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Some properties of analytic univalent functions group

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Abstract

This research investigates the properties of an analytical univalent functions group (S, *) of the following form: $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $a_k \in R$

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Some facts and theorems regarding this group are provided where (*) is the Hadmard product operation (or convolution). More evidence is offered for the isomorphism of the groups (S, *) and (R-{0}, ·).

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1. Introduction

By S. Roman in 2010 see [1], the concepts of group and subgroup were first established. In [3], Koebe (1907) and Bieberbach (1916) studied monovalent analytic functions that define U over the domain with some remarkable geometric properties. Let T be the class of analytical functions in U. For k positive integer $and\ a_k \in \mathbb{R}$, we let $T[x,k]$ be the subclass of T consisting of the form: $f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \cdots, a_k \in \mathbb{R}$

Also, let S be the subclass of T consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \in \mathbb{R} \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$

If $f_1 \in S$ is given by (1.1) and $f_2 \in S$ given by: $f_2(z) = z + \sum_{k=2}^{\infty} b_k z^k$.

The Hamdard product (or convolution) of $f_1\ and\ f_2$ is given by:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z)\ .$$

See [2] S. D. Theyab established the abelian group (S, *), also called the analytic group, in 2022. This group is referred to as an analytic univalent functions group in this study. Additionally, we discover a bijective homomorphism between the groups (S, *) and (R- {0}, ·). We investigate several properties and develop claims and theorems that adhere to this concept [4-6].

2. Some Properties of Analytic Univalent Functions Group

We offer various properties and theorems on an analytical univalent functions group in this part.

To demonstrate the next theorem, we provide this lemma.

Lemma 2.1: *Let* $(S,*)$ *be a group and let* $f, g, h, k \in S$, *then*

$$(f * g) * (h * k) = f * ((g * h) * k).$$

Proof: Let $h * k = f_1$. Then, since the operation $*$ is associative, we have

$$\begin{aligned} f * ((g * h) * k) &= f * (g * (h * k)) \\ &= f * (g * f_1) \\ &= (f * g) * f_1 \\ &= (f * g) * (h * k) \end{aligned}$$

Theorem 2.2: *Let* $(S,*)$ *be a group and* $f, g \in S$, *then* $(f * g)^{-1} = g^{-1} * f^{-1}$.

Proof: Given the definition of inverse, all we need to demonstrate is that:

$$(f * g) * (g^{-1} * f^{-1}) = (g^{-1} * f^{-1}) * (f * g) = I,$$

The group identity is represented by e. The conclusion that would follow from determining the inverse of f*g's uniqueness is

$$(f * g)^{-1} = g^{-1} * f^{-1}.$$

Using the above lemma, we have

$$\begin{aligned}(f * g) * (g^{-1} * f^{-1}) &= f * \left((g * g^{-1}) * f^{-1}\right) \\ &= f * (I * f^{-1}) = f * f^{-1} = I\end{aligned}$$

A similar argument establishes that.

$$(g^{-1} * f^{-1}) * (f * g) = I.$$

Definition 2.3: The center of a group $(S,*)$, denoted by cent S, is the set cent $S = \{f \in S: \quad f * f_1 = f_1 * f \quad \forall f_1 \in S\}$. Thus, cent S consists of those elements which commute with every element of S.

Theorem 2.4: *The group* $(S,*)$ *is abelian if and only if cent* $S = S$.

Theorem 2.5 : *The pair (cent* $S,*$*) is a subgroup of a group* $(S,*)$.

Proof: We first observe that cent $S \neq \phi$, for at very least $I \in centS$.

Suppose that $f, g \in$ cent S. By definition of cent S, we know that $f * f_1 = f_1 * f$ and $g * f_1 = f_1 * g, \ \forall f_1 \in S$.

Thus

$$\begin{aligned}(f * g^{-1}) * f_1 &= f * (g^{-1} * f_1) \\ &= f * \left({f_1}^{-1} * g\right)^{-1} \\ &= f * \left(g * {f_1}^{-1}\right)^{-1} \\ &= f * (f_1 * g^{-1}) \\ &= (f * f_1) * g^{-1} \\ &= (f_1 * f) * \mathrm{g} = f_1 * (f * g^{-1}),\end{aligned}$$

which implies $f * g^{-1} \in centS$, this is a sufficient condition for (cent $S,*$) to be a subgroup of $(S,*)$.

Theorem 2.6: *Let*$(S,*)$*and* $(\mathbb{R} - \{0\},\cdot)$ *be two groups, where (·) as usual, and (*) denote ordinary multiplication and convolution respectively. For* $a_k \in \mathbb{R} - \{0\}$, *and define the function* $\beta: S \to R - \{0\}$ *by:*$\beta(a_k) = f(z)$ *then* β *is isomorphism.*

Proof: $\beta: (\mathbb{R} - \{0\},\cdot) \longrightarrow (S,*), \ \beta(a_k) = f(z)$

- To prove β is well defined.
  Let $a_k, b_k \in \mathbb{R} - \{0\}$ and

$$a_k = b_k$$

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} b_k$$

$$\sum_{k=2}^{\infty} a_k z^k = \sum_{k=2}^{\infty} b_k z^k$$

$$z + \sum_{k=2}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k$$

$$f(z) = g(z)$$

$$\mathrm{B}(a_k) = \mathrm{B}(b_k)$$

- To prove onto:
  Note from $\beta(a_k) = f(z)$, B is onto the function
- To prove one-to-one:
  Let $a_k, b_k \in \mathbb{R} - \{0\}$ and

$$\beta(a_k) = \beta(b_k)$$

$$z + \sum_{k=2}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} b_k z^k$$

$$\sum_{k=2}^{\infty} a_k z^k = \sum_{k=2}^{\infty} b_k z^k$$

$$\sum_{k=2}^{\infty} a_k = \sum_{k=2}^{\infty} b_k$$

$$a_k = b_k$$

To prove $\beta(a_k \,.\, b_k) = \beta(a_k) * \beta(b_k), \forall a_k, b_k \in \mathrm{R}$

$$\beta(a_k \,.\, b_k) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (z + \sum_{k=2}^{\infty} a_k z^k) * (z + \sum_{k=2}^{\infty} a_k z^k) = \beta(a_k) * \beta(b_k)$$

So β is isomorphism.

Definition 2.7: Let β be a homomorphism from the group : $(\mathbb{R} - \{0\}, \cdot)$ into the group $(S, *)$and let I be the identity element of $(S, *)$. The kernel of β, denoted by $\ker(\beta)$, is the set $\ker(\beta) = \{a_k \in \mathbb{R} - \{0\} : \beta(a_k) = \mathrm{I}\}$

Theorem 2.8: *Let β be a homomorphism from the group* : $(\mathbb{R} - \{0\}, \cdot)$ *into the group* $(S, *)$. *Then β is one-to-one if* $ker(\beta) = \{I\}$.

Proof: Suppose the function B is one-to-one. We already know that $1 \in \ker(\beta)$. We aim to show that this is the only element in the kernel. If there existed another element $a_k \in \ker(\beta), a_k \neq 1$, then we would have $\beta(a_k) = \mathrm{I} = \beta(1)$. That is, $\beta(a_k) = \beta(1)$ but $a_k \neq 1$.

This would contradict the hypothesis that β is one-to-one. On the other hand, suppose that $\ker(\beta) = \{1\}$. Let $a_k, b_k \in \mathbb{R} - \{0\}$ and $\beta(a_k) = \beta(b_k)$, then

$$\beta\left(a_k . {b_k}^{-1}\right) = \beta(a_k) * \beta\left({b_k}^{-1}\right) = \beta(a_k) * \beta(b_k)^{-1} = \beta(a_k) * f(a_k)^{-1} = \mathrm{I}$$

which implies $a_k . {b_k}^{-1} \in \ker(\beta)$. But, $\ker(\beta) = \{1\}$. Therefore $a_k . {b_k}^{-1} = 1$ or $a_k = b_k$.

Theoerm 2.9: *If B is a homomorphism from the group* $:(\mathbb{R} - \{0\},\cdot)$ *into the group* $(S,*)$*, then the pair* $(ker(\beta),.)$ *is a subgroup of* $(\mathbb{R} - \{0\},\cdot)$

Proof: We first observe that $\ker(\beta) \neq \phi$, for at very least $1 \in \ker(\beta)$.

Suppose that $a_k, b_k \in \ker(\beta)$ By definition of $\ker(\beta)$, we know that $\beta(a_k) = \mathrm{I}$ and $\beta(b_k) = \mathrm{I}$, $I \in S$.

Thus,

$$\beta\left(a_k . {b_k}^{-1}\right) = \beta(a_k) * \beta\left({b_k}^{-1}\right) = \beta(a_k) * \beta(b_k)^{-1} = \mathrm{I} * \mathrm{I}^{-1} = \mathrm{I}$$

which implies $a_k . {b_k}^{-1} \in \ker(\beta)$, this is a sufficient condition for $(\ker(\beta),.)$ to be subgroup of: $(\mathbb{R} - \{0\},\cdot)$.

References

[1] S. Roman, Fundamentals of Group Theory, Springer New York Dordrecht Heidelberg London (2010).

[2] S. D. Theyab, Various Problems on New Subclasses of Univalent and Bi-Univalent Functions with Some its Analytic Groups, Ph.D. Dissertation University of Kufa, 202

[3] P. Koebe, Uber die uniformisiesung beliebiger analytischeer Kurven, Nach Ges. Wiss. Gottingen, 191-210 (1907).

[4] Shubham Sharma & Ahmed J. Obaid. Mathematical modelling, analysis and design of fuzzy logic controller for the control of ventilation systems using MATLAB fuzzy logic toolbox, *Journal of Interdisciplinary Mathematics*, 23:4, 843-849 (2020), DOI: 10.1080/09720502.2020.1727611.

[5] S.P. Goyal & Pranay Goswami. On sufficient conditions for analytic functions to be Bazilevi [cbreve], *Complex Variables and Elliptic Equations*, 54:5, 485-492 (2009), DOI: 10.1080/17476930902755690.

[6] Zhi-Hong Liu, Yong Sun & Zhi-Gang Wang. On harmonic starlike functions with respect to symmetric, conjugate and symmetric conjugate points, *Quaestiones Mathematicae*, 37:1, 79-90 (2014), DOI: 10.2989/16073606.2013.779960.

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