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Some properties of the first-order partial differential equation in the spaces V_α and L_p

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Abstract

We want to prove the realization of periodic solutions, their density, strong and exponential stability in the L_p, $0 < p < 1$ and V_α spaces of first order partial differential equation.

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1. Introduction

The partial differential equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial t} = \lambda(a)u\,, t \geq 0,\ \ 0 \leq a \leq 1 \tag{1}$$

The elementary condition [1][2][3][4]

$$u(0,a) = v(a)\,, \quad 0 \leq a \leq 1 \tag{2}$$

$\vec{v} \in V$, $(V, \|.\|)$ of $\lambda: [0,1] \to R$ is continuous

Assume $\dot{T}_T$ is semidynamical $\dot{T}_T: V \to V$

And $(\dot{T}_t\vec{v})(a) = \vec{u}(t,a)$

$\vec{u}$ is unique solution of (1) ,(2). Is given by:

$$(\dot{T}_t\vec{v})(a) = \vec{u}(t,a) = e^{G(a)}e^{-G(ae^{-t})}\vec{v}(ae^{-t})\,, a \in [0,1] \tag{3}$$

Where $G(a) = -\int_x^1 \frac{\lambda(s)}{s}ds$ and condition

$$\int_0^1 \frac{\lambda(s)}{s}ds = \infty. \tag{4}$$

Definition 1.1: $\vec{v}_0 \in W$ is a periodical point function of the semigroup $(\dot{T}_t)_{t\geq 0}$, and a period $t_0 \geq 0 \iff \dot{T}_{t_0}\vec{v}_0 = \vec{v}_0$. A number $t_0 > 0$, is named a principal

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period of a periodical point $v^{\rightharpoonup}{}_0 \Leftrightarrow$ the set of all periods (SAP) of $v^{\rightharpoonup}{}_0$ is equal $t_0 N t_0$. [5][6][7]

Definition 1-2: $(\dot{T}_t)_{t\geq 0}$ is powered stable system in $V \Leftrightarrow$ for every $v^{\rightharpoonup} \in V$, $\dot{T}_t v^{\rightharpoonup} = 0$ in V

Definition 1-3: $(\dot{T}_t)_{t\geq 0}$ is exponentially stable system if and only if there exist $D < \infty$ and $\omega > 0$ such that $\|\dot{T}_t\| \leq De^{-\omega t}, for\ t \geq 0$.

It [1] there was described properties of the PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial t} = \gamma u, 0 \leq a \leq 1. \tag{5}$$

the elementary condition

$$u(0,a) = v(a)\,, 0 \leq a \leq 1. \tag{6}$$

The semidynamical system [8][9][10]

$$(T_t v)(a) = u(t,a) = e^{\gamma t} v(ae^{-t}),\ \ a \in [0,1]$$

u is solution of (5) and $u^{\rightharpoonup}$ is solution of (1) then

$$u(t,a) = q(a)u^{\rightharpoonup}(t,a) \tag{8}$$

(8) this is connection between (1) and (5)

$$q(a) = \int_0^a \frac{\lambda(0)-\lambda(s)}{s} ds \text{ and } \gamma = \lambda(0). \tag{9}$$

2. L_p space when $0 < p < 1$

Formula 21: Suppose that $\exists\ C, r > 0\ \ \forall a \in [0,1]\ \ |\lambda(0) - \lambda(a)| \leq Ca^r$

$$u^{\rightharpoonup} \in L_p \ \Leftrightarrow u \in L_p.$$

Proof: Substitution (8) by (2) and assuming that $u^{\rightharpoonup} \in L_p$ when $0 < p < 1$,we will find

$$\|u(t,a)\|^p = \int_0^1 |q(a)u^{\rightharpoonup}(t,a)|^p da \leq \int_0^1 e^{p\int_0^1 \frac{|\lambda(0)-\lambda(a)|}{s} ds} |u^{\rightharpoonup}(t,a)|^p da$$
$$\leq \int_0^1 e^{\frac{Cp}{r}a^r} |u^{\rightharpoonup}(t,a)|^p da \leq e^{\frac{Cp}{r}} \int_0^1 |u^{\rightharpoonup}(t,a)|^p\, da$$
$$= e^{\frac{Cp}{r}} \|u^{\rightharpoonup}(t,a)\|^p < \infty$$

The order direction can be proved in the same way the importance of the previous theorem lies in reaching the next theorem. We were able to use the results of [1,11,12] and generalize them. As of now , we suppose (2-1).

Formula 2.2: $\left\{\forall\lambda(0) : \exists -\frac{1}{p} < \lambda(0)\,,\ \exists v^{\rightharpoonup}\right\}$ $v^{\rightharpoonup}$ is a periodical solution of (1)

Proof: $\forall t_0$, the function $v^{\rightharpoonup}$ defined as:

$$v^{\rightharpoonup}(x) = e^{G(a)} e^{-G(ae^{nt_0})} \omega^{\rightharpoonup}(ae^{nt_0}), \quad a \in \left[e^{-(n+1)t_0}, e^{-nt_0}\right]$$
$$v^{\rightharpoonup}(0) = 0$$

Where $\vec{\omega} \in L_p, 0 < p < 1$ this function not no appointment. we proved that the function $v^{\rightharpoonup}$ is periodic solution , this is enough to prove that $v^{\rightharpoonup} \in L_p, v^{\rightharpoonup}$ is solution of equation (1), we use the function v as solution to (5) , $v^{\rightharpoonup}(a) = \frac{v(a)}{q(a)}$. $\tilde{v}$ is a periodic solution and $\tilde{v} \in L_p$ for $\gamma > -\frac{1}{p}$,as proved by Brzezniak and one of the authors [1]. By hypothesis and theorem 2.1 we draw a similar result for the function $v^{\rightharpoonup}$.

Formula 2.3: If $\lambda(0) > -\frac{1}{p}$, then the set of periodical points (SPP) is dense in $L_p, 0 < p < 1$.

Proof: let $\omega^{\rightharpoonup} \in L_P$ and $\epsilon > 0$. Fix $t_0$such that $\left[\int_0^{-t_0} |\omega^{\rightharpoonup}(a)|^p da\right]^{\frac{1}{p}} < \frac{\varepsilon}{2}$ and $e^{\frac{\varepsilon}{r}}\|\tilde{v}\| < \frac{\varepsilon}{2}$, $C > 0$, v is a periodical solution of (5).

(2-2) the formula defined through the function $v^{\rightharpoonup}$. By theorem (2-2), $v^{\rightharpoonup}$ belong to the set of periodic point. it is enough estimate $\|v^{\rightharpoonup} - \omega^{\rightharpoonup}\|$. Since $v^{\rightharpoonup}(a) = \omega^{\rightharpoonup}(a)$ for $x \in [e^{-t_0}, 1]$, 0bviously $\|v^{\rightharpoonup} - \omega^{\rightharpoonup}\| = \|(v^{\rightharpoonup} - \omega^{\rightharpoonup})1_{[0,e^{-t_0}]}\|$, where $1_{[0,e^{-t_0}]}$ denotes the indicator of the set $[0, e^{-t_0}]$. Compensating for (8) and applying theorem (2-1) it can be confirmed that

$$\|v^{\rightharpoonup} - \omega^{\rightharpoonup}\| \le \|v^{\rightharpoonup}1_{[0,e^{-t_0}]}\| + \|\omega^{\rightharpoonup}1_{[0,e^{-t_0}]}\| < \epsilon$$

Formula 2.4: If $\lambda(0) \le -\frac{1}{p}$, and $v^{\rightharpoonup} \in L_p$ then $\|\dot{T}_t v^{\rightharpoonup}\| = 0$

Over and above , $\forall\lambda(0) < -\frac{1}{p}$,$\left(\dot{T}_t\right)_{t\ge 0}$ is exponentially stable system on L_p.

Proof: Let $v^{\rightharpoonup} \in L_p$ be an arbitrary function.

$$\|\dot{T}_t v^{\rightharpoonup}\|^p = \int_0^1 |u^{\rightharpoonup}(t,a)|^p da = \int_0^1 \left|\frac{u(a)}{q(a)}\right|^p da$$
$$= \int_0^1 \left|\frac{1}{q(a)}\left(\dot{T}_t v\right)(a)\right|^p da \le e^{\frac{Cp}{r}}\|\dot{T}_T v\|^p$$

When $C > 0$.and you know that $\|\dot{T}_t v\| \to 0$, as $t \to \infty$ [1] , which proves the first segment of the formula. from [1] you know that $\|\dot{T}_t v\|^p \le e^{(\gamma p+1)t}\|v\|^p$. From it we get that the system is exponentially stable $\left(\dot{T}_t\right)_{t\ge 0}$ with $D = e^{\frac{C}{r}}$ and $\omega = -\frac{1}{p}(\lambda(0) + 1)$.

3. The V_α Space

For every α belong to the interval (0,1] and for every $A \subset [0,1]$, assume function on [0,1], define

$$H_{A,\alpha}(v^{\rightharpoonup}) = sup_{a,b\in A, a\ne b}\frac{|v^{\rightharpoonup}(a) - v^{\rightharpoonup}(b)|}{|a-b|^\alpha}$$

where $H_{A,\alpha}(v^{\rightharpoonup}) < \infty$ a Holder continuous (HC). formulate $H_\alpha = H_{[0,1],\alpha}$

Definition 3.1: Let V_α space of all (HC) function $v^{\rightharpoonup}$ on $[0,1]$ together α, approach at zero and the following condition is fulfilled $H_{[0,1],\alpha}(v^{\rightharpoonup}) = 0$

$\forall \gamma > \alpha$, there exist periodic solution of (1.5) as well as (SAP) is dense in V_α . power stability and EXP stability spoke on condition that $\gamma \leq \alpha$ and $\gamma < \alpha$.

Formula 3.2: Let $|\lambda(0) - \lambda(a)| \leq Ca, C > 0, a \in [0,1]$ (10)
verified. Then $u^{\rightharpoonup} \in V_\alpha$ if and only if $u \in V_\alpha$.

Proof: The assume $u^{\rightharpoonup} \in V_\alpha$, it leads to that $u^{\rightharpoonup}$ is a (HC) function together with α, as well as $lim_{a\to 0} H_{[0,1],\alpha}(u^{\rightharpoonup}) = 0$. Gives us

$$H_{[0,1],\alpha}(u) = sup_{a.b\in[0,1],a\neq b} \frac{|u(t,a)-u(t,b)|}{|a-b|^\alpha}$$
$$= sup_{a.b\in[0,1],a\neq b} \frac{|q(a)u^{\rightharpoonup}(t,b)-k(b)u^{\rightharpoonup}(t,b)|}{|a-b|^\alpha}$$
$$\leq sup_{a\in[a,x]}|q(a)|.\, sup_{a.b\in[0,1],a\neq b} \frac{|u^{\rightharpoonup}(t,a)-u^{\rightharpoonup}(t,b)|}{|a-b|^\alpha}$$
$$+sup_{a\in[a,x]}|u^{\rightharpoonup}(t,b)|.\, sup_{a.b\in[0,1],a\neq b} \frac{|q(a)-q(b)|}{|a-b|^\alpha}$$

Take advantage of (9) and (10) , you obtain:

$$|q(a)| \leq e^{\int_0^a \frac{\lambda(0)-\lambda(s)}{s}ds} \leq e^{\int_0^a Cds} = e^{Ca}.$$

k is (HC) with the exponent α. if it $k^{\frac{1}{\alpha}}$ turns out to be a Lipschitz function, For $a \in [0, x]$

$$\left|\left(q^{\frac{1}{\alpha}}(x)\right)'\right| = \left|e^{\frac{1}{\alpha}\int_0^a \frac{\lambda(0)-\lambda(s)}{s}ds}.\frac{1}{\alpha}.\frac{\lambda(0)-\lambda(a)}{a}\right| \leq \left|e^{\frac{1}{\alpha}\int_0^a Cds}.\frac{C}{\alpha}\right|$$
$$= \left|e^{\frac{Ca}{\alpha}}.\frac{C}{\alpha}\right| \leq e^{\frac{Cx}{\alpha}}.\frac{C}{\alpha}$$
$$H_{[0,x],\alpha}(u) \leq e^{Cx}\left(H_{[0x],\alpha}(u^{\rightharpoonup}) + sup_{b\in[0,x]}|u^{\rightharpoonup}(t,b)|.\left(\frac{C}{\alpha}\right)^\alpha\right).$$

For this, $(u) = 0$ and $u \in V_\alpha$. In the same way ,we prove that $u^{\rightharpoonup}$, suppose that $u \in V_\alpha$.

Formula 3.3: If $\lambda(0) > \alpha \in (0,1]$, For any t_0, there exists such $v^{\rightharpoonup}{}_0 \in V_\alpha$ that

$$T_{t_0} v^{\rightharpoonup}{}_0 = v^{\rightharpoonup}{}_0 \tag{11}$$

And

$$\dot{T}_{t_0} v^{\rightharpoonup}{}_0 = v^{\rightharpoonup}{}_0 \Leftrightarrow t = nt_0 ,n \geq 0. \tag{12}$$

Proof: Let $\omega^{\rightharpoonup}$ be not on assignment (HC) function for exponent α, and defined on the $[e^{-t_0}, 1]$ with the conditions is fulfilled:

$$e^{-G(e^{-t_0})}\omega^{\rightharpoonup}(e^{-t_0}) = \omega^{\rightharpoonup}(1) \tag{13}$$

$$\forall t \in (0, t_0)\ e^{-g(e^{-t})}\omega^{\rightharpoonup}(e^{-t}) \neq \omega^{\rightharpoonup}(1) \quad (14)$$

the following function is defined $v^{\rightharpoonup}$ on (0,1]:

$$v^{\rightharpoonup}(a) = e^{G(a)}e^{-G(ae^{nt_0})}\omega^{\rightharpoonup}(ae^{nt_0}) \text{ for } a \in \left[e^{-(n+1)t_0}, e^{-nt_0}\right]$$

$$(0,1] = \cup_{n=0}^{\infty}(e^{-(n+1)t_0}, e^{-nt_0}] \text{ is whole interval.}$$

Since is supposition of the continuity of $\omega^{\rightharpoonup}$ on $[e^{-t_0}, 1]$, its bounded, i.e $\exists\, M > 0,\quad \exists\, |\omega^{\rightharpoonup}(a)| \leq M,\quad \forall a \in [e^{-t_0}, 1]$. previously, for $x \in \left[e^{-(n+1)t_0}, e^{-nt_0}\right]$,we have the following checks:

$$|v^{\rightharpoonup}(a)| = e^{G(a)}e^{-g(ae^{nt_0})}|\omega^{\rightharpoonup}(ae^{nt_0})| \leq Me^{G(a)}.\,sup_{a\in[e^{-t_0,1}]}e^{-G(a)} \leq M_1e^{G(a)}$$

When $M_1 = M.\,sup_{a\in[e^{-t_0,1}]}e^{-G(a)}$. Using (1.4) , $e^{G(a)} = 0$, we conclude form this $v(0) = 0$. Thus we will get function v exact on the whole interval $[0,1]$. Property (11) from (12), as well as property (13) from (14).

Assume $\lambda(0) > \alpha$ leads to $v \in V_\alpha$, see [1]. From theorem (12), we see now $v^{\rightharpoonup} \in V_\alpha$.

Formula 3.4: $\forall\lambda(0): \alpha < \lambda(0)$, (SPP) of (1) is dense in V_α.

Proof: Let $v^{\rightharpoonup}$ be a function not no assignment $v^{\rightharpoonup} \in V_\alpha$.

$$\omega^{\rightharpoonup}(a) = e^{G(a)}\left(e^{-G(ae^{(nt_0)})}v^{\rightharpoonup}(ae^{nt_0}) - \sum_{k=n+1}^{\infty} \quad me^{-G(ae^{kt_0})}\right)$$

When $m = e^{-G(e^{t_0})}v^{\rightharpoonup}(e^{-t_0}) - v^{\rightharpoonup}(1)$ and $x \in \left[e^{-(n+1)t_0}, e^{-nt_0}\right]$. It sufficient to make the following observation to prove the correctness of the definition.

$$e^{G(e^{-(n+1)t_0})}\left(e^{-G(e^{-(n+1)t_0}e^{nt_0})}v^{\rightharpoonup}(e^{-(n+1)t_0})e^{nt_0} - \sum_{k=n+1}^{\infty} \quad me^{-G(e^{-(n+1)t_0}e^{kt_0})}\right)$$

$$= e^{G(e^{-(n+1)t_0})}\left(v^{\rightharpoonup}(1) + m - \sum_{k=n+1}^{\infty} \quad me^{-G(e^{-(n+1)t_0}e^{kt_0})}\right)$$

$$= e^{G(e^{-(n+1)t_0})}\left(e^{-g(e^{-(n+1)t_0}e^{(n+1)t_0})}v^{\rightharpoonup}(e^{-(n+1)t_0})e^{(n+1)t_0} - \right)$$

$$\sum_{k=n+1}^{\infty} \quad me^{-G(e^{-(n+1)t_0}e^{kt_0})}$$

$\omega^{\rightharpoonup}$ is a consequence of (3-6). continuous vanishes at zero .

Let $\epsilon > 0$. Since $v^{\rightharpoonup}, \omega^{\rightharpoonup} \in V_\alpha$, $\exists$ t_0 that $H_{[0,e^{-t_0}],\alpha}(v^{\rightharpoonup}) < \frac{\varepsilon}{4}$ and $H_{[0,e^{-t_0}],\alpha}(\omega^{\rightharpoonup}) < \frac{\epsilon}{4}$. we know that $\omega^{\rightharpoonup}(a) = \frac{\omega(a)}{q(a)}$ from (1.8), ω is periodic solution of PDE (1.5). $\forall\lambda(0) > \alpha$ (SPP) of PDF (1.5) is dense in V_α [1],and also so $H_\alpha(v^{\rightharpoonup} - \omega) < \frac{\varepsilon}{4}$ and $H_\alpha(\omega^{\rightharpoonup} - \omega) < \frac{\varepsilon}{4}$.

Therefore $H_\alpha(v^{\rightharpoonup} - \omega^{\rightharpoonup}) \leq H_{[e^{-t_0},1],\alpha}(v^{\rightharpoonup} - \omega^{\rightharpoonup}) + H_{[0,e^{-t_0}],\alpha}(v^{\rightharpoonup} - \omega^{\rightharpoonup})$

$$\leq H_\alpha(v^{\rightharpoonup} - \omega) + H_\alpha(\omega - \omega^{\rightharpoonup}) + H_{[0,e^{-t_0}],\alpha}(v^{\rightharpoonup}) + H_{[0,e^{-t_0}],\alpha}$$

Formula 3.5: If $\lambda(0) \leq \alpha$ and $v^{\rightharpoonup} \in V_\alpha$, then $H_\alpha\left(\dot{T}_t v^{\rightharpoonup}\right) = 0$

Over and above, if $\lambda(0) < \alpha$, then $\left(\dot{T}_t\right)_{t\geq 0}$ is exponentially stable system.

Proof: Taking advantage of theorem 2-2 and the proof in the same way, we find

$$H_\alpha\left(\dot{T}_t v^{\rightharpoonup}\right) = sup_{a,b\in[0,1],a\neq b}\frac{|u^{\rightharpoonup}(t,a)-u^{\rightharpoonup}(t,b)|}{|a-b|^\alpha} = sup_{a,b\epsilon[o,1],a\neq b}\frac{\left|\frac{u(t,a)}{q(a)}-\frac{u(t,b)}{q(b)}\right|}{|a-b|^\alpha}$$

$$\leq sup_{a\in[0,1]}\left|\frac{1}{q(a)}\right|.sup_{a,b\in[0,1],a\neq b}\frac{|u(t,a)-u(t,b)|}{|a-b|^\alpha}$$

$$+sup_{b\epsilon[0,1]}|u(t,b)|.sup_{a,b\in[0,1],a\neq b}\frac{\left|\frac{1}{q(a)}-\frac{1}{q(b)}\right|}{|a-b|^\alpha}$$

$$\leq e^C.\left(H_\alpha(T_t v) + sup_{b\in[0,1]}|u(t,b)|.\left(\frac{C}{\alpha}\right)^\alpha\right)$$

As we know $T_T v \to 0$ in V_α for every $v \in V_\alpha$. This requirement $T_\alpha(T_t\tilde{v}) = 0$ is return to the results of paper [1]. We draw the following from the like, source

$$\|T_t v\| \leq e^{(\gamma-\alpha)t}\|v\|.$$

$$|u(t,b)| = |(T_t v)(b)| = |(T_t v)(b) - (T_t v)(0)| \leq H_\alpha(T_t v)b^\alpha \leq H_\alpha(T_t v)$$

since $\lambda(0) < \alpha$,this gives exponential stability of the system $\left(\dot{T}_t\right)_{t\geq 0}$ together with $D = e^C\left(1+\left(\frac{C}{\alpha}\right)^\alpha\right)$ as well $\omega = \alpha - \lambda(0)$.

4. Conclusion

we describe a system $\left(\dot{T}_t\right)_{t\geq 0}$ generated through the first order partial differential equation (1-1).we found that some properties were realized in the space $L_p, 0 < p < 1$, such as periodic solutions , their density for $\lambda(0) > -\frac{1}{p}$, and exponentially stable in this space when $\lambda(0) \leq -\frac{1}{p}$. We also saw their realization in the Holder continuous functions, if $\lambda(0) > \alpha \in (0,1]$ then periodic solutions and density of (1.1), and also $\lambda(0) \leq \alpha$ achieved exponentially stable .So these properties were deduced in two spaces $L_p, 0 < p < 1$ and V_α.

References

[1] Braga E. J. and Marcelo J. S. de Lemos, " Simulation of turbulent natural convection in porous cylindrical annulus using a macroscopic two- equation model", *International Journal of heat and Mass transfer* 49, 4340-4351, (2006).

[2] Dawidowics A. L. and Haribash A. ," On the dense trajectory of Lasota equation", Univ. Iagell. Acta Math 43, 61-66, (2005).

[3] A.L. Dawidowicz A. L. ," On the existence of an invariant measure for a quasi-linear partial differential equation", Zeszyty Nankowe UJ, Parce Matematyczne 23, 117-123, (1982).

[4] Dawidowicz A. L., "On the existence of an invariant measure for the dynamical system generated by partial differential equation", Ann. Polon. Math. XLI, 129-137,(1983).

[5] Dawidowicz A. L., Haribash N., "On the periodic solution of von Foerster type equation", Universitatis Iagellonicae Acta Mathematica 37, 321-324, (1999).

[6] Matsui M. ,Takeo F. "Chaotic semigroup generated by certain differential operators of order 1", SUT J, Math 37(1), 51-67, (2001).

[7] Lasota A. , Pianigiani G. , "Invariant measures on topological spaces", Boll. Un. Mat. Ital, (5) 15-B, 592-603, (1977).

[8] Lasota A., Szark T. , "Dimension of measure invariant with respect to Wazewska partial differential equation", *J. Differential Equations* 196, 448-465, (2004).

[9] Lasota K. , "Turbulent solution of first order partial differential equation", *J. Differential Equation* 58, 1, 1-14 (1985).

[10] Rudnick R. , "Invariant measures for the flow a first order partial equation", Ergodic Theory and Dynamical System, 5, 3, 437-443 (1985).

[11] Abbood, Mohaimen M. , Ebrahim, Hassan H., Al-Fayadh, Ali & Obaid, Ahmed J. () Measure defined on Γ–algebra and some of their generalizations, *Journal of Interdisciplinary Mathematics*, 1-7, DOI: 10.47974/JIM-1502.

[12] Al-Ameedee, Sarah A. & Obaid, Ahmed J. Sandwich results of meromorphic multivalent functions defined by multiplier transform, *Journal of Interdisciplinary Mathematics, OnlineFirst : Latest Articles*, 1-8 (2023), DOI: 10.47974/JIM-1484.

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