Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 27 (2024), No. 4, pp. 737–743
DOI : 10.47974/JIM-1763

Some numerical methods for solving fractional partial difference equations in the discrete domain

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Abstract

This paper proposes a method for solving fractional partial difference equations using discrete Sumudu transformation. Some properties of discrete Sumudu transformation were used. The applications of the test examples for the initial value problems of fractional partial

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

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difference equations were tested using two methods: the successive approximation method. The second, homotopy perturbation, was proven efficient with the proposed method.

Subject Classification: *35R11, 35R10.*

Keywords: *Discrete calculus, Sumudu transform, Fractional partial difference equation, Homotopy perturbation method.*

1. Introduction

A difference equation or recurrence equation is an equation like this. The equations show up in many different places and ways, both in mathematics and the fields of statistics, computer science, electrical circuit analysis, biology, and other areas where math is used [1-3,8]. This article aims to accomplish its purpose of providing an efficient, frequent relationship in the interest of resolving these issues [4,13,14]. In this paper, we have solved the partial difference equations using a combination of two essential methods, the Sumudu Transformations and the successive approximation method, and we have provided an iterative process to obtain the final solution.

2. Simple Model Including Quantity Demand

This section presents a few definitions and axioms of the Sumudu Transform related to this paper.

Lemma 2.1 [5]: The discrete Sumudu transform for some elementary functions for $\left|\frac{u+1}{u}\right| > \aleph$ are as follows:

i) $\mathbb{S}\{\sum_{m=0}^{k-1} f(m)\} = u\mathbb{S}\{f(t)\}$ (1)

ii) $\mathbb{S}\{t^{\underline{n}}\} = n!\,u^n$, where $t^{\underline{n}} = t(t-1)\cdots(t-n+1),\ n \in N$

Definition 2.2: Falling factorial function [12]: The falling factorial power $t^{\underline{\nu}}$ (read t to ν falling) is defined as follows: $t^{\underline{\nu}} = t(t-1)(t-2)\cdots\big(t-(\nu-1)\big) = \prod_{k=0}^{\nu-1}(t-k) = \frac{\Gamma(t+1)}{\Gamma t(+1-\nu)}$, Where $\nu \geq 0$

Lemma 2.3 [11]: If $\left|\frac{u+1}{u}\right| > \aleph$, then for the same values of u,

(i) $\mathbb{S}\{\Delta f(k)\}(u) = \frac{1}{u}[F(u) - f(0)]$ (2)

and in general,

(ii) $\mathbb{S}\{\Delta^n f(k)\}(u) = u^{-n}\left[F(u) - \sum_{i=0}^{n-1} u^i \left|\Delta^i f(k)\right|_{k=0}\right],$ (3)

3. Sumudu Transform

In this subsection, some properties of the Sumudu transform of the discrete domain have introduced.

Theorem 3.2 [7]: S*uppose* $f\colon \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$ *and let* $v > 0$ *with* $N-1 < v \leq$ *Then for all* $u \in \mathbb{C} \setminus \{-1,0\}$ *such that* $|(u+1)/u| > r$, $\mathbb{S}_{a+N-v}\{{}^{C}\Delta_a^v f\}(u) = \frac{(u+1)^{N-v}}{u^N}[\mathbb{S}_a\{f\}(u) - \sum_{k=0}^{N-1} u^k \Delta^k f(a)]$.

Theorem 3.3 [7]: *Suppose* $f\colon \mathbb{N}_a \to \mathbb{R}$ *is of exponential order* $r \geq 1$ *and let* $v > 0$ *with* $N-1 < v \leq N$. *Then for all* $u \in \mathbb{C} \setminus \{-1,0\}$ *such that* $|(u+1)/u| > r$,

i) $\mathbb{S}_{a+v}\{\Delta_a^{-v} f\}(u) = (u+1)^v \mathbb{S}_a\{f\}(u)$,

ii) $\mathbb{S}_{a+v-N}\{\Delta_a^{-v} f\}(u) = \frac{u^N}{(u+1)^{N-v}} \mathbb{S}_a\{f\}(u)$.

4. Fractional Sum Operator and Fractional Difference Operator

Definition 4.1 (the Sum Fractional) [10]: If $\digamma\colon \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ is given, then the α^{th} order a fractional sum of $\digamma$ is defined by:$(\Delta_a^{-\alpha}\digamma)(t) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\underline{\alpha-1}} \digamma(s)$ for $t \in \mathbb{N}_{a+\alpha}$,where $(t - \sigma(s))^{\underline{\alpha-1}}$ is the generalized falling function.

5. Composing Fractional Sum and Difference Operators with some properties

Theorem 5.3 (Composing a Difference with Sum) [6]: *Let* $\digamma : \mathbb{N}_a \to \mathbb{R}$ *be given, and suppose* $\alpha, \mu > 0$ *with* $\Upsilon - 1 < \alpha \leq \Upsilon$*.Then we have,* $\Delta_{a+\mu}^{\alpha} \Delta_a^{-\mu} \digamma(t) = \Delta_a^{\alpha-\mu} \digamma(t)$, *for* $t \in \mathbb{N}_{a+\mu+\Upsilon-\alpha}$

Definition 5.4 [9]: Let. $\Upsilon - 1 < \alpha < \Upsilon$ the α th fractional sum with respect to t of $\digamma(x, t)$ is defined by ${}_t\Delta^{-\alpha} \digamma(x, t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{\Upsilon-\alpha} (t - s - 1)^{\underline{-\alpha-1}} \digamma(x, s)$.

6. Main Results

This section introduces the discrete Sumudu transform with successive approximation methods. We present the solution of the type partial difference equation using a combination involving the Sumudu transformation with the successive approximation method.

Problem 6.1.1: Consider the fractional partial difference equation

$$ {}_t^c\Delta_{a+v-N}^v y(x, t) - {}_x\Delta_a^n y(x, t) = g(x, t) \; x, t \in \mathbb{N}_0 , \tag{6}$$

$$ \Delta^i y(x, a+v-N) = \boldsymbol{y_k}(x, 0) = \boldsymbol{f_k}(x, \mathbf{0}) \tag{7}$$

Proof: By taking Sumudu transform for Equation (6),

$\mathbb{S}\{{}_t^c\Delta_{a+v-N}^v y(x, t)\} - \mathbb{S}\{{}_x\Delta_a^n y(x, t)\} = \mathbb{S}\{g(x, t)\}$, Using Theorem 3.3,

$$ y(x, t) = \left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x, 0)\right\} + \mathbb{S}_{a+v-N}^{-1}\left\{\frac{u^N}{(u+1)^{N-v}} \mathbb{S}_a\left\{{}_x\Delta_a^n y(x, t) + g(x, t)\right\}\right\}$$

Now, applying the method of successive approximation method,

$y_m(x,t) = \left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,0)\right\} + +\mathbb{S}_{a+v-N}^{-1}\{\frac{u^N}{(u+1)^{N-v}} \mathbb{S}_a\{ {}_x\Delta_a^n y_{m-1}(x,t) + g(x,t)\}\}\}, m = 1,2,3, \ldots$ (8)

If $= 1$, $y_1(x,t) = \sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,0) + {}_t\Delta^{-v}\ {}_x\Delta^n \boldsymbol{f_o}(x,0) + {}_t\Delta^{-v} g(x,t)$

Then, $y_m(x,t) = \left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,0)\right\} + \Big\{ {}_t\Delta^{-v}\ {}_x\Delta_a^n \left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,0)\right\} + \Big\{\Big({}_t\Delta^{-2v}\ {}_x\Delta_a^{2n} \sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,0) + {}_t\Delta^{-3v}\ {}_x\Delta^{3n} \boldsymbol{f_o}(x,0) + \cdots + {}_t\Delta^{-mv}\ {}_x\Delta^{mn} \boldsymbol{f_o}(x,0) + {}_t\Delta^{-mv}\ {}_x\Delta_a^{(m-1)n} g(x,t)\Big)\Big\} + {}_t\Delta^{-v} g(x,t)\Big\}$

Then the solution $y(x,t) = \lim_{m\to\infty} y_m(x,t)$

Now, we solve the problem (8)-(9) by using the method Sumudu transform with Homogony perturbation.

Theorem 6.1.2: *The solution of equation (8)-(9) is given by*

$$y(x,t) = \lim_{p\to 1} \sum_{j=0}^{\infty} p^j \vartheta_j(x,t)$$

Proof: By taking Sumudu transform for Equation (6), using Theorem 3.2 ,

$$[\mathbb{S}\{y(x,t)\}(u) - \sum_{k=0}^{N-1} u^k \boldsymbol{f_k}(x,0)] = \frac{u^N}{(u+1)^{N-v}} \mathbb{S}\{ {}_x\Delta_a^n y(x,t) + g(x,t)\}$$

Using Lemma 2.1 (ii),

$$y(x,t) = \left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,0)\right\} + \mathbb{S}_{a+v-N}^{-1}\{\frac{u^N}{(u+1)^{N-v}} \mathbb{S}_a\{ {}_x\Delta_a^n y(x,t) + g(x,t)\}\}$$

We construct Homotopy in the way that is described below:

$$\{\vartheta(x,t)\}(p) = y_0(x,t) - p y_0(x,t)(u) + \mathrm{p}\left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,t)\right\}$$

$$+p\mathbb{S}_{a+v-N}^{-1}\{\frac{u^N}{(u+1)^{N-v}} \mathbb{S}_a\{ {}_x\Delta_a^n y(x,t) + g(x,t)\} \qquad (20)$$

We assume

$$\{\vartheta(x,t)\}(p) = \sum_{j=0}^{\infty} p^j \vartheta_j(x,t) \qquad (21)$$

Substituting equation (21) into (20), then

$(p^0\vartheta_0(x,t) + p^1\vartheta_1(x,t) + p^2\vartheta_2(x,t) + \cdots) = y_0(x,t) - p y_0(x,t)(u) + p\left\{\sum_{k=0}^{N-1} \frac{t^k}{k!} \boldsymbol{f_k}(x,t)\right\} + p\ {}_t\Delta^{-v}\Big\{\{ {}_x\Delta_a^n (p^0\vartheta_0(x,t) + p^1\vartheta_1(x,t) + p^2\vartheta_2(x,t) + \cdots)\} + \{g(x,t)\}\Big\}$, When we do this, we make sure to compare the coefficients of the term with the same power of p, then,p^0: $\vartheta_0(x,t) = y_0(x,t) = \boldsymbol{f_o}(x,0)$then,$p^m$: $\vartheta_m(x,t) = {}_t\Delta^{-v}\ {}_x\Delta_a^n \vartheta_{m-1}(x,t) = {}_t\Delta^{-v}\ {}_x\Delta_a^n \vartheta_{m-1}(x,t)$

Then the solution$y(x,t) = \lim_{p\to 1} \sum_{j=0}^{\infty} p^j \vartheta_j(x,t)$.

7. Implementations

In these sections, the test examples of initial value problems of partial difference equations are implemented to examine the efficiency of the proposed method.

Example 7.1: Consider the initial value problem difference equation

$$ {}_t\Delta_0^{0.8} y(x,t) + {}_x\Delta^2 y(x,t) - y(x,t) = \Gamma(0.8)2^x t^{\underline{1}} $$

$$ x,t \in N_0 \ , = 2 \ , g(x,t) = \Gamma(0.8)2^x t^{\underline{1}} \tag{12} $$

$$ y(x,0) = \boldsymbol{y_o}(x,t) = \boldsymbol{f_o}(x,t) = 2^x \tag{13} $$

then the exact solution is $y(x,t) = 2^x t^{\underline{-0.2}}$, then

$$ y(x,t) = 2^x + \mathbb{S}_{0.2}^{-1}\big[\frac{(u+1)^{0.2}}{u^1}\mathbb{S}\{ {}_x\Delta^2 y(x,t) - y(x,t) + \mathbb{S}\{\Gamma(0.8)2^x t^{\underline{1}}\}\}\}\big] \tag{14} $$

Now, applying the method of successive approximation method Hence,

$$ y_m(x,t) = 2^x + \mathbb{S}_{0.2}^{-1}\big[\frac{(u+1)^{0.2}}{u^1}\mathbb{S}\{ {}_x\Delta^2 y_{m-1}(x,t) - y_{m-1}(x,t) + \{\Gamma(0.8)2^x t^{\underline{1}}\}\}\}\big] $$

IF m=1, then,

$$ y_1(x,t) = 2^x + \mathbb{S}_{0.2}^{-1}\big[\frac{(u+1)^{0.2}}{u^1}\mathbb{S}\{ {}_x\Delta^2 2^x - 2^x + \{\Gamma(0.8)2^x t^{\underline{1}}\}\}\}\big], $$

Then, $y_1(x,t) = 2^x + {}_t\Delta^{-1}\{\Gamma(0.8)2^x t^{\underline{1}}\} = 2^x + 2^x t^{\underline{1}} = 2^x[1 + t^{\underline{-0.2}}]$

If m=2, then, $y_2(x,t) = 2^x[1 + t^{\underline{-0.2}}]$, if m=3, then, $y_3(x,t) = 2^x[1 + t^{\underline{-0.2}}]$.

The solution $y(x,t) = \lim_{m\to\infty} y_m(x,t)$, $y(x,t) = 2^x[1 + t^{\underline{-0.2}}]$

Now, solve the problem by Homotopy perturbation Sumudu transform Method.

In equation (14), we have, $y(x,t) = 2^x + \mathbb{S}_{0.2}^{-1}\big[\frac{(u+1)^{0.2}}{u^1}\mathbb{S}\{ {}_x\Delta^2 y(x,t) - y(x,t) + \mathbb{S}\{\Gamma(0.8)2^x t^{\underline{1}}\}\}\}\big]$,We construct Homotopy in the as following :$\{\vartheta(x,t)\}(p) = 2^x$

$$ +p\mathbb{S}_0^{-1}\{\frac{u^1}{(u+1)^{0.2}}\mathbb{S}_a\{ {}_x\Delta_a^2 y(x,t) - y(x,t) + \Gamma(0.8)2^x t^{\underline{1}}\} \tag{15} $$

We assume

$$ \{\vartheta(x,t)\}(p) = \sum_{j=0}^{\infty} p^j \vartheta_j(x,t) \tag{16} $$

Substituting equation (16) into (15), then

$$ (p^0\vartheta_0(x,t) + p^1\vartheta_1(x,t) + p^2\vartheta_2(x,t) + \cdots) = $$

$$ 2^x + p\ {}_t\Delta^{-v}\Big\{\{ {}_x\Delta_a^2(p^0\vartheta_0(x,t) + p^1\vartheta_1(x,t) + p^2\vartheta_2(x,t) + \cdots)\} - $$

$$ \{(p^0\vartheta_0(x,t) + p^1\vartheta_1(x,t) + p^2\vartheta_2(x,t) + \cdots)\} + \Gamma(0.8)2^x t^{\underline{1}}\Big\}, $$

We compare the coefficients of the term with the identical power of p,

$$ p^0 : \vartheta_0(x,t) = y_0(x,t) = \boldsymbol{f_o}(x,0) = 2^x, $$

then, $p^m: \vartheta_m(x, t) = {}_t\Delta^{-0.8}\ {}_x\Delta_a^2 \vartheta_{m-1}(x, t) = 2^x t^{-0.2}$, Then the solution $y(x, t) = \lim_{p \to 1} \sum_{j=0}^{\infty} p^j \vartheta_j(x, t) = \vartheta_0(x, t) + \vartheta_1(x, t) + \vartheta_2(t) + \vartheta_3(x, t) + \cdots = 2^x$

In Example 7.1, two methods were used to test the applications of the test examples for the initial value problems of fractional partial difference equations.

8. Discussion and Conclusion

This paper suggests a method based on the discrete Sumudu transformation for solving fractional-order partial difference equations. The discrete Sumudu conversion features were put to use. First, the successive approximation method was used to examine how well the test instances applied to the initial value problems of the fractional partial difference equations as shown in the following Figures 1 to 4.

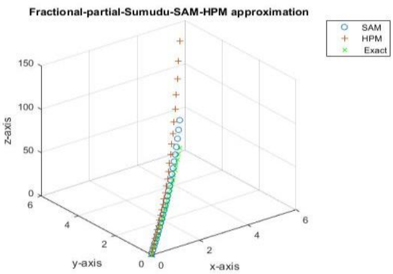

Figure 1
FPΔE - SAM-HPM [0,5], m=10

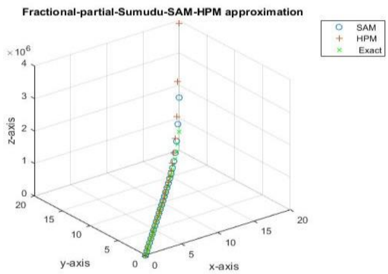

Figure 2
FPΔE - SAM-HPM [0,20], m=10

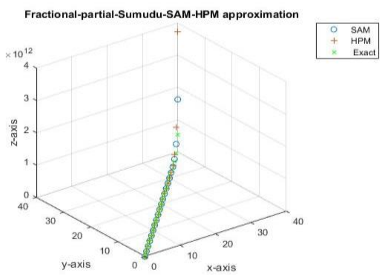

Figure 3
FPΔE - SAM-HPM [0,40], m=10

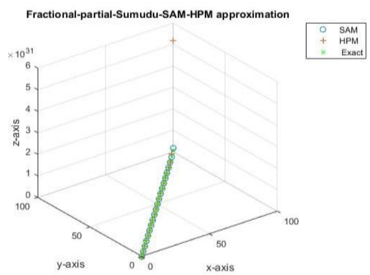

Figure 4
FPΔE - SAM-HPM [0,100], m=10

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Received October, 2023