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Studying the upper bound of Beurling zeta function in the strip (0,1)

Safa Mohsen Abd Alzahra *
Faez A. Al-Maamori †
Department of Mathematics
College of Education for Pure Sciences
University of Babylon
Babylon
Iraq

Abstract

During the third decade of the last century, Arne Beurling stated the sense of the generalized prim systems. He said that any positive, increasingly real sequence started from any number greater than one precisely, and this sequence is called Beurling primes (or the generalized prim). They also mentioned that the fundamental Theorem of arithmetic gives the generalized integers. In the seventeenth decade of the last century, Diamond modified all the related counted functions of prime π_g (x) and integers Ng (x) and the generalized Riemann zeta function. This work addresses an overview of the behavior of the generalized counting function N_g (x) where the change happens in the error term of N_g (x). The changing in E(x) = $|N_g$ (x)-ρx$|$ implies changing in the real part of (s = σ + it) and hence affects the behavior of the generalized ζ_g (x).

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Introduction

The B-prime system of Beurling's work opened several hard works that had been done since the middle of the last century. It's appeared from the numerous papers that any information on the B-counting function of primes then leads to inform us about the B-counting function of integers and vice versa. Of course, this involved the generalized zeta function $\zeta_g(x)$, the meaning of the

* *E-mail:* **safa.alzahra.pure440@student.uobylon.edu.iq (Corresponding Author)**
† *E-mail:* **faez@it.netuobabylon.edu.iq**

word "information" about the behaviors of $N_g(x)$ as $x\to\infty$ say means $\lim_{x\to\infty} |N_g(x) - \rho x|$ =E(x) [1-4].

Where E(x) is the error term. Therefore, concentrating on the changing of E(x) gives changing on the real part of (s=σ+it) and then changing in the behaviors of $\zeta_g(x)$. The reason for what is mentioned is related to:

$$\zeta_g(s) = \int_1^\infty x^{-s}\, dN_g(x) \tag{1}$$

$$\frac{-\zeta'_g(s)}{\zeta(s)} = \int_1^\infty x^{-s}\, d\Psi_g(x) \tag{2}$$

Moreover, $\zeta_g(s) = \int_1^\infty x^{-s} dN_g(x) = \exp\{\int_1^\infty x^{-s}\, d\prod_g(x)\}$ where $\prod_g(x) = \sum_{k=1}^\infty \frac{\pi(x^{\frac{1}{k}})}{k}$

As a result, the generalized zeta function establishes a relationship between the counting functions of primes and integers. There is no loss of the generality of mention of the R H: The $\zeta_q(\sigma + it)$ is a constant in the plane where its real part >1, and then it does not have any zeros there. At the same time, this function has "nontrivial zeros" in the plane where Re(s) ≥0. In fact, $\zeta_q(\sigma + it)$ = zero with σ =-2n. So, the remaining zeros of $\zeta_q(\sigma + it)$ lie on the (1/2+t) that is the RH meaning. In the complex plane, that was defended as "the critical line." Therefore, the (RH) is where the zeta function's nontrivial zeros are all located. On the other sense, the Riemann Hypothesis means that the following holds:

$\pi(x) = li(x) + O(x^{\frac{1}{2}-\epsilon})$ ∀∈>0,⇔ N(x) = x + O(1). As known from the definitions π(x) and N(x) are counting functions related to each other by ζ(s), and hence if (R H) is true, this means that the zeros of (s) have σ =1/2. Notably, the vertical line R(s) =1/2 contains all of the nontrivial zeros of (s). The literature provides some known upper bounds for the actual zeta function, which are as follows:

(i) $\zeta(1/2 + it) = O(t^{\frac{1}{6}} \log t)$

(ii) In (2005), Huxley showed that $\zeta(1/2 + it) = O(t^{\frac{32}{20c}} \log t)$ in the case of certain

(iii) In (1963), Richert proved that $(1 + it) = O(\log\ t)^{\frac{2}{3}}$, he also established for $1/2 \le \sigma \le 1$ that: $|\zeta(\sigma+it)| \le A(1 + t^{B(1-\sigma)^{\frac{3}{2}}}(\log t)^{2/3}$, with B = 100.

The data provided above is connected to the function $\zeta_g(\sigma + it)$ upper bound. Therefore, increasing this function's upper bound is the difficult part here. Daimond developed the generalization of the Riemann-zeta function in the seventeenth decade of the previous century, about thirty years after Beurling defined "the generalized prime system." Beurlig extended the meaning of prime numbers as follows:

g – prime is a sequence of positive reals $p_1, p_2, p_3, \ldots$ satisfying $1 \le p_1 \le p_2 \le p_3 \le, \ldots \le p_n \le \ldots$ and for which $p_n \to \infty$ as $n \to \infty$. The numbers $\{p_n\}$ are called g- primes (or B- primes). The g- integers (or B- integers) can be determined by using the arithmetic f-Theorem. Through the g-zeta function ζ_g (s), the g-counting functions of integers and primes are connected to one another as follows:

$$\zeta_g\,(s)= \int_{-1}^{\infty} x^{-s}\, d\, Ng(x) \qquad (*)$$

Whereas
$$\frac{-\zeta' g(s)}{\zeta g(s)}=\int_{-1}^{\infty} x^{-s}\ d\Psi_g(x) \qquad (**)$$

When one of the counting functions is defined (as x goes to infinity), the relationship between $\Psi g(x)$, $Ng(x)$ *and* $\zeta g(x)$ exists, we may determine the other. Thus, the literature informs us that understanding $\Psi_g(x)$ leads to understanding the behaviours of the integers $N_g(x)$ Additionally, the B-zeta function is involved, as is of course the case $\zeta_g(s)=\sum_{n\geq 1} n^{-s}$ with $s=\sigma$+it.

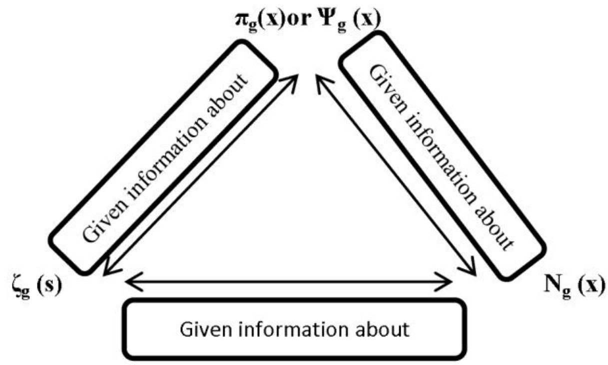

Figure 1
Shows the relationship between the triangles

Diamond defined the generalisation of ζ (s) for a complex variable s = σ + it in his paper (see [5]) in the seventh decade of the last century. This generalisation is applicable for σ> 1, so that:

$$\zeta_q\,(s)= \sum_{n\geq 1} n^{-s} =\prod_p\,(1\text{-}1\tfrac{1}{p^s})^{-1} \qquad (2)$$

Here, p is notified as a prime for which the product is running over all primes. Clearly the above equation iszaClearly the above equation isza converges absolutely for σ >1 (see [5]).

Define the space of arithmetical functions V. For any v in V, v: R→¢

is right-continuous and of nearby bounded variety, given that x is inside the extend of 1 to limitlessness.

Let $V^+\subseteq V$ and for any v in V, m is increasing function. So, if $V_0\subseteq V$ (where $V_0=\{v\in V: v(1)=0\}$), then $V_0^+= V\cap V+$.

In order to let the reader see some easy and difficult Beurling's prime systems:

For instance, one can take the sequence of B-primes apart from 2. This informs us that the function of counting B-primes asymptoticly to $\frac{x}{\log x}$-1.

whereas, applying the fundamental Theorem of arithmetic, one could get a sequence of B-integers numbers for which the set of Beurling's integers has no even numbers. Here, the counting function of B-integers behaves as an asymptotic of $O(\frac{x}{2})$.

So, in the above example means there is a B- primes system such that for some k >0 $\lim_{x\to\infty}\left|\frac{(\pi_q(x)-1)\log x}{x}\right|=1$ and $\lim_{x\to\infty}\left|N_q(x)-\frac{1}{2}x\right|\le k$.

This actually said a discrete B-prime system (πq, Nq) , Nq depends on Beurling's integers produced from the same g-integers and πq reflects the B-primes.

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It's worth letting the reader keep in mind that it is not always building the B-integers sequence from the B-primes sequence since it is usually difficult and maybe impossible. The below equations reflect Figure 1 above:

$\zeta_g(s)=\int_{1-}^{\infty} x^{-s}\, dN_g(x)$ and $\frac{-\zeta_g{}'(s)}{\zeta_g(s)}=\int_{1-}^{\infty} x^{-s} d\psi_g(x)$ Where $\pi_g(x)\sim\psi_g(x)$ for $\sigma>1/2$ (see [5]).

So with the above it is obvious that:

$V_0^+=\{v: v$ is increasing function with $v(1)=0\}$ and $V_1=\{v: v$ is increasing function with $v(1)=1\}$ $N_q(x)=e^{\pi_q}$. Then the order pair (π_q, N_q) can be referred to as Beurling's prime system [5].

This work is concentrated on studying different error teems of $|N_g(x)-\rho x|$ in order to see the changing of the real part of $(s=\sigma+it)$ in the way that $\zeta_g(s)$ still has the upper bound $O(t^c)$ for some c<1.

This paper adapted the following Theorem from [2] to see the changing error term of $|N_g(x)-\rho x|$ and its affection on the upper bound of the generalized zeta function [6-9].

Theorem (*) :[2] *Let U has an analytic continuation to $h_\gamma\backslash\{1\}$, and $\gamma<1$ with residue ρ and $U(\sigma\pm it)$ "e^t as $t\to\infty$ for $\sigma>\gamma$ Assume furthermore that $U(x)=\rho x+O(xe^{k(x)})$, where u is positive and strictly increasing such that u(ex)/x is decreasing. Then ∃ a,c > 0 s-t $U(\sigma\pm it)$ "t^c as $t\to\infty$, Around the area where $\sigma>1-\frac{u(e^{at})}{at}$*

So, in order to explain the size of the error term of $\lim_{x\to\infty}|N(x)-\rho x|$, one could take for instance, with

$u_1(x)=\sqrt{\log x\,\log\log x}$ this shows that for $\sigma>1-/\frac{1}{\sqrt{t}}$ Beurling zeta function still has the form $O(t^c)$. Moreover, for $u_2(x)=\frac{\log x}{\log\log x}$ which means that the Beurling zeta function still has the same upper bound for $\sigma>1-\frac{1}{\log t}$. The reader could take more applications such that $u_3=\sqrt[5]{\log x}$ or $u_4=\sqrt[4]{(x-\log x)^3}$. All he above gives the size of $\lim_{x\to\infty}|N(x)-\rho x|$ for which the Beurling zeta function still has the form $O(t^c)$. However, the difference between $u_i(x)$ under the conditions of Theorem (*) shows the difference in the place "means the real part of s" that the Beurling zeta function has the same upper bound but in a very tiny strip. See the figure below from what is mentioned above, one can see that the estimation of N_q (x) (or π_q (x)) gives information about Beurling Zeta function behavior in the strip (0,1). So, this also told us that whenever $U_i(x)$ as a function of x goes to infinity so sharp (as in the example one above) then the Beurling zeta function still of the form $O(t^c)$ [10-15].

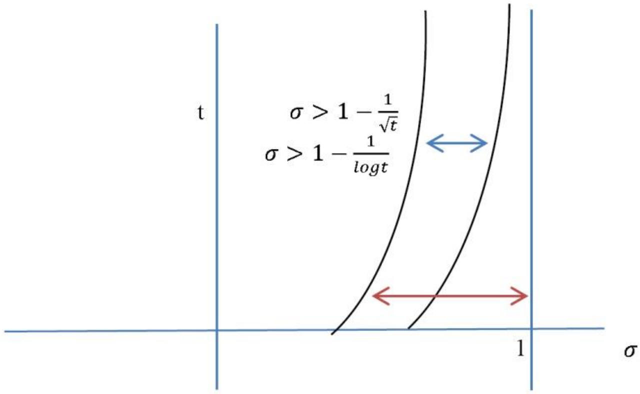

Figure 2
Shows the change in E(x).

σ is very close to line 1, and vice versa.

Now, Suppose u(x) to be $\frac{a\log x \log\log\log x}{\log\log x}$ where $\lim_{x\to\infty} \left| \frac{N(x)-\rho x}{x\exp\{-u(x)\}} \right| \leq k$ (***), for some positive constants a,k.

The question is "Does Beurling zeta function still of the form $O(t^c)$, and does the c must be less than one or could be bigger than one!! as shown in Figure 2 above.

In order to answer this, the reader must have the necessary information and be familiar with the paper [2]. In fact, as a special case with (**) and assuming $\Psi_g(x)+1=x-\{x\}$, $x\geq1$ then $(\log\zeta'_g$ (s) $+1=\zeta(s)$. Here ζ_g is the Beurling zeta function, whereas ζ on the LHS is the actual Riemann-zeta function. In this case with (***) hold, then one could show that the Beurling zeta function is also of the form $O(t^c)$ for some c>0 with $1-\sigma\geq a \sqrt[3]{\log x\,(\log\log x)^{-1})^2}$. This opened a so many windows for research, one of these is the upper bound of Beurling zeta function ζ_g $(\sigma+it)$ when σ gets closer to the real line ½. Indeed the upper bound of ζ_g $(\sigma+it)$ should be $O(e^t)$.

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