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Solution homogeneous and non-homogeneous fractional ordinary differential equations by ZMA-transform

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Abstract

In this view, a modern vision to discuss homogeneous and non-homogeneous fractional ordinary differential equations (HFODEs)- (NHFODEs) with initial condition (IC) by employment ZMA-transform (ZMAT). Out of exhibit basic properties for theorems of ZMA-transform (ZMAT) and fractional calculus(FC). Moreover presented ZMA-transform (ZMAT) of Riemann-Liovill derivative the consequent application of ZMA-transform (ZMAT) of fractional derivatives and to understand the task more, some applications has been solved.

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Keywords: *Fractional, ZMA-transform, Derivative, Differential equations, Homogeneous.*

1. Introduction

In the past, there are many literature intersect topic of (FC) in widely fields of science and engineering. Furthermore offered famous integral transform it is effective tool for settling various differential equations (DE),integral equations (IE) like Sumudu, Kamal, and ZMA transform method in [6], [8-12], and activate it to convert system of (DE), (IE) to algebraic equations easily solved [1-4], [7]. The objective of survey, the aim to discovery exact solution of (HFODE) and (NHFODE) by using (ZMAT). Also present elementary properties of (FC) and (ZMAT) [8-9], [11]. Also show the transform efficiency ,applicability and attractively by discuses theorems and settling applications of (HFODEs) and (NHFODEs).

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2. Methodology of (FC) and (ZMAT)

2.1 *Major Acquaintance of (FC)*

In this part, recall some of the primary properties of the (FC). Fractional derivatives (and integrals as well) definitions may diversified, in [12] Abel-Riemann (A-R) is most commonly used .Closer the terminology [3],a derivative of fractional order in the A-R sense realized as Anywhere $m \in z^{+}$ and $\alpha \in \mathcal{R}^{+}$ [12]. D^{α} is the derivative operator:

$$D^{\alpha}\left[\mathrm{N}(t)\right] = \begin{cases} \frac{1}{\Gamma[m-\alpha]}\frac{d^m}{dt^m}\int_0^t \frac{\mathrm{N}(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\tau \; m-1 < \alpha \le m \\ \frac{d^m}{dt^m}\,\mathrm{N}(t) \; \alpha = m \end{cases} \quad ...(1)$$

$$D^{-\alpha}\left[\mathrm{N}(t)\right] = \frac{1}{\Gamma[\alpha]}\int_0^t ((t-\tau)^{\alpha-1}\mathrm{N}(t)\, d\tau \,, t > 0 \,, 0 < \alpha \le 1 \quad ...(2)$$

From A-R ,J^{α} is the integral operator

$$J^{\alpha}[\mathrm{N}(t)] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\mathrm{N}(\tau)\, d\tau \,, t > 0 \,, \alpha > 0 \quad ...(3)$$

In [5] a mong share primary properties of fractional as:

$$J^{\alpha}\left[t^{n}\right] = \frac{\Gamma[1+n]}{\Gamma[1+n+\alpha]}\, t^{n+\alpha} \quad ...(4)$$

$$D^{\alpha}[t^{n}] = \frac{\Gamma[1+\mathrm{n}]}{\Gamma[1+n-\alpha]}\, t^{t-\alpha} \quad ...(5)$$

The fractional derivative of that Caputo has main definition in[1,2] , Caputo fractional derivative [4] as:

$$^{\mathrm{c}}\mathrm{D}^{\alpha}\left[\mathrm{N}(t)\right] = \begin{cases} \frac{1}{\Gamma[m-\alpha]}\int_0^t \frac{\mathrm{N}^{m}(\tau)}{(t-\tau)^{\alpha-m+1}}\, d\,\tau \,, m-1 < \alpha < m \\ \frac{d^m}{dt^m}\,\mathrm{N}(t) \,, \alpha = m \end{cases} \quad ...(6)$$

$$J^{\alpha}[\,^{\mathrm{c}}\mathrm{D}^{\alpha}\left[\mathrm{N}(t)\right] = \mathrm{N}(t) - \sum_{\mathrm{k}=0}^{\infty}\mathrm{N}^{(k)}\;(0^{+})\frac{t^{k}}{k!} \quad ...(7)$$

2.2 *Major Facts of (ZMAT)*

The (ZMAT) defined the functions in [8] like:

$$\Lambda\{\wp(t)\} = \left\{\not{f}(t) \colon \exists M, \Omega_1, \Omega_2 > 0, |\wp(t)| < Me^{-\frac{t}{sv}}\, if\; t \in (-1)^{j} \times [0 \times [0, \infty)\right\} \quad ...(8)$$

Where $\wp(t)$ exposure lik:

$$\mathbb{Z}_{MA}(v; s) = \Lambda\{\not{f}(t)\} = \frac{1}{s}\int_0^{\infty} \not{f}(vt)e^{-\frac{t}{s}}dt \,, 0 \le t \,, \Omega_1, \Omega_2 > 0 \quad ...(9)$$

Or

$$\mathbb{Z}_{\mathrm{MA}}(v; s) = \Lambda\{\not{f}(t)\} = \frac{1}{sv}\int_0^{\infty} \not{f}(t)e^{-\frac{t}{sv}}dt \,, 0 \le t \,, sv \in (\Omega_1, \Omega_2) \quad ...(10)$$

In the following some results of (ZMAT) [8-9]:

Theorem 2.1: *The (ZMAT) of $\mathscr{f}(t)$ is $\mathbb{Z}_{MA}(v;s)$ then the(1^{st}, 2^{nd} and n^{th}) derivatives of $\mathscr{f}(t)$ by (ZMAT) [8] as:*

$$\Lambda\,[\mathscr{f}'(t)] = \frac{1}{sv}\,\mathbb{Z}_{MA}(v;s) - \frac{1}{sv}\,\mathscr{f}(0) \qquad ...(11)$$

$$\Lambda\,[\mathscr{f}''(t)] = \frac{1}{(sv)^2}\,\mathbb{Z}_{MA}(v;s) - \frac{1}{(sv)^2}\,\mathscr{f}(0) - \frac{1}{sv}\,\mathscr{f}'(0) \qquad ...(12)$$

So on n^{th} order like: $\Lambda\,[\mathscr{f}^n(t)] = \frac{1}{(sv)^n}\,\mathbb{Z}_{\mathrm{MA}}(v;s) - \sum_{k=0}^{n-1}\frac{1}{(sv)^n}\,\mathscr{f}^{\mathrm{n}}(0)$ ...(13)

Theorem 2.2: *The (ZMAT) of the Riemann- Liouville fractional derivative as introduce :*

$$\Lambda\,[D^{\alpha}\mathscr{f}(t)] = (sv)^{-\alpha}\,[\,\mathbb{Z}_{\mathrm{MA}}(v;s) - \sum_{k=1}^{n}\,(sv)^{\alpha-k}\,[\mathrm{D}^{\alpha-k}(\mathscr{f}(t))]_{t=0}\,],$$
$$-1 < n-1 \le \alpha < n \qquad ...(14)$$

Theorem 2.3: *Let $\mathscr{f}(t)$ is afunction and $t > 0$,then by make (ZMAT)as:*

$$\Lambda\{\mathrm{t}^{n}\} = \mathrm{n!\,s^n v^n} = \Gamma(\mathrm{n}+1)\mathrm{s^n v^n} \qquad ...(20)$$

$$\Lambda\{\mathrm{t}\mathscr{f}(t)\} = \mathrm{sv}^2\frac{\mathrm{d}}{\mathrm{dv}}(\mathbb{Z}_{\mathrm{MA}}(\mathrm{v,s})) + \mathrm{sv}\mathbb{Z}_{\mathrm{MA}}(\mathrm{v,s}) \qquad ...(21)$$

$$\Lambda\{t\mathscr{f}(t)'(\mathrm{t})\} = \mathrm{sv}^2\frac{\mathrm{d}}{\mathrm{dv}}\left(\frac{1}{\mathrm{sv}}\mathbb{Z}_{\mathrm{MA}}(\mathrm{v,s})\right) + \mathbb{Z}_{\mathrm{MA}}(\mathrm{v,s}) \qquad ...(22)$$

$$\Lambda\{\mathrm{t}\mathscr{f}(t)''(\mathrm{t})\} = \frac{1}{\mathrm{s}}\frac{\mathrm{d}}{\mathrm{dv}}(\mathbb{Z}_{\mathrm{MA}}(\mathrm{v,s})) - \frac{1}{\mathrm{sv}}\mathbb{Z}_{\mathrm{MA}}(\mathrm{v,s}) + \frac{1}{\mathrm{sv}}\mathscr{f}(0) \qquad ...(23)$$

Theorem 2.4: *If $\mathscr{f}(t)$ is a piece wise periodic function and period $\boldsymbol{\wp}$, then by*

$$\text{(ZMAT)}:\Lambda\{\mathscr{f}(t)\} = \frac{1}{sv\,(1-e^{-\frac{\wp}{sv}})}\int_0^{\wp}\,\mathscr{f}(t)e^{-\frac{t}{sv}}dt\,, sv > 0 \qquad ...(24)$$

3. Applications

Application 3.1: Consider the general linear (FODE) as:

$$\frac{\partial^{\alpha}\mathcal{H}(\mathrm{t})}{\partial \mathrm{t}^{\alpha}} = \frac{\partial^2\mathcal{H}(\mathrm{t})}{\partial \mathrm{t}^2} + \frac{\partial\mathcal{H}(\mathrm{t}))}{\partial \mathrm{t}} + \mathcal{H}(\mathrm{t}) + \mathrm{c} \qquad ...(25)$$

With (IC) $\qquad \mathcal{H}(0) = \mathscr{y}(0)$ ...(26)

Solution: Take (ZMAT) of (25), after some steps find:

$$\rightarrow\ \mathcal{H}(t) =\ \Lambda^{-1}\left[\frac{1}{1-(sv)^{\alpha-2}-(sv)^{\alpha-1}-(sv)^{\alpha}} \times (\mathfrak{P}\,(t) -\ (sv)^{\alpha-1}\,\mathscr{y}(0) - (sv)^{\alpha-2}\,\mathscr{y}(0) + (sv)^{\alpha}\,c\,)\right] \qquad ...(28)$$

Application 3.2: Examine the (NHFODEs)

$$D^{\alpha}[\mathcal{H}(t)] = -\mathcal{H}(t) + \frac{2}{\Gamma[3-\alpha]}t^{2-\alpha} - \frac{1}{\Gamma[2-\alpha]}t^{1-\alpha} + t^2 - t\,, t > 0, 0 < \alpha \le 1 \qquad ...(29)$$

With $\qquad \mathcal{H}(0) = 0$ ...(30)

Solution: by (ZMAT) of (29),$k = 1$and use (IC) → $\mathcal{H}(t) = 2\,t^2 - t$ ...(31)

Application 3.3: Investigate the (NHFODEs):

$$\mathrm{D}^{0.5}[\mathcal{H}(t)] + \mathcal{H}(t) = \mathrm{t}^2 + \frac{\Gamma[3]}{\Gamma[2.5]}\mathrm{t}^{1.5}\,, \mathrm{t} > 0 \qquad ...(32)$$

Subject to (IC) $[\mathrm{D}^{0.5}(\mathcal{H}(t)|_{t=0})] = 0$ ...(33)

Solution: Occupy the (ZMAT) of (32) , $k = 1 \rightarrow \mathcal{H}(t) = t^2$ ...(34)

Application 3.4: Look the (NHFODEs):

$$D^{\frac{3}{2}}\mathcal{H}(t) + D\mathcal{H}(t) = 1 + t \qquad ...(35)$$

With (IC) $\mathcal{H}(0) = \mathcal{H}'(0) = 0$ and $\left[\mathrm{D}^{-\frac{1}{2}}(\mathcal{H}(t)|_{t=0})\right] = 0$ ...(36)

Solution: Utilize the (ZMAT) of(35) ,$k = 1 \rightarrow \mathcal{H}(t) = t + \frac{t^2}{2!}$ ...(37)

Application 3.5: Look the (NHFODEs):

$$D^{\frac{1}{2}}\mathcal{H}(t) + \mathcal{H}(t) = \frac{1}{2}t + t^{\frac{1}{2}} \qquad ...(38)$$

With (IC) $\left[\mathrm{D}^{-\frac{1}{2}}(\mathcal{H}(t)|_{t=0})\right] = 0$ ...(39)

Solution: Apply the (ZMAT) of (38), take $k = 1$, using (IC) and take inverse

$$\text{(ZMAT)} \rightarrow \mathcal{H}(t) = \frac{1}{2}\,t \qquad ...(40)$$

Application 3.6: Let the (HFODEs):

$$\frac{d\mathcal{H}(t)}{dt} + \frac{d^{1/2}\mathcal{H}(t)}{dt^{1/2}} - 2\mathcal{H}(t) = 0 \qquad ...(41)$$

With (IC) $\frac{d^{1/2}\mathcal{H}(0^+)}{dt^{1/2}} = c, t > 0\,, \mathcal{H}(0^+) = 0$ ...(42)

Solution: Apply (ZMAT) of (41), $k = 1$, using (IC) after that take usage(ZMAT) obtain analytical solution:

$$\mathcal{H}(t) = \frac{2c}{3}\left[e^{4\,t} - e^{4\,t}\,\mathrm{erf}\big(2\sqrt{2\,t}\big) + \frac{c}{3}\left[e^t + e^t\,\mathrm{erf}(\sqrt{t})\right]\right] \qquad ...(43)$$

5. Discussion and Conclusion

In this paper , proposed anew efficient transform named the (ZMAT) to settle (HFODEs) and (NHFODEs) with (IC) in the Riemann-Liovill derivative sense is easily computable and directly apply to present the analytical solutions of initial value problem. The method used here is attractive and effective tool in terms of finding exact solutions. In the future, hope to extend the (ZMAT) method to solve boundary value problems with consideration for Higher fractional order differential equations.

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