mu))^\lambda c_0 = \beta \tau_1, \quad (24)$$

$$(1 - (\delta - \mu))^{2n} c_0^2 + 2c_1 = \frac{1}{2}\beta(\beta - 1)\tau_1^2 + \beta \tau_2 \quad (25)$$

$$\begin{aligned} (1 - (\delta - \mu))^{3n} c_0^3 + 3(1 - (\delta - \mu))^n c_0 c_1 + 3(1 - (2\delta\mu + \delta - \mu))^n c_2 \\ + 3(1 - (2\delta\mu + \delta - \mu))^n c_0 c_1 \\ = \frac{1}{6}\beta(\beta - 1)(\beta - 2)\tau_1^3 + \beta(\beta - 1)\tau_1 \tau_2 + \beta \tau_3. \end{aligned} \quad (26)$$

From (20) and (23), it follows that

$$c_0 = -\beta v_1 = \beta \tau_1 (v_1 = -\tau_1) \quad (27)$$

and

$$c_0^2 = \frac{\beta^2(v_1^2 + \tau_1^2)}{2(1 - (\delta - \mu))^{2n}}. \quad (28)$$

The function $v\left(\frac{1}{z}\right) \in \mathcal{P}$ as $\Re(v(z)) > 0$ in $\mathbb{U}^*$. Similar to that, $\tau\left(\frac{1}{w}\right) \in \mathcal{P}$. The inequality of Lemma [4] is therefore satisfied by the $v(z)$ and $\tau(w)$ coefficients the . Applications of the triangle inequality lead to the following results in (29):

$$|c_0| \leq \frac{2\beta}{(1 - (\delta - \mu))^n}.$$

Furthermore, by using (23) and (26), we have the ability to determine the bound on $|c_1|$. that is,

$$|c_1| \leq \frac{2\sqrt{5}\beta^2}{3},$$

Finally, consider the sum of (22) and (24) with $v_1 = -\tau_1$ to find the constraints on c_2 .

$$c_0 c_1 = \frac{\beta(\beta-1)v_1(v_2-\tau_2)+\beta(v_3+\tau_3)}{6(1-(\delta-\mu))^n + 3(1-(2\delta\mu+\delta-\mu))^n}. \quad (29)$$

Subtracting (27) from (24) with $v_1 = -\tau_1$, we have

$$\begin{aligned} -6(1-(2\delta\mu+\delta-\mu))^n c_2 &= 2(1-(\delta-\mu))^{3n} c_0^3 + 3(1-(2\delta\mu+\delta-\mu))^n c_0 c_1 \\ &\quad + \frac{1}{3}\beta(\beta-1)(\beta-2)v_1^3 + \beta(\beta-1)v_1(v_2+\tau_2) + \beta(v_3-\tau_3) \end{aligned} \quad (30)$$

Putting (26) and (28) in (29) and applying Lemma [1] to the previously mentioned equation, we have

$$|c_2| \leq \frac{2\beta}{3(1-(2\delta\mu+\delta-\mu))^n} \left[2 \left\{ \frac{(6(1-(\delta-\mu))^{3n-1}\beta^2+3\beta-2)}{3} \right\} + 3 - 2\beta \right]$$

This represents the desired c_2 estimations provided by (14). Theorem 1, $k = 1$ assumption leads to the outcomes shown below.

Corollary 1: Let $h \in \Sigma_{\delta\mu}^*(k, \beta)$. Then

$$|c_0| \leq 2\beta, \quad |c_1| \leq \sqrt{5}\beta^2, \quad \text{and} \quad |c_2| \leq \frac{2\beta}{3} \left[2 \left\{ \frac{5\beta^2+3\beta-2}{3} \right\} + 3 - 2\beta \right].$$

5. Discussion and Conclusion

This study establishes a new subclass of meromorphic bi-univalent associated with the Raducanu-Orha operator, denoted by the symbols $\Sigma_{\delta\mu}^*(k, \beta)$ and defined on $\mathbb{U}^* = \{z \in \mathbb{C}; 1 < |z| < \infty\}$. Each of the new subclass coefficient issues has been introduced and researched. We have gotten estimates for the coefficients $|c_0|$, $|c_1|$ and $|c_2|$ are discovered for this subclass each of the meromorphic functions is bi-univalent.

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