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Semi-complement Co-Hopfian *R-mods*

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Abstract

This study presents a semi-complement Co-Hopfian *R-module* (si-Cp Co-Hop. mod). The proposal is a monomorphic endomorphism of *X* with a si-comp image, which is a generalization of the Co-Hop. Mod. This article introduces some properties of the si-Cp Co-Hop, mod, and an investigation with examples. Furthermore, the current study deals with this class of mods and gives several fundamental properties related to this concept.

Subject Classification: *11J93, 11G09.*

Keywords: *Semi-complement submod, Co-Hopfian module, Semi-complement co-Hopfian module.*

1. Introduction

Let R be a commutative ring with unity, and X be a unitary left R-mod unless otherwise specified. Co-Hop. Mods have been studied by many authors since 1960. The mod X is called Co-Hop. if any monomorphic(mon.) endomorphism(end.) of X is an isomorphism(iso.)[12]. For instance, we can say that any Artinian mod is Co-Hop. Also, a sub-mod A of a mod X is called a complement(comp). For a submod B of X if it is maximal concerning the property that A∩B=0 [11]. This paper introduces a semi-comp sub-mod as a generalization of the comp submodel.

A sub-mod A of an R-mod X is called a semi-cp for a sub-mod B of X if it is semi-maximal concerning the property that A∩B=0. Also, the si-Cp Co-Hop mod has been presented as a generalization of the Co-Hop mod. A mod X is said to be a si-Cp Co-Hop if any mon end of X has a semi-cp image. The intersection

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scheme of significant sub-mods of M, indicated by G(M), is an unguided simple graph that its vertices are in one-to-one correspondence with all significant sub-mods of M and two featured vertices are adjacent if and only if the intersection of corresponding sub-mods is a non-zero sub-mod of M[18]. The main goal of our proposal is to study these concepts and present some of their basic properties, characterizations, and examples. In addition to the above, their relationship with other mods classes is studied.

2. Basic Properties of Semi-Cp Co-Hop. Mods

This section presents the concept of semi-Cp Co-Hop mods and the basic properties of this type of module.

Definition 1: A submod A of a mod X is named semi-comp for a submodel B of X if it is semi-maximal concerning the property that $A \cap B=0$. Since each maximal submod is semi-maximal [13], we can get the following remarks.

Remark 1: Each comp submodel is a semi-comp.

Remark 2: If A is a semi-comp sub-mod of X such that $[A:B]=[A:X]$ for every sub-mod B of X containing A, then A is a comp. sub-mod of X.

Definition 2: A mod X is named a semi-Cp Co-Hop mod if for any mon $f \in End(X)$, then $Im\, f$ is a semi-comp submod of A in X for some submod A of X.

Example 1: Z_6 is a semi-Cp Co-Hop. Z-mod**.**

Remark 3: Each Co-Hop mod is a semi-Cp Co-Hop Mod, But the converse is not generally valid, as we see in the following example**.**

Example 2: **X**$=Q^n = Q \oplus Q \oplus \ldots \oplus Q$ as a Z-mod is a semi-Cp Co-Hop mod but not a Co-Hop mod.

The following prop shows the converse of (remark 3) held in the class of Dedekind finite mods.

Recall that a mod X is named Dedekind finite mod if $X=X \oplus A$ for some non-zero submod A of X [6].

Proposition 1: Let X be a Dedekind finite mod if X is a semi-Cp Co Hop mod such that for each submod A of X $[A{:}E]=[A{:}X]$ for every submod E of X containing A. Then X is a Co-Hop mod.

***Proof*:** Let X be a mon mod; since X is a semi-Cp Co-Hop mod, then $Im\, f$ is semi-comp, sub mod for A in X for sub-mod A, so it is comp. (remark 2), also, it is a direct summand [14]. Thus $X=Im\, f \oplus B$, for some submod B of X. Define a homomorphism $g: X \oplus B \to B$ by $g(x,b)=f(x) + b$ where it is clear that g is well-define. Since X is a Dedekind finite implies $B=0$, Thus, $Im\, f =X$. Hence, f is an isom. Therefore, X is a Co-Hop mod. By the same argument in the proof of Prop.1, the following corollary are hold.

Corollary 1: *Let X be a Dedekind finite mod, such that for each submod A of X $[A:E]=[A:X]$ for every submod E of X containing A. Then X is a semi-Cp Co-Hop mod if and only if X is a Co-Hop mod.*

The following result is true since a simple mod has no proper submods [11].

Proposition 2: Each simple mod is a semi-Cp Co-Hop mod**.**

Example 3: Z_6 is a semi-Cp Co-Hop. Z-mod, but it is not simple.

Before we give our next result, we need to prove the following lemma.

Lemma 1: *Each direct summand sub-mod is a semi-comp sub-mod*

***Proof*:** Let X be a mod and let A be a direct summand submod of X. Thus, A is a closed submod by [14]. Since each closed submod is a comp by [9], then by(remark (1)) A is a semi-comp submod of X.

It is well known that each sub-mod of a semi-simple mod is a direct summand[16]. The only sub-mods of simple mod X are 0 and X . $X=0\oplus X$, so the simple mod is a semi-simple.

Proposition 3: Each semi simple mod is si-Cp Co-Hop mod .

Remark 4: Any sub-mod of a semi-simple mod is a semi-Cp Co-Hop

Proposition 4: Each comp. submod of semi-Cp Co-Hop mod is semi-Cp Co-Hop mod**.**

***Proof*:** Let X be a semi-Cp Co-Hop mod and let A be a comp sub-mod of X, then by [14], A is a direct summand sub-mod of X, hence $X=A\oplus B$ for some sub-mod B of X. Let $f: A\to A$ be a mon function and let $g: X\to X$ be a function defined by $g(a,b) = f(a)+b$ and it is a mon. Since X is a semi-Cp Co-Hop mod, thus Img is a semi-comp. submod of A in X. Since $Im\ g = X = A \oplus B = Im\ f \oplus B$, $Im\ f$ is a direct summand submod of A, so $Im\ f$ is a comp. submod of A [14]. So $Im\ f$ is a semi-comp. (remark1). Therefore X is a semi-Cp Co-Hop mod. By remark 1, each comp sub-mod is a semi-comp; hence, the following result is hold.

Since each comp is a direct summand by [14] and each direct summand submod is closed by [14]. Also, each direct summand sub-mod is pure by [16], so the following corollary is true.

Corollary 2: *Each semi-comp. (direct summand), (closed), (pure) submod of semi-Cp Co-Hop R-mod is a semi-Cp Co-Hop R-mod***.**

Proposition 5: X is semi-Cp Co-Hop mod if and only if *a submod A of X* is a semi-comp such that $A \cong X$.

Proof **:** Let A be a sub-mod of X such that $A \cong X$, then there exist

$f{:}X \rightarrow A$ is an is om, hence $X \xrightarrow{f} A \xrightarrow{i} X$, where $i : A \rightarrow X$ is the inclusion mapping, and that implies $i_{o}f \in End\ (X)$, $i_{o}f\ (X)$ is mon. So $(i_{o}f)\ (X)$ is a semi-comp submod in X . Thus $(i_{o}f)(X) = i\ (f(X)) = i(A)=A$ is a semi-comp in X.

Let $f \in End\ (X)$, f is mon. Hence, $Im\ f \cong X$, so by the assumption, $Im\ f$ is si-comp submod in X. Therefore, X is a si-Cp Co-Hop mod.

Proposition 6: Let X be a mod, and A is a si-comp sub-mod of X. If A is a semi-Cp Co-Hop mod, X is a semi-Cp Co-Hop mod.

Proof: Suppose that X is not a semi-Cp Co-Hop mod. Then there exists a sub-mod A of X such that $X \cong A$(prop. 5), A is not a semi-comp sub-mod of X, then A is not a semi-Cp Co-Hop mod; this is a contradiction. Therefore, X is a semi-Cp Co-Hop mod.

Corollary 3: *Let A be a semi-comp sub-mod of a mod X . Then A is a semi-Cp Co-Hop mod if and only if X is a semi-Cp Co-Hop mod.*

A sub-mod A of a mod X is essential if every non-zero submod of X has a non-zero intersection with A [10]. A mod X is called extending if every sub-mod A of X is essential in a direct summand of X [7]. Equivalently, every closed sub-mod is a direct summand of X.

Proposition 7: Let X be an extending mod, and A is a direct summand submod of X. If A is a semi-Cp Co-Hop mod, X is a semi-Cp Co-Hop mod.

Proof: Suppose that X is not a semi-Cp Co-Hop mod; then there exists a sub-mod A of X such that $X \cong A$. If A is not a semi-comp sub-mod of X, since X is an extending mod, which implies that A is not a direct summand, then A is not a semi-Cp Co-Hop mod by corollary (6), which is a contradiction. Therefore, X is a semi-Cp Co-Hop mod.

Corollary 4: *Let A be a direct summand of an extending mod X. Then A is a semi-Cp Co-Hop mod if and only if X is a semi-Cp Co-Hop mod.*

4. Fully Stable and Duo Mods with Semi-Cp Co-Hop Mod

This section studies fully stable and duo mods with semi-Cp Co-Hop mods. First, the following definitions are needed.

A sub-mod A of a mod X is called stable if $f(A) \subseteq A$, for each homomorphism $f{:}\ A \rightarrow X$ [9]. A mod is called fully stable if each sub-mod is stable[1].

A sub-mod A of a mod X is called invariant (fully invariant) if for every end $f: X \to X, f(A) \subseteq A$ [15]. A mod X is called a duo if every sub-mod is fully invariant [15] .

Proposition 8: Let X be a mod with $X= X_1 \oplus X_2$ where X_1 and X_2 are stable X submods of X. If X1 and X2 are semi-Cp Co-Hop mods, X is a semi-Cp Co-Hop mod.

***Proof*:** Let $f: X_1 \to X_2$ be a mon such that $f_1 = f|X_1$ and $f_2 = f|X_2$**.** Since f is mon, it implies that f_1 and f_2 are mon. Since X_1 and X_2 are stable sub-mods.Then $f_1(X_1) \subseteq X_1$ and $f_2(X_2) \subseteq X_2$. Since X_1 and X_2 are semi-Cp Co-Hop mods, *Im* $f_{1,}$ and *Im* f_2 are semi-comp submods of X_1 and $X_{2,}$ respectively. Therefore, X_1 and X_2 are semi-Cp Co-Hop mods.

Corollary 5: *Let X be a mod with* $X= X_1 \oplus X_2 \oplus \dots$ *where* X_i *are (fully stable) submods of X for all i=1,2,…. Then X is si-Cp Co-Hop mod if* X_i *is semi-Cp Co-Hop mod.*

5. The Relationship between Semi-Cp Co-Hop Mod and Some Generalizations of Co-Hop Mod

This section introduces a relationship between semi-Cp Co-Hop mod and some generalization of Co-Hop mods. To start this section, some definitions of some generalizations of Co-Hop mod as (semi-Co-Hop, weakly Co-Hop, densely Co-Hop, pseudo-Co-Hop, and quasi Co-Hop) mods are defined. Note that X is a semi-Co-Hop mod if every mon $f: X \to X, f(X)$ is a direct summand of X [5]. Any Co-Hop mod is semi-Co-Hop. But the converse is not true in general [5].

Proposition 9: Each semi Co-Hop mod is a semi-Cp Co-Hop mod

Proof: Let X be a semi-Co-Hop R-mod and let $f: X \to X$ be a mon such that *Im f* is a direct summand, by [5] *Im f* is a comp submod of X, so it is semi-comp (remark(2)). Therefore, X is a semi-Cp Co-Hop mod.

The converse of prop.(10) is not true in general, as seen in the following example :

Example 4: Consider the statement mod $X = Z_8 \oplus Z_2$ as a Z-mod. Let $A = ((2,1))$ be a submod generated by $(2,1)$. $A = \{(0,0),(2,1),(4,0),(6,1)$. Then A is not a direct summand but a semi-comp submod in X.

Proposition 10: If X is an extending and semi-Cp Co Hop mod, then X is semi Co-Hop mod. We can get the following corollary from Proposition (9) and Proposition (10).

Corollary 6: *Let X be an extending mod. Then X is a semi-Cp Co-Hop mod if and only if X is a semi-Co-Hop mod.*

Proposition 11: If X is a mod, then X is a semi-Cp Co-Hop mod if and only if Any submod A of X such that $A \cong X$, A is a semi-comp submod of X.

Proof: Let A be a sub-mod of X such that $A \cong X$, then there exist

$f : X \to A$ is an is om, hence $X \xrightarrow{f} A \xrightarrow{i} X$, where $i : A \to X$ is the inclusion mapping, implies that $i \circ f \in End(X)$, $i \circ f$ is mon. $(i \circ f)(X)$ is a semi-comp in X. Thus $(i \circ f)(X) = i(f(X)) = i(A) = A$ is a semi-comp in X.

Let $f \in End(X)$, f is mon. Hence $Imf \cong X$, so by our assumption, Imf is semi-comp in X. Therefore, X is semi-Cp Co-Hop mod.

Next, we introduce the relationship between the descending chain condition and the semi-Cp Co-Hop mod.

Proposition 12: If X is a mod with descending chain condition on a semi-comp submod of X, Then X is a semi-Cp Co-Hop mod.

Proof: Suppose that X is not a semi-Cp Co-Hop mod. Then there exists A_1 (which is not a semi-comp submod of X) such that $A_1 \cong X$ (prop 11); hence A_1 is not semi-Cp Co-Hop mod, so there exists A_2 is a sub-mod of A_1, which is not semi-comp with $A_1 \cong A_2$. By repeating this argument, we have strictly descending chain $A_1 \supseteq A_2 \supseteq \dots$. Moreover, A_i is not a semi-comp sub-mod in X for $i = 1,2 \dots$ To prove A_2 is not semi-comp in X, if A_2 is semi-comp in X implies A_1 is semi-comp in X, which is a contradiction. Similarly, A_i is semi-comp in X for $i = 3,4,\dots$. Thus $A_1 \supseteq A_2 \supseteq \dots$ is a strictly descending chain of non-semi-comp submod of X, which is a contradiction. Therefore, X is a semi-Cp Co-Hop mod.

Corollary 7: *If X is a mod with descending chain condition on semi-Cp Co-Hop submods, X is a semi-Cp Co-Hop mod.*

Proof: The proof is similar to Proposition 12.

Recall that a mod X is called weakly Co-Hop mod if f(X) is an essential submod of X for any injective end of X [3]. Co-Hop mods are weak and are Dedekind finite [3]. But these mods classes are equivalent to the direct injective mods class [17]. The relation between semi-Cp Co-Hop mod and weakly Co-Hop mod is introduced[19].

Proposition 13: Let X be a Dedekind finite mod such that for each submod A of X $[A:B]=[A:X]$ for every submod B of X containing A. Then X is a semi-Cp Co-Hop mod if X is a weak Co-Hop mod.

Proof: Let X be a semi-Cp Co-Hop mod, and since X is Dedekind finite mod, then it is Co-Hop mod (Prop 1), so it is weakly Co-Hop mod by [14].

Conversely, let X be a weakly Co-Hop mod and let $f \in End(X)$, which is a mon. Since X is a weakly Co-Hop mod, $Im\, f$ is an essential sub-mod of X. By [8], we get *Im f* as a comp submod of *X*, so *Im f* is a semi-comp sub-mod of *X*(remark (1)). Therefore, *X* is a semi-Cp Co-Hop mod.

A mod *X* is called a direct injective (or C_2) if every submod isomorphic to a summand of *X* is itself a summand[17].

Proposition 14: Let *X* be a direct injective mod. Then *X* is a semi-Cp Co-Hop mod if *X* is a weak Co-Hop mod.

A mod *X* is called a singular provided $Z(X)=X$, and is called a nonsingular mod if $Z(X) =0$ [8]. A sub-mod *A* of a mod *X* is called dense if $\frac{X}{A}$ is Goldie torsion. A mod *X* is called Goldie torsion (or Z_2-torsion) if $Z_2(X)=X$, where $Z_2(X)=\{x \in X$: the ascending chain condition *ann(x)* is dense ideal in $R\}$ [2]. An R-mod *X* is called a Densely Co-Hop mod if, for all mon *End f of X, f(X)* is a dense sub-mod of M [2]. Since each dense sub-mod is an essential sub-mod [8], we can say that each densely Co-Hop mod is weak. If *M* is a nonsingular mod, then each essential submod is dense [2]. Thus, the following results are held.

Proposition 15 [2]: Let *X* be a nonsingular mod such that for each submod *A* of *X* $[A:B]=[A:X]$ for every submod *B* of *X* containing A. If *X* is a weakly Co-Hop mod, then *X* is a densely Co-Hop mod.

Proposition 16: Let *X* be a nonsingular direct injective mod. Then *X* is a semi-Cp Co-Hop mod if *X* is a densely Co-Hop mod.

***Proof*:** Since *X* is a semi-Cp Co-Hop mod, then *X* is a weakly Co-Hop mod by propositions (5,9). Thus, *Im f* is an essential sub-mod of *X*. Since *X* is a nonsingular mod, and by[2], *Im f* is a dense sub-mod of *X*. Therefore, *X* is a densely Co-Hop mod.

Conversely, since *X* is a densely Co-Hop mod, *Im f* is a dense sub-mod of *X*, so it is an essential sub-mod. Then *X* is a weak Co-Hop mod, and by proposition (15), we get *X* is a semi-Cp Co-Hop mod.

A mod *X* is pseudo-Co-Hop if for each mon $f \in End(X)$. *f(X)* is closed submod of *X*[16]. Also, any Co-Hop mod is a pseudo-Co-Hop mod[16]. But the converse is not true in general, but if *X* is an extending Dedekind finite, we get pseudo-Co-Hop mod is Co-Hop [16]. The following prop gives the relation between the semi-Cp Co-Hop mod and the pseudo-Co-Hop mod.

Proposition 17: Let *X* be a Dedekind finite mod for each sub-mod *A* of *X* $[A:B]=[A:X]$ for every sub-mod *B* of X containing *A*. Then *X* is a semi-Cp Co-Hop mod if *X* is a pseudo-Co-Hop mod.

Proof: Let *X* be a semi-Cp Co-Hop mod, since *X* is a Dedekind finite mod, *X* is a semi-complement Co-Hop mod (prop(1)), so it is comp. submod of *X* (remark(2)). *Im f* is direct summand [14]. Hence, it is closed. Therefore, *X* is a pseudo-Co-Hop mod. Conversely, Let *X* be a pseudo-Co-Hop mod implies *Im f* is a closed sub-mod of *X*. By [5], *Im f* is comp. [14], so it is semi-comp.

(remark(1)). Therefore, X is a semi-Cp Co-Hop mod. A mod X is called quasi Co-Hop if $X/f(X)$ is singular for all injective ends of X [4].

5. Discussion and Conclusion

This manuscript aims to introduce a new generalized of Co-Hopfian R-mod, which is semi-Cp Co-Hopfian, that each Co-Hopfian R-mod is semi-Cp Co-Hopfian. Still, the converse holds when the module is Dedekind finite. Furthermore, in semi-Cp Co-Hop R-mod, proofed for some concepts via properties such as complement, closed, essential, direct summand and pure submods…etc. Additionally, it shows the relationship between semi-Cp Co-Hop R-mod with some generalizations of Co-Hop R-mods as semi-Co-Hop, weakly Co-Hop, purely Co-Hop, …etc. We recommend studying and developing the concepts of generalization semi-co-hop mods and semi-prime, semi-primary sub-mods on semi-comp Co-Hop mods.

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