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Subclass of bi-starlike function associated with touchard polynomials

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Abstract

The current paper aims to study bi- starlike functions. It presents a new class using Touchard polynomial, which fulfills the subordination conditions. In addition, an estimation of the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ was found.

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Keywords: *Convex function, Starlike function, Touchard polynomial, Bi- univalent function, Bi-starlike function.*

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1. Introduction

In (1939) Jacques Touchard studied a polynomial called the Touchard polynomial or an exponential polynomial or Bell polynomial and it is a polynomial with binomial sequence. Let y be a random variable of Poisson distribution and an expected value δ and a kth moment is $E(y^k) = T_k(\delta)$, and this lead to $T_k(z) = e^{-\zeta} \sum_{k=0}^{\infty} \frac{\zeta^k k^\ell}{k!} z^k$. [1] After a period of time Touchard introduced the coefficient of the polynomial after the second force as follows:

$$\mathfrak{X}_\zeta^\ell(z) = z + \sum_{k=2}^{\infty} \frac{\zeta^{k-1}(k-1)^\ell}{(k-1)!} e^{-\zeta} z^k, z \in \mathbb{D}$$

where $\zeta > 0$ and $\ell \geq 0$, by the ratio test the radii of convergence of the series is infinite.[1] we define the function $\mathfrak{S}_\zeta^\ell(z)$ in the following form: $\mathfrak{S}_\zeta^\ell(z) = z^{-1}\mathfrak{X}_\zeta^\ell(z), z \neq 0$.The function $\mathfrak{S}_\zeta^\ell$ can be simplified as follows: $\mathfrak{S}_\zeta^\ell(z) = 1 + (\zeta e^{-\zeta})z + (\zeta^2 2^{\ell-1} e^{-\zeta}) z^2 + \left(\frac{\zeta^3 3^\ell}{3!} e^{-\zeta}\right) z^3 + \cdots, (\zeta e^{-\zeta} > 0)$. We note that the function $\mathfrak{S}_\zeta^\ell$ is analytic with non- negative and non- zero parts in UD, with $\mathfrak{S}_\zeta^\ell(0) = 1, (\mathfrak{S}_\zeta^\ell)'(0) > 0$, and $\mathfrak{S}_\zeta^\ell(UD)$ is symmetric w.r.t the real axis. Let $\wp$ signify the class of holomorphic functions in the open unit disc $UD = \{z: z \in \mathbb{C} \; and \; |z| < 1\}$ and normalized as $\chi(0) = 0$ and $\chi'(0) = 1$ which is in the following form

$$\chi(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

If both the function χ and its inverse χ^{-1} in UD are univalent then it is said to be $\chi \in \wp$ bi$-$ univalent in UD and let £ denotes functions of the bi$-$ univalent class in unit disc UD and this class is not empty because it contains a set of functions. Since $\chi \in$ £ given by (1) has the Maclaurin series , after computation we find that the invers of the function $\xi = \chi^{-1}$ has the following expansion see[2-5]

$$\xi(w) = \chi^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots \tag{2}$$

For convex and starlike functions are subordinate to a superordinate function Ma and Minda define a subclass, they consider the holomorphic function m with non- negative and non- zero parts in the UD, $m(0) = 1$ and $m'(0) > 0$, and m maps UD onto a space starlike w.r.t 1 and is symmetric w.r.t the real axis. So, it is supposed that m is an holomorphic function with non- negative and non- zero parts in UD with $m(0) = 1$ and $m'(0) > 0$, and $m(UD)$ is symmetric w.r.t real axis.These functions are as follows $m(z) = 1 + m_1 z + m_2 z^2 + m_3 z^3 + \cdots, (m_1 > 0)$. [6]İf both χ and χ^{-1} are Ma$-$ Minda bi$-$ convex or starlike then χ is convex of Ma$-$ Minda type or bi$-$starlike of Ma$-$ Minda type. Previously, Brannan and Taha presented certain sub – classes of £ explicity bi$-$starlike of order $\mathcal{O}(0 \leq \mathcal{O} \leq 1)$ denoted by $\mathcal{S}_£^*(\mathcal{O})$, as well as bi$-$ convex functions of order $\mathcal{O}$ which is represented by $\mathcal{C}_£$. For $\chi \in \mathcal{S}_£^*(\mathcal{O})$ and $\chi \in \mathcal{C}_£$ which are non- sharp estimates on the first established two Taylor$-$Maclaurin coefficients $|a_2|$ and $|a_3|$. Nonetheless for each subsequent

Taylor−Maclaurin coefficients solve the coefficients problem. $|a_n|(n \in N\backslash\{1,2\}; N = \{1,2,3, \dots\})$ is still a work in progress.

Definition 2: Let $\chi \in £$ be defined by (1) and $\chi \in \Im_£(\mu, \mathfrak{S}_\zeta^\ell)$. If the following two conditions holds:

$$\frac{z\chi'(z)}{(1-\mu)\chi(z)+\mu z\chi'(z)} \prec \mathfrak{S}_\zeta^\ell(z), \tag{3}$$

$$\frac{w\xi'(w)}{(1-\mu)\xi(w)+\mu w\xi'(w)} \prec \mathfrak{S}_\zeta^\ell(w), \tag{4}$$

where $\ell \geq 0, \zeta > 0$ and $0 \leq \mu \leq 1$, $z, w \in UD$ and $\xi = \chi^{-1}$ given by (2).

Remark 3: For a function $\chi \in £$ assumed by (1) and $\mu = 0$, then $\Im_£(0, \mathfrak{S}_\zeta^\ell)$ satisfies the conditions:

$$\frac{z\chi'(z)}{\chi(z)} \prec \mathfrak{S}_\zeta^\ell(z) \text{ and } \frac{w\xi'(w)}{\xi(w)} \prec \mathfrak{S}_\zeta^\ell(w),$$

where $z, w \in UD$ and $\xi = \chi^{-1}$ given by (2).

4. Coefficients estimates for the functions in the class $\Im_£(\mu, \mathfrak{S}_\zeta^\ell)$

Lemma 4.1: [7] *If* $\chi(z) = \sum_{k=1}^{\infty} c_k z^k$, $z \in UD$, *is analytic function in* UD *such that* $|\chi(z)| < 1$ *for all* $z \in UD$, *then* $|c_1| \leq 1, |c_k| \leq 1 - |c_1|^2, k = 2,3, \dots$.

The upper bounds of the first two coefficients of the functions belonging to the calss are found in the following results.

Theorem 4.2: *Let* χ *given by* (1) *be in the class* $\Im_£(\mu, \mathfrak{S}_\zeta^\ell)$, $0 \leq \mu \leq 1$ *and* $\ell \geq 0, \zeta > 0$. *Then,*

$$|a_2| \leq \frac{\zeta e^{-\zeta}\sqrt{3\zeta e^{-\zeta}}}{\sqrt{\left|(\mu-1)^2(\zeta^2 2^{\ell-1} e^{-\zeta}) + 3(\mu^2-4\mu+3)\left(\zeta e^{-\zeta}\right)^2\right|}}, \tag{5}$$

and

$$|a_3| \leq \frac{2\zeta e^{-\zeta}}{|\mu-1|} + \frac{6\left(\zeta e^{-\zeta}\right)^2 |\mu-3|}{(\mu-1)^2}, \tag{6}$$

where $\mu \neq 1$.

Proof: Let $\chi \in \Im_£(\mu, \mathfrak{S}_\zeta^\ell)$ and $\xi = \chi^{-1}$. From (3) and (4), we have

$$\frac{z\chi'(z)}{(1-\mu)\chi(z)+\mu z\chi'(z)} = \mathfrak{S}_\zeta^\ell(s(z)) \tag{7}$$

$$\frac{w\xi'(w)}{(1-\mu)\xi(w)+\mu w\xi'(w)} = \mathfrak{S}_\zeta^\ell(t(w)), \tag{8}$$

and the functions s, t are of the form

$$s(z) = e_1 z + e_2 z^2 + \cdots, \tag{9}$$

$$t(w) = b_1 w + b_2 w^2 + \cdots, \tag{10}$$

$s(z)$ and $t(w)$ are analytic in UD with $s(0) = t(0) = 0$, and $|s(z)| < 1, |t(w)| < 1$ for all $z, w \in UD$. By using Lemma (4.1), we find that

$$|e_i| \leq 1 \text{ and } |b_i| \leq 1, \text{ for all } i \in N. \tag{11}$$

By substituting (9) and (10) in (7) and (8), respectively, we get

$$\frac{1+2a_2z+3a_3z^2+\cdots.}{1+(3-\mu)a_2z+(4-\mu)a_3z^2+\cdots} = 1 + \left(\varsigma\, e^{-\varsigma}\right)s(z) + \left(\varsigma^2 2^{\ell-1}\, e^{-\varsigma}\right) s^2(z) + \cdots,$$

$$\frac{1-2a_2w+3\left(2a_2^2-a_3\right)w^2+\cdots}{1-(\mu+1)a_2w+(2\mu+1)\left(2a_2^2-a_3\right)w^2+\cdots} = 1 + \left(\varsigma\, e^{-\varsigma}\right)t(w) + \left(\varsigma^2 2^{\ell-1}\, e^{-\varsigma}\right) t^2(w) + \cdots$$

$$1 + (\mu-1)a_2z + [(\mu-1)a_3 - (\mu^2-4\mu+3)a_2^2]\, z^2 + \cdots$$
$$= 1 + \left(\varsigma\, e^{-\varsigma}\, e_1\right)z + \left[\varsigma\, e^{-\varsigma}\, e_2 + \varsigma^2 2^{\ell-1}\, e^{-\varsigma}\, e_1^2\right] z^2 + \cdots,$$

and

$$1 + (\mu-1)a_2w + [2(\mu-1)a_3 + (\mu^2-4\mu+3)a_2^2]w^2 + \cdots$$
$$= 1 + \left(\varsigma\, e^{-\varsigma}\, b_1\right)w + \left[\varsigma\, e^{-\varsigma}\, b_2 + \varsigma^2 2^{\ell-1}\, e^{-\varsigma}\, b_1^2\right] w^2 + \cdots$$

This leads to the following relationships

$$(\mu-1)a_2 = \varsigma\, e^{-\varsigma}\, e_1, \tag{12}$$

$$(\mu-1)a_3 - (\mu^2-4\mu+3)a_2^2 = \varsigma\, e^{-\varsigma}\, e_2 + \varsigma^2 2^{\ell-1}\, e^{-\varsigma}\, e_1^2\,, \tag{13}$$

and

$$(\mu-1)a_2 = \varsigma\, e^{-\varsigma}\, b_1, \tag{14}$$

$$2(\mu-1)a_3 + (\mu^2-4\mu+3)a_2^2 = \varsigma\, e^{-\varsigma}\, b_2 + \varsigma^2 2^{\ell-1}\, e^{-\varsigma}\, b_1^2. \tag{15}$$

From (12) and (14), we have

$$e_1 = b_1, \tag{16}$$

and $-(\mu-1)^2a_2^2 = \left(\varsigma\, e^{-\varsigma}\right)^2[b_1^2 - 2e_1^2]$,

$$a_2^2 = \frac{-\left(\varsigma\, e^{-\varsigma}\right)^2[b_1^2 - 2e_1^2]}{(\mu-1)^2} \tag{17}$$

Multiplying (13) by (2) and subtracting (15), using (17), we get

$$a_2^2 = \frac{\left(\varsigma\, e^{-\varsigma}\right)^3[\,b_2-2e_2]}{(\mu-1)^2\left(\varsigma^2 2^{\ell-1}\, e^{-\varsigma}\right)+3(\mu^2-4\mu+3)\left(\varsigma\, e^{-\varsigma}\right)^2} \tag{18}$$

Applying (11) for the coefficients e_2 and b_2 we obtain (5).

Now by subtracting (13) from (15), using (16) and (17), we have

$$a_3 = \frac{\varsigma\, e^{-\varsigma}(b_2-e_2)}{\mu-1} - 2(\mu-3)a_2^2 \tag{19}$$

$$= \frac{\varsigma\, e^{-\varsigma}(b_2-e_2)}{\mu-1} + \frac{2\left(\varsigma\, e^{-\varsigma}\right)^2(\mu-3)\left(2e_1^2-b_1^2\right)}{(\mu-1)^2}.$$

Applying (11) for the coefficients e_1, e_2, b_1 and b_2, we deduce (6).

Remark 4.3: Let χ assumed by (1) be in the class $\Im_{£}\left(0, \mathfrak{S}_{\varsigma}^{\ell}\right)$. Then,

$$|a_2| \leq \frac{\zeta\, e^{-\zeta}\sqrt{3\zeta\, e^{-\zeta}}}{\sqrt{\left|\left(\zeta^2 2^{\ell-1}\, e^{-\zeta}\right)+9\left(\zeta\, e^{-\zeta}\right)^2\right|}}, \text{ and } |a_3| \leq 2\zeta\, e^{-\zeta} + 18\left(\zeta\, e^{-\zeta}\right)^2$$

Remark 4.4: Let χ assumed by (1) be in the class $\mathfrak{I}_{\pounds}\left(\mu, \mathfrak{S}_{\zeta}^{1}\right)$. Then,

$$|a_2| \leq \frac{\zeta\, e^{-\zeta}\sqrt{3\zeta\, e^{-\zeta}}}{\sqrt{\left|(\mu-1)^2\left(\zeta^2\, e^{-\zeta}\right)+3(\mu^2-4\mu+3)\left(\zeta\, e^{-\zeta}\right)^2\right|}}, |a_3| \leq \frac{2\zeta\, e^{-\zeta}}{|\mu-1|} + \frac{6\left(\zeta\, e^{-\zeta}\right)^2|\mu-3|}{(\mu-1)^2},$$

where $\mu \neq 1$.

5. Discussion and Conclusion

Through the use of Touchard polynomial, the study first defines a new class and finds the initial Tayler coefficients. Hence, to define another class, a linear combination with Touchard polynomial is utilized.

References

[1] S. Porwal and G. Murugusundaramoorthy, "Univalent functions with positive coefficients involving Touchard polynomials," *Al-Qadisiyah Journal of Pure Science*, vol. 25, pp. 1-8 (2020).

[2] G. I. Oros and L.-I. Cotîrlă, "Coefficient estimates and the Fekete–Szegö problem for new classes of m-fold symmetric bi-univalent functions," *Mathematics*, vol. 10, p. 129 (2022).

[3] A. R. S. Juma and M. H. Saloomi, "Generalized differential operator on bistarlike and biconvex functions associated by quasi-subordination," in *Journal of Physics: Conference Series*, p. 012046 (2018).

[4] M. M. Gour, S. Joshi, P. Goswami, and S. Purohit, "New classes of bi-univalent functions," *Journal of Interdisciplinary Mathematics*, vol. 23, pp. 583-590 (2020).

[5] K. A. Jassim, R. O. Rasheed, and R. H. Jassim, "Generalized subclass of analytic bi-univalent functions defined by differential operator," *Journal of Interdisciplinary Mathematics*, vol. 24, pp. 961-970 (2021).

[6] A. Alburaikan, G. Murugusundaramoorthy, and S. M. El-Deeb, "Certain Subclasses of Bi-Starlike Function of Complex Order Defined by Erdély–Kober-Type Integral Operator," Axioms, vol. 11, p. 237 (2022).

[7] Al-Ameedee, Sarah A. & Obaid, Ahmed J. Sandwich results of meromorphic multivalent functions defined by multiplier transform, *Journal of Interdisciplinary Mathematics*, OnlineFirst : Latest Articles, 1-8 (2023), DOI: 10.47974/JIM-1484.

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