*Journal of Interdisciplinary Mathematics*
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 4, pp. 729–735
DOI : 10.47974/JIM-1762

# The numerical solutions for a class of fifth-order partial differential equations via generalized RKM methods

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**Abstract**

This paper's RKM method is explicitly constructed for solving particular fifth-order ordinary differential equations (ODEs) [1]. The proposed method has been used to solve the system of fifth-order (ODEs) numerically by converting the fifth-order partial differential equations. The algorithm of the proposed method is introduced and then used to study

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

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the numerical problems, which showed that the approximated solutions using the proposed approach agree well with the exact solutions.

***Subject Classification:*** *97F20, 32W50.*

***Keywords:*** *Method of lines (MOL), System of fifth-order ODEs, RKM method, Fifth-order PDEs.*

## 1. Introduction

It is common practice to use numerical methods for solving evolutionary issues posed by partial differential equations (PDEs) in cases where doing so is of real importance, so many researchers are interested in employing numerical methods; for example, Stepnicka and Valasek in 2005 developed an F-transform-based computational technique for a family of PDEs [2], and in 2017 Dlamini and Khumalo proposed a novel approach to the solving of PDEs [3]. As well as, Han et al. invention of techniques to solve high-dimensional (PDEs) [4], and Sun et al. in 2019 presented a novel approach to solving the differential equation, one that makes use of the extreme learning machine algorithm and the Bernstein neural network model [5], in addition to [6], [7] and [8] solved Different types of PDEs were solved using various methods. In contrast, Wang et al. in 2021 applied numerical solutions to differential equations that can be posed as inference problems amenable to rigorous statistical approach [9-11]. This paper is organized as follows: the first part contains the introductory information. In part 2 the proposed RKM method is given in Specifics. To prove the reliability of the suggested method, we conducted numerical experiments in part 3 and contained some final ideas about the topic.

## 2. Proposed RKM-Methods for Solving Fifth-Order PDEs

We define the class of quasi-linear partial differential equations of fifth-order as follows:

$$U_{ttttt} = \mathrm{f}(x, t, U, U_x, U_{xx}, U_{xxx}, U_{xxxx}, U_{xxxxx}), \quad a \le x \le \mathrm{b}, \quad 0 < t < T, \tag{1}$$

with the initial conditions (I.C.s):

$$U(x,0) = \mathrm{f}_1(x),\ U_x(x,0) = \mathrm{f}_2(x),\ U_{xx}(x,0) = \mathrm{f}_3(x),\ U_{xxx} = \mathrm{f}_4(x),\ U_{xxxx} = \mathrm{f}_5(x), \tag{2}$$

and the boundary conditions (B.C.s),

$$U(a,t) = \mathrm{g}_1(t),\ U(b,t) = \mathrm{g}_2(t) \tag{3}$$

where $f$, $f_i$ and $\mathrm{g}_j(t)$ are given functions, $i = 1, \dots, 5, j = 1,2$.

For solving problems (1) with (2) and (3), the RKM method was developed by combining with MOL.

2.1. *The Direct Numerical RKM method [1]*

Consider the following fifth-order ODE

$$y^{(5)}(x) = \mathrm{f}(x, z), x \ge x_0, \tag{4}$$

with I. C.s

$$y^{(i)}(x_0) = \gamma^i, i = 0,1, \dots ,4. \tag{5}$$

Let $w_n \cong y(x_n), n = 1,2, \dots.$

The general form of the RKM method with $s$-stage for initial value problem (4)-(5) is as follows:

$$w_{n+1} = w_n + hw_n' + \frac{h^2}{2} w_n'' + \frac{h^3}{6} w_n^{(3)} + \frac{h^4}{24} w_n^{(4)} + h^5 \sum_{i=1}^{s} b_i K_i, \tag{6}$$

$$w_{n+1}' = w_n' + hw_n'' + \frac{h^2}{2} w_n^{(3)} + \frac{h^3}{6} w_n^{(4)} + h^4 \sum_{i=1}^{s} b_i' K_i, \tag{7}$$

$$w_{n+1}'' = w_n'' + hw_n^{(3)} + \frac{h^2}{2} w_n^{(4)} + h^2 \sum_{i=1}^{s} b_i'' K_i, \tag{8}$$

$$w_{n+1}^{(3)} = w_n^{(3)} + hw_n^{(4)} + h^2 \sum_{i=1}^{s} b_i^{(3)} K_i, \tag{9}$$

and

$$w_{n+1}^{(4)} = w_n^{(4)} + h \sum_{i=1}^{s} b_i^{(4)} K_i, \tag{10}$$

where,

$$K_1 = f(x_n, w_n), \tag{11}$$

$$K_i = f\left(x_n + c_i h, w_n + hc_i w_n' + \frac{h^2}{2} c_i^2 w_n'' + \frac{h^3}{6} c_i^3 w_n^{(3)} + \frac{h^4}{24} c_i^4 w_n^{(4)} + h^5 \sum_{j=1}^{i-1} a_{ij} K_j\right),$$

for $i = 2,3, \dots, s$. The RKM method assumes the following real values for the parameters $c_i, a_{ij}, b_i, b_i', b_i'', b_i^{(3)}$, and $b_i^{(4)}$ for $i, j = 1,2, \dots, s$. It is an explicit method if and only If $a_{ij} = 0$ for $i \le j$, and an implicit method otherwise. The following tables of coefficients Table 1 can be used to describe the RKM method in Butcher notation:

**Table 1**
**The Butcher Tableau method for the RKM method of Three-Stage, Fifth-Order**

| | | | |
|---|---|---|---|
| $0$ | $0$ | | |
| $\frac{3}{5} - \frac{\sqrt{6}}{10}$ | $\frac{1}{2}$ | $0$ | |
| $\frac{3}{5} + \frac{\sqrt{6}}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ |
| | $1$ | $0$ | $\frac{-119}{120}$ |
| | $-\frac{1}{40} - \frac{\sqrt{6}}{360}$ | $\frac{1}{60} + \frac{\sqrt{6}}{360}$ | $0$ |
| | $\frac{1}{18}$ | $\frac{1}{18} - \frac{\sqrt{6}}{48}$ | $\frac{1}{18} + \frac{\sqrt{6}}{48}$ |
| | $\frac{1}{9}$ | $\frac{7}{36} - \frac{\sqrt{6}}{48}$ | $\frac{7}{36} - \frac{\sqrt{6}}{48}$ |
| | $\frac{1}{9}$ | $\frac{7}{36} - \frac{\sqrt{6}}{48}$ | $\frac{7}{36} - \frac{\sqrt{6}}{48}$ |

2.2. *The Proposed Numerical Method*

In this subsection, we introduce a computational strategy for solving type IV of $5^{\text{th}}$ −order PDEs (1) - (3) using a hybrid of the method of lines and the RKM method.

Specifically, we presume that the x-axis interval [a,b] and the t-axis interval [0,T] correspond to the numerical solution, with h=(b-a)/N and k= T/M, where N and M are the number of points in the x-direction in [a, b] and t-direction in [0, T], respectively. Combining the method of lines (MOL) and the RKM method, we can solve problem (1) with given initial and boundary conditions (2), (3) by the following algorithm.

**Algorithm:**

1. Do the steps 2-6 while $1 \leq i \leq N$.
2. PDE (1) can be transformed into the following equation by fixing the value of $x = x_i$ at the point $(x, t)$, where $x_i = a + ih$.

$$U_i^{(5)}(t) = f\left(x, t, U(x,t), \frac{\partial U(x,t)}{\partial x}, \frac{\partial^2 U(x,t)}{\partial x^2}, \frac{\partial^3 U(x,t)}{\partial x^3}, \frac{\partial^4 U(x,t)}{\partial x^4}, \frac{\partial^5 U(x,t)}{\partial x^5}\right)\Bigg|_{x=x_i}, \quad (12)$$

3. The derivatives on the right-hand side of ODE (12), when substituted with finite difference formulas of orders 1, 2, 3, 4 and 5, yield a system of ODEs at the fifth-order.

$$U_i^{(5)}(t) = f(x_i, t, U_{i-3}(t), U_{i-2}(t), U_{i-1}(t), U_i(t), U_{i+1}(t), U_{i+2}(t), U_{i+3}(t)), \quad (13)$$

for $i = 1,2,\dots,N$, the fundamental finite differences for orders 1, 2, 3, 4 and 5:

$$\frac{\partial U(x,t)}{\partial x}\bigg|_{(x,t)=(x_i,t_j)} \cong \frac{U_{i+1,j}-U_{i-1,j}}{2h},$$

$$\frac{\partial^2 U(x,t)}{\partial x^2}\bigg|_{(x,t)=(x_I,t_j)} \cong \frac{U_{i+1,j}-2U_{i,j}+U_{i-1,j}}{h^2},$$

$$\frac{\partial^3 U(x,t)}{\partial x^3}\bigg|_{(x,t)=(x_[,t_j)} \cong \frac{U_{i+2,j}-2U_{i+1,j}+2U_{i-1,j}+U_{i-2,j}}{3h^3},$$

$$\frac{\partial^4 U(x,t)}{\partial x^4}\bigg|_{(x,t)=(x_i,t_j)} \cong \frac{U_{i+2,j}-4U_{i+1,j}+6U_{i,j}-4U_{i-1,j}+U_{i-2,j}}{h^4},$$

$$\frac{\partial^5 U(x,t)}{\partial x^5}\bigg|_{(x,t)=(x_i,t_j)} \cong \frac{-U_{i+3,j}+2U_{i+2,j}-5U_{i+1,j}+5U_{i-1,j}-2U_{i-2,j}+U_{i-3,j}}{2h^5}.$$

4. Starting conditions (with j =1)

$$U_i(0) = f_1(x_i),\ U_i'(0) = f_2(x_i),\ U_i''(0) = f_3(x_i),\ U_i^{(3)}(0) = f_4(x_i),\ U_i^{(4)}(0) = f_5(x_i). \tag{14}$$

If $2 \leq j \leq M$, so the (I.C.s) are

$$\begin{aligned} U_i(t_{j-1}) &= U(x_i, t_{j-1}),\\ U_i'(t_{j-1}) &= \left.\frac{dU(x,t_{j-1})}{dx}\right|_{x=x_l},\\ U_i''(t_{j-1}) &= \left.\frac{d^2U(x,t_{j-1})}{dx^2}\right|_{x=x_l},\\ U_i^{(3)}(t_{j-1}) &= \left.\frac{d^3U(x,t_{j-1})}{dx^3}\right|_{x=x_l},\\ U_i^{(4)}(t_{j-1}) &= \left.\frac{d^4U(x,t_{j-1})}{dx^4}\right|_{x=x_l}, \end{aligned} \tag{15}$$

5. Put (B.C.s):

$$U_{0,j} = U(a, t_j) = g_1(t_j),\ U_{n,j} = U(b, t_j) = g_2(t_j), \tag{16}$$

6. By using the RKM method, solve the system of fifth-order ODEs in Equation (13), given (I.C.s) (14) or (15) and (B.C.s) (16), at $t = t_j$.

## 3. Implementations (Numerical Examples)

To test the proposed method, we used some PDEs of fifth-order in the following examples:

**Problem 3.1:** Consider the following 5th order quasi-linear PDEs:

$$y_{ttttt} = -y,\ 0 \leq x \leq 1,\ t > 0,$$

with I.C.s:

$$\begin{aligned} y(x,0) &= e^{-x},\ y_x(x,0) = -e^{-x},\ y_{xx}(x,0) = e^{-x},\\ y_{xxx}(x,0) &= -e^{-x}, y_{xxxx}(x,0) = e^{-x}, \end{aligned}$$

and B.C.s:

$$y(a,t) = e^{-a}e^{-t},\ y(b,t) = e^{-b}e^{-t},$$

The exact solution is $y(x,t) = e^{-t}e^{-x}$.

**Problem 3.2:** Consider the following 5th order quasi-linear PDEs:

$$y_{ttttt} = 4y_{xx} - y_{xxxx},\ 0 \leq x \leq 1,\ t > 0,$$

with I.C.s:

$$\begin{aligned} y(x,0) &= \sin(2x),\ y_x(x,0) = 2\cos(2x),\ y_{xx}(x,0) = -4\sin(2x),\\ y_{xxx}(x,0) &= -8\cos(2x), y_{xxxx}(x,0) = 16\sin(2x), \end{aligned}$$

and B.C.s:

$$y(a,t) = e^{-2t}\sin(2a),\ y(b,t) = e^{-2t}\sin(2b),$$

The exact solution is $y(x,t) = \mathrm{e}^{-2t}\sin(2x)$.

**Problem 3.3:** Consider the following 5th order quasi linear PDEs:

$$y_{ttttt} = 4y_x - y_{xxxx} + (-2)^5\sin(2t),\ 0 \le x \le 1,\ t > 0,$$

with I.C.s:

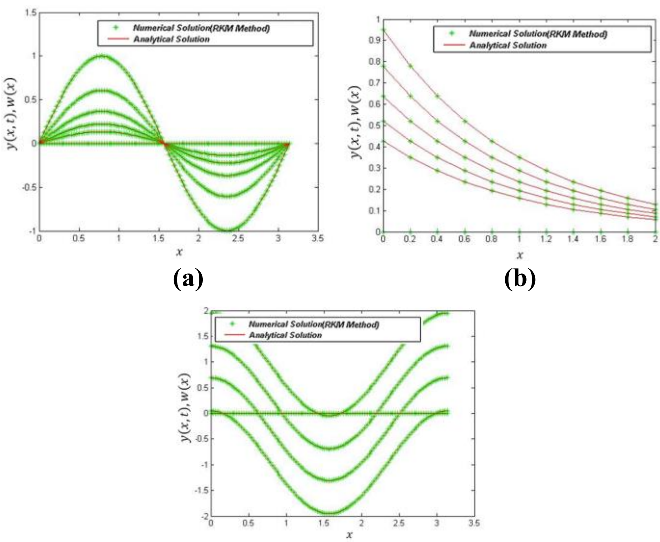
$$\begin{aligned} y(x,0) &= 1 + \sin(2x)\,,\ y_x(x,0) = 2\cos(2x)\,,\ y_{xx}(x,0) = -4\sin(2x), \\ y_{xxx}(x,0) &= -8\cos(2x)\,, y_{xxxx}(x,0) = 16\sin(2x), \end{aligned}$$

and B.C.s:

$$y(a,t) = \cos(2t) + \sin(2a),\ y(b,t) = \cos(2t) + \sin(2b)\,,$$

The exact solution is $y(x,t) = \cos(2t) + \sin(2x)$.

A comparison between the numerical solutions $w(x)$ evaluated by Generalized RKM method versus the exact solutions $y(x,t)$ for the above problems and for 5 Lines of $t$ is shown in Figure 1.




**(a)** **(b)**

**(c)**

**Figure 1**

**A Numerical Comparison for (a) Problem 3.1. (b) Problem 3.2. and (c) Problem 3.3. via Analytical Solutions for Five Lines of t in the Domain.**

## Conclusion

The RKM method, constructed by [1] for solving a special fifth-order ODEs, is combined with the method of lines to solve special fifth-order PDEs. This proposed method was used to solve the fifth-order PDEs by converting them to the system of fifth-order ODEs. The test problems showed that the proposed approach agrees well with the exact solutions. The proposed method proved more accurate and efficient due to the direct method.

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*Received October, 2023*